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Selfish cops and active robber: Multi-player pursuit evasion on graphs ☆

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ABSTRACT

We introduce and study the game of "Selfish Cops and Active Robber" (SCAR) which can be seen as a multiplayer variant of the "classic" two-player Cops and Robbers (CR) game. In classic CR all cops are controlled by a single player, who has no preference over which cop captures the robber. In SCAR, on the other hand, each of $N-1$ cops is controlled by a separate player, and a single robber is controlled by the N-th player; and *the capturing cop player receives a higher reward than the non-capturing ones*. Consequently, SCAR is an *N-player* pursuit game on graphs, in which each cop player has an increased motive to be the one who captures the robber. The focus of our study is the existence and properties of SCAR *Nash Equilibria* (NE). In particular, we prove that SCAR always has one NE in deterministic positional strategies and (for $N \geq 3$) another one in, generally, deterministic nonpositional strategies. Furthermore, we study conditions which, at equilibrium, guarantee either capture or escape of the robber and show that (because of the antagonism between the "selfish" cop players) the robber may, in certain SCAR configurations, be captured later than he would be in classic CR, or even not captured at all. Finally we define the *selfish cop number* of a graph and study its connection to the classic cop number.

1. Introduction

In this paper we introduce and study the game of "*Selfish Cops and Active Robber*" (SCAR) which can be seen as an *N-player variant* of the "classic" two-player *cops and robbers* (CR) game [2,22].

The rules of SCAR are similar to those of CR: $N-1$ cops and a robber take turns moving along the edges of an undirected finite simple connected graph; the robber is *captured* if at the end of a turn he is located in the same vertex as one or more cops.

However each cop in SCAR is a separate player (while in CR a single player controls all cops). Furthermore, *payoffs* are quite different from those of CR. A complete description will be given in Section 2; the gist of the matter (and the SCAR novelty) is that *in SCAR the capturing cops receive a higher reward than the remaining, non-capturing cops.* As a result, one cop's win is another cop's *partial* loss (as well as the robber's complete loss).

In other words, while in SCAR (as in CR) the robber will try to maximize capture time, each cop has a motive to minimize capture time and an additional motive for the capture to be effected by himself; depending on some game parameters, situations will arise in which a cop will enforce a longer capture time to ensure that he (rather than another cop) captures

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the robber. Hence cop cooperation cannot be taken for granted (the cops are *selfish*); in this respect, SCAR differs essentially from classic CR where a *team* of $N-1$ cops chase a single robber.

SCAR belongs to the extensively studied family of *stochastic games*.[1] For two-player stochastic games see [8] and for the N-player case see [20,28]. More specifically, SCAR is an *N-player pursuit evasion game on graphs, where the interests of each of the N players are in (partial or total) conflict with those of the remaining players*. To the best of our knowledge such games have not been previously studied.

The prototypical pursuit/evasion game played on graphs is the classic CR game introduced in [22,24]; for an extensive recent overview of the subject see the book [2]. While the connection between graph pursuit games and game theory is a natural one, relatively few publications [14–17,19] exploit the "game theoretic approach".[2] In particular, we are aware of only two previous publications (by ourselves) on graph pursuit games involving selfish pursuers [14,17]. There is also some related work [1] involving selfish *searchers*.

Graph pursuit games are also related to several other research areas: reachability games [4,18], recursive games [7,28], combinatorial games (see [11,23] and especially [3]) and *differential* pursuit games [13]. It is worth noting that the idea of selfish pursuers has been occasionally (but not extensively) explored in studies of differential pursuit games [10,25,27].

The rest of the paper is organized as follows. In Section 2 we present the necessary preliminaries (rules, notation etc.) for the analysis of the *three*-player (two cops, one robber) SCAR. In Section 3 we briefly present a game theoretic formulation of a slightly modified version of the classic CR game; this formulation will be useful in the analysis of SCAR presented in later sections. In Section 4 we prove that three-player SCAR admits *Nash equilibria* in deterministic strategies and at least one of these is an equilibrium in *positional* strategies; we also prove several additional properties, regarding the connection of SCAR capturability to the classic *cop number*. In Section 5 we extend our results to N-player SCAR (with $N \geq 2$), and we also define the *selfish cop number* of a graph and study its connection to the classic cop number. We conclude, in Section 6, by presenting variants and extensions of SCAR which can be the subject of future work.

2. Preliminaries

We denote the SCAR game played by N players on G by $\Gamma_N(G|s_0, \gamma, \varepsilon)$; s_0 is the *initial position* and γ, ε are *game parameters* which will be discussed later. Sometimes we simplify the notation to $\Gamma_N(G|s_0)$ and/or $\Gamma_N(G)$. The main task of this section is a rigorous definition of *three-player* SCAR $\Gamma_3(G)$; the generalization to $\Gamma_N(G)$ will appear in Section 5.

"Iff" means "if and only if". The cardinality of set A is denoted by $|A|$; the set of elements of A which are not elements of B is denoted by $A \backslash B$. We use the following sets of integers:

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

Given a graph $G = (V, E)$, for any $x \in V$, $N(x)$ is the *neighborhood* of x: $N(x) = \{y : \{x, y\} \in E\}$; $N[x]$ is the *closed neighborhood* of x: $N[x] = N(x) \cup \{x\}$.

$\Gamma_3(G)$ is played on an undirected, finite, simple connected graph $G = (V, E)$. The *first player* is the cop C_1, the *second player* is the cop C_2 and the *third player* is the robber R. Thus the *player set* is $I = \{C_1, C_2, R\}$ or, for simplicity, $I = \{1, 2, 3\}$.

The game is played in *turns*, numbered by $t \in \mathbb{N}_0$. At the zero-th turn the player initial positions are given; at each subsequent turn, a single player moves. Any player can have the first move and they play in "cyclical" order $\ldots \rightarrow C_1 \rightarrow C_2 \rightarrow R \rightarrow \ldots$. The game ends if the robber is captured, i.e., if at the end of a turn he is in the same vertex as one or more cops; otherwise it continues indefinitely.

A *game position* or *game state* has the form $s = (x^1, x^2, x^3, p)$ where $x^n \in V$ is the position (vertex) of the n-th player and $p \in \{1, 2, 3\}$ is the number of the player who has the next move. The set of *nonterminal* states is

$$S' = V \times V \times V \times \{1, 2, 3\}.$$

We will also need a *terminal state* τ. Hence the full state set is $S = S' \cup \{\tau\}$.

We partition the state set as follows. Define (for each $n \in I$) the set S^n of states in which the n-th player has the next move[3]:

$$S^n = \left\{ s : s = \left(x^1, x^2, x^3, n \right) \in S \right\}.$$

Then the full state set can be partitioned as follows:

[1] Actually all elements of SCAR are *deterministic*; the term "stochastic games" denotes a general game family which contains, as a special case, games deterministically evolving in time.

[2] By this we mean an approach which involves a payoff, defined in terms of strategy functions, and in which the existence of game value (from optimal, minimax strategies) and/or NE is investigated.

[3] Formally speaking and given that SCAR is a stochastic game (as noted before) at every turn all players make a move. There is however at every turn a single player who can choose from a non-singleton set of moves and this is what (for reasons of brevity) we mean by the expression "the player who has the next move".

$$S = S' \cup \{\tau\} = S^1 \cup S^2 \cup S^3 \cup \{\tau\}. \tag{1}$$

An alternative partition of the state set is effected as follows. We define *capture state* sets:

$$S_C^1 = \left\{s : s = (x^1, x^2, x^3, n) \text{ with } x^1 = x^3, x^2 \neq x^3\right\}, \text{ where } R \text{ is captured by } C_1;$$
$$S_C^2 = \left\{s : s = (x^1, x^2, x^2, n) \text{ with } x^1 \neq x^3, x^2 = x^3\right\}, \text{ where } R \text{ is captured by } C_2;$$
$$S_C^{12} = \left\{s : s = (x^1, x^2, x^2, n) \text{ with } x^1 = x^3, x^2 = x^3\right\}, \text{ where } R \text{ is captured by both } C_1 \text{ and } C_2.$$

Now define:

$$S_C = S_C^1 \cup S_C^2 \cup S_C^{12}, \text{ the set of all capture states;}$$
$$S_{NC} = S' \backslash S_C, \text{ the set of all non-capture, non-terminal states.}$$

Then the state set can be partitioned as follows:

$$S = S' \cup \{\tau\} = S_{NC} \cup S_C \cup \{\tau\}. \tag{2}$$

We define $A^n(s)$, the n-th player's *action set* when the game state is $s = \left(x^1, x^2, x^3, m\right)$, by

$$A^n(s) = \begin{cases} N\left[x^n\right] & \text{for } s \in S^n \cap S_{NC}, \\ \left\{x^n\right\} & \text{for } s \in S^m \cap S_{NC} \text{ with } n \neq m, \\ \{\lambda\} & \text{for } s \in S_C, \text{ where } \lambda \text{ is the null move} \\ \{\lambda\} & \text{for } s = \tau. \end{cases}$$

The players' action sets have the following implications on state-to-state transitions:

1. when the n-th player has the move at a non-capture state, he can stay at his current vertex or move to any neighboring vertex, thus producing the next state of the game;
2. when another player has the move at a non-capture state, the n-th player can only stay in his current vertex (trivial move);
3. when the game is in a capture state, every player has only the null move and the game moves to the terminal state;
4. when the game is in the terminal state, every player has only the null move and the game moves to (actually stays in) the terminal state.

State-to-state transitions are described by the *transition function* $\mathbf{T}(s, a)$ which gives the game state resulting when the game is at a position $s \in S$ and the actions profile is $a = (a^1, a^2, a^3)$. The behavior of $\mathbf{T}(s, a)$ is illustrated by some examples as follows:

$$\text{for } s = \left(x^1, x^2, x^3, 1\right) \in S^1 \cap S_{NC} \quad : \mathbf{T}\left(s, (a^1, a^2, a^3)\right) = \left(a^1, x^2, x^3, 2\right),$$
$$\text{for } s = \left(x^1, x^2, x^3, 2\right) \in S^2 \cap S_C \quad : \mathbf{T}(s, (\lambda, \lambda, \lambda)) = \tau,$$
$$\text{for } s = \tau \quad : \mathbf{T}(\tau, (\lambda, \lambda, \lambda)) = \tau.$$

Often we use the following simplified notation: if at t the game state is $s_t = \left(x_t^1, x_t^2, x_t^3, n\right) \in S^n$ and at $t+1$ the n-th player's action is a_{t+1}^n, we write

$$s_{t+1} = \mathbf{T}\left(s_t, a_{t+1}^n\right).$$

A *game history* is a sequence $h = (s_0, s_1, s_2, \ldots)$, where s_t is the state of the game at the t-th turn ($t \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$). We define the following history sets:

$$\text{histories of length } k : H_k = \{h = (s_0, s_1, s_2, \ldots, s_{k-1})\},$$
$$\text{histories of finite length} : H_* = \cup_{k=1}^{\infty} H_k,$$
$$\text{histories of infinite length} : H_\infty = \{h = (s_0, s_1, \ldots, s_t, \ldots)\}.$$

Histories of infinite length can be further partitioned as $H_\infty = H_C \cup H_{NC}$ where

$$\text{histories where capture occurs} : H_C = \{h = (s_0, s_1, \ldots) \in H_\infty : \exists s_t \in S_C\},$$
$$\text{histories where the robber evades indefinitely} : H_{NC} = \{h = (s_0, s_1, \ldots) \in H_\infty : \nexists s_t \in S_C\}.$$

Furthermore, we define the *capture time of history* $h \in H_\infty$ as

$$T_C(h) = \begin{cases} \min\{t : s_t \in S_C\} & \text{if } h \in H_C \\ \infty & \text{if } h \in H_{NC}. \end{cases}$$

We will often use the simpler notation T_C, when the history is clear from the context. Now consider the following cases.

1. If $T_C = 0$ then the initial state is a capture state and $s_t = \tau$ for every $t \in \mathbb{N} = \{1, 2, ...\}$.
2. If $0 < T_C < \infty$ then:
   (a) at the 0-th turn the game starts at some preassigned state $s_0 \in S_{NC}$;
   (b) at the t-th turn (for $t \in \{1, 2, ..., T_C - 1\}$), the game moves to some state $s_t \in S_{NC}$;
   (c) at the T_C-th turn the game moves to some capture state $s_{T_C} \in S_C$;
   (d) at the $(T_C + 1)$-th turn the game moves to the terminal state and stays there for all subsequent turns (for every $t > T_C$, $s_t = \tau$ and the game effectively ends at time T_C).
3. If $T_C = \infty$ then $s_t \in S_{NC}$ for every $t \in \mathbb{N}_0$.

The case $s_0 = \tau$ is uninteresting and hence excluded from consideration.

A *pure, or deterministic* strategy is a function $\sigma^n : H_* \to A^n$ which assigns a move in the player's action set to each finite-length history.[4] That is,

$$\forall h = (s_0, s_1, s_2, ..., s_t) \in H_* \; \exists a^n \in A^n(s_t) : \sigma^n(h) = a^n.$$

We call σ^n *positional* (or *Markovian stationary*) if the next move depends only on the current state of the game (but not on previous states or current time). That is,

$$\forall h = (s_0, s_1, s_2, ..., s_t) \in H_* : \sigma^n(h) = \sigma^n(s_t).$$

A *strategy profile* is a triple $\sigma = (\sigma^1, \sigma^2, \sigma^3)$. We define $\sigma^{-n} = (\sigma^m)_{m \in I \setminus \{n\}}$; for instance, $\sigma^{-1} = (\sigma^2, \sigma^3)$. A profile is *positional* if $\sigma^1, \sigma^2, \sigma^3$ are positional. Otherwise (i.e., if at least one of the σ^n's is not positional) we call the profile *non-positional*. If $(\sigma^1, \sigma^2, \sigma^3)$ applied to the game $\Gamma_3(G|s_0, \gamma, \varepsilon)$ results (resp. does not result) in a capture, we call $(\sigma^1, \sigma^2, \sigma^3)$ a *capturing* (resp. *non-capturing*) profile in $\Gamma_3(G|s_0, \gamma, \varepsilon)$.

We complete the description of $\Gamma_3(G)$ by defining *payoff functions* for the players. Each player will try to maximize his payoff; the payoffs will encapsulate the following facts.

1. The longer the capture time, the less the cops gain and the less the robber loses.
2. The capturing cop gains at least as much as the non-capturing one.

We fix a constant $\varepsilon \in \left[0, \frac{1}{2}\right]$ and define *turn payoffs* as follows. For $n \in \{1, 2\}$, C_n's payoff is

$$q^n(s) = \begin{cases} 1 - \varepsilon & \text{if } s \in S_C^n, \\ \varepsilon & \text{if } s \in S_C^m \text{ with } n \neq m, \\ \frac{1}{2} & \text{if } s \in S_C^{12}, \\ 0 & \text{else.} \end{cases} \tag{3}$$

R's turn payoff is

$$q^3(s) = \begin{cases} -1 & \text{if } s \in S_C, \\ 0 & \text{else.} \end{cases} \tag{4}$$

Next, we fix a *discounting factor* $\gamma \in (0, 1)$ and, for $n \in \{1, 2, 3\}$, we define the n-th player's *total payoff* function by

$$Q^n(s_0, s_1, s_2, ...) = \sum_{t=0}^{\infty} \gamma^t q^n(s_t), \tag{5}$$

In the rest of the paper we will assume, unless explicitly stated otherwise, that

$$(\gamma, \varepsilon) \in \Omega^{(3)} = (0, 1) \times \left[0, \frac{1}{2}\right].$$

Since a history is fully determined by the initial position $s_0 = (x^1, x^2, x^3, p)$ and the strategy profile $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, we can write the n-th player's payoff in any one of the equivalent forms $Q^n(s_0, s_1, s_2, ...)$, $Q^n(s_0, \sigma)$ and $Q^n(s_0, \sigma^1, \sigma^2, \sigma^3)$.

[4] We only consider *legal* strategies i.e., they never produce moves outside the player's closed neighborhood.

To understand the consequences of (3)-(5), let us first consider the case where (i) $T_C < \infty$ (finite capture time), (ii) $\varepsilon < \frac{1}{2}$ and (iii) capture is effected by a single cop. Then the players receive the following payoffs:

1. the capturing cop receives $(1-\varepsilon)\gamma^{T_C}$;
2. the non-capturing cop receives $\varepsilon\gamma^{T_C}$;
3. the robber receives $-\gamma^{T_C}$ (i.e., loses γ^{T_C}).

The total cops' reward is equal to the robber's loss, but, since $\varepsilon < \frac{1}{2}$, the capturing cop receives more than the other one (unless both cops simultaneously capture the robber). Since $\gamma \in (0,1)$, the robber's loss is decreasing with capture time T_C and he will play so as to maximize T_C. Conversely, the cops have a motive to minimize T_C. But, since $(1-\varepsilon)\gamma^{T_C} > \varepsilon\gamma^{T_C}$, there is an additional motive for each cop to be the capturing one; there will exist combinations of γ, ε and T_C for which a cop may choose to delay the robber capture in order to ensure that it is effected by himself (an example is given in Section 4.3). The SCAR game $\Gamma_3(G|s_0, \gamma, \varepsilon)$ as defined above is a three-player, perfect information discounted stochastic game.

Let us consider briefly some additional scenaria obtained for particular values of T_C, γ, ε.

1. If the robber can avoid capture *ad infinitum*, i.e., if $T_C = \infty$, then all players receive zero payoff. Clearly this is the best outcome for R.
2. Since a single player moves at each turn, it is only possible to have a "double capture" if R, on his turn, moves into a vertex which is occupied by both C_1 and C_2. In this case each cop will receive equal payoff of $\gamma^{T_C}/2$.
3. When $\varepsilon = \frac{1}{2}$ (and for any $\gamma \in (0,1)$) each cop receives the same payoff whether he captures R or not; hence one might expect the two cops to collaborate to effect capture in the shortest possible time as in classic CR played by two cops against one robber; however this is not always the case, as we shall see in Section 4.

A basic question in classic CR is the existence of *winning* and/or *(time) optimal* strategies. In SCAR, which is an N-player game, we look for *equilibrium strategy profiles*. The prevalent definition of equilibrium is the one due to Nash [21], which we now present in the general context of N-player stochastic games; the application to three-player SCAR (and N-player SCAR, as we will see in Section 5) is immediate.

Consider an N-player perfect-information stochastic game starting at state s_0. When the players use the (deterministic) strategy profile $\sigma = \left(\sigma^1, \sigma^2, ..., \sigma^N\right)$ they receive (total) payoffs $Q^1(s_0, \sigma)$, $Q^2(s_0, \sigma)$, ..., $Q^N(s_0, \sigma)$. We say that $\sigma_* = \left(\sigma_*^1, \sigma_*^2, ..., \sigma_*^N\right)$ is a *Nash equilibrium* (NE) iff

$$\forall n, \forall \sigma^n : Q^n(s_0, \sigma_*) \geq Q^n\left(s_0, \sigma^n, \sigma_*^{-n}\right). \tag{6}$$

What (6) says is that, when the rest of the players stick to their equilibrium strategies, no player can improve his payoff by *unilaterally* changing his own; for example, if players 2, 3, ..., N play $\sigma_*^{-1} = \left(\sigma_*^2, ..., \sigma_*^N\right)$, then the first player cannot increase his payoff by switching from σ_*^1 to some other σ^1.[5] The following points must be emphasized.

1. A game may possess no NE, or exactly one, or more than one.
2. A NE is a *strategy* profile; different NE may yield the same *payoffs* to the players.
3. Different NE may yield different payoffs. The fact that σ_* is a NE does not imply that the corresponding payoff is the best a player can achieve; if *more than one* players change their strategies, they may achieve better payoffs than the ones implied by a NE. In other words, a NE is not necessarily an optimal solution.

3. Modified CR from a game theoretic point of view

Before embarking on the study of SCAR, we present and study a *modified* CR game, which will be used in later sections.

1. The game is played by two players: the *cop player* controls $N-1$ cop *tokens* (with $N \geq 2$) and the *robber player* controls a single robber token.
2. States, movement rules, histories and capture time are the same as those of SCAR.
3. The same is true for strategies except for the fact that the cop player's strategy is of the form $\left(\sigma^1, ..., \sigma^{N-1}\right)$, i.e., it contains one strategy for each of his tokens.
4. The cop (resp. robber) player's payoff is γ^{T_C} (resp. $-\gamma^{T_C}$) (with $\gamma^\infty = 0$).

This is a *two-player, zero-sum, discounted* stochastic game which differs from the classic CR game (with $N-1$ cops and one robber) only in the following.

[5] The definition of NE can be extended to games of *non*-perfect information, provided the σ^n's are understood as *probabilistic* strategies and the Q^n's as *expected* payoffs. We will not need these generalizations in the current paper.

1. In modified CR, the cop (resp. robber) player tries to maximize (resp. minimize) γ^{T_C}; this is obviously equivalent to classic CR, where the cop (resp. robber) player tries to minimize (resp. maximize) T_C.
2. In modified CR, time is counted in turns, while in classic CR it is counted in *rounds*, where each round consists of one move for each player. This is roughly equivalent to a rescaling of time by the factor $1/N$.
3. In classic CR, the players *select* their initial positions, while in modified CR the initial position is *predetermined*. It is easy to recover this aspect of classic CR by adding to modified CR a "placement turn" for each player; this change has no major impact on the essential features of the game such as the existence of value and optimal strategies.

Using standard results [8, Section 4.3] we see that modified CR has a *value*, which in fact is γ raised to the capture time under optimal play,[6] and both players have *optimal positional* strategies.

As already mentioned, we can assume that the cop (resp. robber) player tries to minimize (resp. maximize) the capture time. Let $T_N(G|s_0)$ be the capture time when the cop player has $N-1$ tokens and both players play optimally (note the dependence on the initial position s_0). Hence, if $T_N(G|s_0)$ is finite (resp. infinite) then the cop's (resp. robber's) optimal strategies are winning (for the respective player). We denote the maximum value of optimal capture time over all starting positions by

$$T_N(G) = \max_{s_0} T_N(G|s_0).$$

Assuming the game is played with $N-1$ cops and one robber, it is easily seen that:

1. if $T_N(G) < \infty$, then the cop player has an (optimal) winning strategy for *every* starting position;
2. for every starting position, assuming subsequent optimal play by the cop player (but not necessarily by the robber player), the capture time is less than or equal to $T_N(G)$.

The *cop number* of a graph G is denoted by $c(G)$ and defined to be the smallest number of cop tokens which guarantees finite capture time (i.e., one less than the smallest N for which $T_N(G) < \infty$). We call G *cop-win* if capture time is finite for CR on G with one optimally played cop token (i.e., $T_2(G) < \infty$ or, equivalently, $c(G) = 1$).

The above remarks show that all essential aspects of the classic CR are captured by the modified CR. *In the rest of the paper, the term "CR game" will denote the modified game* (unless we specifically use the term "classic CR").

Finally note that, when $N = 3$ (two cops vs. one robber) the modified two-cops CR is *path-equivalent* to $\Gamma_3(G)$, i.e., both games produce the same infinite history when strategies $\sigma^1, \sigma^2, \sigma^3$ are applied (starting from the same position s_0) to:

1. $\Gamma_3(G)$, with σ^n being the strategy of the n-th player;
2. the modified two-cops CR, with σ^1 (resp. σ^2) being the strategy the cop player uses for his first (resp. second) token, and σ^3 being the strategy the robber player uses.

4. Three-player SCAR

In this section we study $\Gamma_3(G)$ and prove that it always has both positional and non-positional NE; we also study the connection between classic cop number and existence of capturing NE.

4.1. Existence of a positional NE

First we prove the existence of at least one positional NE in *deterministic* strategies for $\Gamma_3(G)$.

Theorem 4.1. *For every graph G and for every $s_0 \in S$, $(\gamma, \varepsilon) \in \Omega^{(3)}$ the game $\Gamma_3(G|s_0, \gamma, \varepsilon)$ has a deterministic positional NE. More specifically, there exists a deterministic positional profile $\sigma_* = (\sigma_*^1, \sigma_*^2, \sigma_*^3)$ such that*

$$\forall n, \forall s_0, \forall \sigma^n : Q^n\left(s_0, \sigma_*^n, \sigma_*^{-n}\right) \geq Q^n\left(s_0, \sigma^n, \sigma_*^{-n}\right). \tag{7}$$

For every s and n let $u^n(s) = Q^n(s, \sigma_*)$. Then the following equations are satisfied

$$\forall n, \forall s \in S^n : \sigma_*^n(s) = \arg\max_{a^n \in A^n(s)} \left[q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, a^n\right)\right)\right], \tag{8}$$

$$\forall n, m, \forall s \in S^n : u^m(s) = q^m(s) + \gamma u^m\left(\mathbf{T}\left(s, \sigma_*^n(s)\right)\right). \tag{9}$$

[6] And, since $\gamma \in (0, 1)$, is a decreasing function of capture time.

Proof. The existence, in every N-player discounted stochastic game, of a positional NE in *probabilistic* strategies has been proved by Fink in [9]. In addition, Fink proves that at equilibrium the following equations[7] are satisfied (i.e., they have at least one solution) for all $m \in \{1, 2, 3\}$ and $s \in S$:

$$\mathrm{u}^m(s) = \max_{\pi^m(s)} \sum_{a^1 \in A^1(s)} \sum_{a^2 \in A^2(s)} \sum_{a^3 \in A^3(s)} \pi^1\left(a^1|s\right)\pi^2\left(a^2|s\right)\pi^3\left(a^3|s\right)\left[q^m(s) + \gamma \sum_{s'} \Pr\left(s'|s, a^1, a^2, a^3\right)\mathrm{u}^m\left(s'\right)\right], \tag{10}$$

where

1. $\mathrm{u}^m(s)$ is the expected value of $u^m(s)$;
2. $\pi^m(a^j|s)$ is the probability that, given the current state is s, the m-th player plays a^j;
3. $\pi^m(s) = \left(\pi^m(a^m|s)\right)_{a^m \in A^m(s)}$ is the vector of all probabilities (i.e., for all available actions);
4. $\Pr\left(s'|s, a^1, a^2, a^3\right)$ is the probability that the next state is s', given the current state is s and the players actions a^1, a^2, a^3.

As mentioned above, (10) applies to the general game, with *simultaneous* moves by all players and probabilistic strategies and state transitions. Our task now is to prove that in SCAR the above equations yield a *deterministic* NE.

Choose any n and any $s \in S^n$. For all $m \neq n$, the m-th player has a single move, i.e., we have $A^m(s) = \left\{a^m\right\}$, and so $\pi^m(a^m|s) = 1$. Also, since transitions are deterministic,

$$\sum_{s'} \Pr\left(s'|s, a^1, a^2, a^3\right)\mathrm{u}^n\left(s'\right) = \mathrm{u}^n\left(\mathbf{T}\left(s, a^n\right)\right).$$

Hence, for $m = n$, (10) becomes

$$\mathrm{u}^n(s) = \max_{\pi^n(s)} \sum_{a^n \in A^n(s)} \pi^n\left(a^n|s\right)\left[q^n(s) + \gamma \mathrm{u}^n\left(\mathbf{T}\left(s, a^n\right)\right)\right]. \tag{11}$$

Furthermore let us define $\sigma_*^n(s)$ (for the specific s and n) by

$$\sigma_*^n(s) = \arg\max_{a^n \in A^n(s)} \left[q^n(s) + \gamma \mathrm{u}^n\left(\mathbf{T}\left(s, a^n\right)\right)\right]. \tag{12}$$

If more than one action satisfy (12), we set $\sigma_*^n(s)$ to one of these actions arbitrarily. Then, to maximize the sum in (11) the n-th player must set $\pi^n\left(\sigma_*^n(s)|s\right) = 1$ and $\pi^n\left(a^n|s\right) = 0$ for all $a^n \neq \sigma_*^n(s)$. Since this is true for all states and players (i.e., every player can, without loss, use deterministic strategies) we also have $\mathrm{u}^n(s) = u^n(s)$. Hence (12) and (11) become respectively

$$\sigma_*^n(s) := \arg\max_{a^n \in A^n(s)} \left[q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, a^n\right)\right)\right] \tag{13}$$

and

$$u^n(s) = \max_{a^n \in A^n(s)} \left[q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, a^n\right)\right)\right] = q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, \sigma_*^n(s)\right)\right). \tag{14}$$

For $m \neq n$, the m-th player has no choice of action (i.e., $\sigma_*^m(s)$ is the unique element of $A^m(s)$) and (11) becomes

$$u^m(s) = q^m(s) + \gamma u^m\left(\mathbf{T}\left(s, \sigma_*^n(s)\right)\right). \tag{15}$$

We recognize that (13)-(15) are (8)-(9). Also, (13) defines $\sigma_*^n(s)$ for every n and s and so we have obtained the required deterministic positional strategies $\sigma_* = \left(\sigma_*^1, \sigma_*^2, \sigma_*^3\right)$. $\square$

Note that the initial state s_0 plays no special role in the system (8)-(9). In other words, using the notation $u(s) = \left(u^1(s), u^2(s), u^3(s)\right)$ and $\mathbf{u} = (u(s))_{s \in S}$ (with the G dependence suppressed) we see that $\mathbf{u}$ and σ_* are the same for every starting position s_0 (i.e., for every $\Gamma_3(G|s_0)$). Also note that, because of the structure of the payoffs, if $\Gamma_3(G|s_0)$ at some time t_1 reaches state s_1, the "remainder" game which is played from t_1 onward is equivalent (modulo a payoff rescaling) to $\Gamma_3(G|s_1)$. From these observations follows that, if the players use σ_* in $\Gamma_3(G|s_0)$ and at some time $t_1 > 0$ the game reaches s_1, then σ_* is an positional NE for both $\Gamma_3(G|s_0)$ and $\Gamma_3(G|s_1)$; the payoffs to the players are $u(s_0)$ in the former and $u(s_1)$ in the latter.

[7] We have adapted Fink's notation to our own, so as to fit the $\Gamma_3(G)$ context.

Let us also note that Theorem 4.1 in fact holds for any $\varepsilon \in [0,1]$; we have confined attention to the case $\varepsilon \in \left[0, \frac{1}{2}\right]$ to represent the intuition that the capturing cop's reward should be at least as large as that of the non-capturing one's. In fact, the theorem holds for *any* family of bounded payoff functions q^n. On the other hand, the theorem depends essentially on the players' moving sequentially; in case all players played simultaneously, they would not have *perfect information* and hence they would, in general, benefit from the use of probabilistic strategies.

Finally, Theorem 4.1 (as well as Fink's result) do not address the *computation* of the NE; it is well known that, in general, the computation of NE in multi-player games is a hard problem.

4.2. Existence of non-positional NE

Now we will construct an additional deterministic NE of $\Gamma_3(G|s_0)$, which will, generally, be non-positional.[8] This NE is based on the use of *threat strategies* [5,6,26].

Let us introduce (for $n \in \{1,2,3\}$) auxiliary games $\Gamma_3^n(G|s_0)$; these are two-player, zero-sum, perfect-information games with states, action sets, movement sequence, capturing conditions etc. being the same as in $\Gamma_3(G|s_0)$. However, in $\Gamma_3^n(G|s_0)$ player P_n controls token n and has payoff Q^n; and player P_{-n} controls tokens $\{1,2,3\} \setminus \{n\}$ and has payoff $-Q^n$. More specifically, the following hold.

1. $\Gamma_3^3(G|s_0)$ (played on G with initial state s_0) is the game where P_3, controlling R, plays against P_{-3}, controlling C_1 and C_2; P_{-3} has reward (and P_3 has penalty) equal to

$$\gamma^{T_C}: \text{ when either } C_1 \text{ or } C_2 \text{ captures } R,$$
$$0: \text{ when } R \text{ is not captured.}$$

It is easily seen that $\Gamma_3^3(G|s_0)$ is the two-cops, one-robber modified CR game.

2. $\Gamma_3^1(G|s_0)$ is the game in which P_1, controlling C_1, plays against P_{-1}, controlling R and a "robber-friendly" C_2; P_1 receives reward (and P_{-1} receives penalty) equal to

$$(1-\varepsilon)\gamma^{T_C}: \text{when } C_1 \text{ captures } R,$$
$$\varepsilon\gamma^{T_C}: \text{when } C_2 \text{ captures } R,$$
$$0: \text{when } R \text{ is not captured.}$$

3. $\Gamma_3^2(G|s_0)$ is similar to $\Gamma_3^1(G|s_0)$, with the roles of C_1 and C_2 interchanged.

It can be seen that in $\Gamma_3^1(G|s)$ an optimal action plan for P_{-1} is

1. when $c(G)=1$: C_2 and R meet in the longest possible time but before R is caught by C_1;
2. when $c(G)>1$ and C_1 cannot alone capture R (when the game starts at s_0): C_2 always avoids R and R always avoids both C_1 and C_2.

Hence, for every $n \in \{1,2,3\}$ and $s_0 \in S$, $\Gamma_3^n(G|s_0,\gamma,\varepsilon)$ is a two-player, zero-sum discounted stochastic game with perfect information and standard results [8, Section 4.3] give the following.

Lemma 4.2. *For each $m \in \{1,2,3\}$, $s_0 \in S$ and $(\gamma,\varepsilon) \in \Omega^{(3)}$, the game $\Gamma_3^m(G|s_0,\gamma,\varepsilon)$ has a value and the players have optimal deterministic positional strategies.*

Definition 4.3. *For $m,n \in \{1,2,3\}$, we define ϕ_m^n to be the optimal strategy regarding the n-th token in the game $\Gamma_3^m(G|s_0,\gamma,\varepsilon)$.*

We return to $\Gamma_3^m(G|s_0,\gamma,\varepsilon)$ and introduce the threat strategies. The n-th player plays the strategy ϕ_n^n which is optimal for P_n in $\Gamma_3^m(G|s_0,\gamma,\varepsilon)$, as long as the other players do the same. If at some point the m-th player deviates[9] from the above, then the n-th player (with $n \in \{1,2,3\} \setminus \{m\}$) adopts the strategy ϕ_m^n which is the part of P_{-m}'s optimal strategy regarding the n-th token in $\Gamma_3^m(G|s_0,\gamma,\varepsilon)$.

In other words, the threat strategy $\overline{\sigma}^n$ for the n-th player is "composed" by the strategies ϕ_m^n as follows:

$$\overline{\sigma}^n = \begin{cases} \phi_n^n & \text{as long as every player } m \in \{1,2,3\} \setminus n \text{ follows } \phi_m^m; \\ \phi_m^n & \text{as soon as some player } m \in \{1,2,3\} \setminus n \text{ "deviates" from } \phi_m^m. \end{cases} \tag{16}$$

[8] The non-positionality is further discussed at the end of this section.

[9] We say that a player "deviates" from a strategy if he plays a move different from the one prescribed by this strategy; since the game has perfect information, this deviation will be immediately detected by the other players.

Since the ϕ_n^m's are positional they do not depend on the starting state s_0; in fact the same ϕ_n^m is optimal for every s_0 and corresponding game $\Gamma_3^n(G|s_0)$. We now show that, for every s_0, $\overline{\sigma} = (\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3)$ is a NE of $\Gamma_3(G|s_0)$.

Theorem 4.4. *For every graph G, $(\gamma, \varepsilon) \in \Omega^{(3)}$ and $s_0 \in S$ in the game $\Gamma_3(G|s_0, \gamma, \varepsilon)$ we have*

$$\forall n \in \{1, 2, 3\}, \forall \sigma^n : Q^n\left(s_0, \overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right) \geq Q^n(s_0, \sigma^n, \overline{\sigma}^{-n}) \tag{17}$$

where $\overline{\sigma}^n$ (for $n \in \{1, 2, ..., N\}$) is a strategy of the form defined in (16).

Proof. Recall that we can write payoffs in any of the equivalent forms: $Q^n(s_0, \sigma)$, $Q^n(s_0, s_1, s_2, ...)$, $Q^n(h)$ (where $h = \left(s_0, s_1, s_{2,} ...\right)$).

We choose some initial state s and fix it for the rest of the proof. Now let us prove (17) for the case $n = 1$. In other words, we need to show that

$$\forall \sigma^1 : Q^1(s, \overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3) \geq Q^1(s, \sigma^1, \overline{\sigma}^2, \overline{\sigma}^3). \tag{18}$$

We take any σ^1 and let

$\overline{h} = (\overline{s}_0, \overline{s}_1, \overline{s}_2, ...)$ be the history produced by $(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3)$ and initial state $\overline{s}_0 = s$,

$\widetilde{h} = (\widetilde{s}_0, \widetilde{s}_1, \widetilde{s}_2, ...)$ be the history produced by $(\sigma^1, \overline{\sigma}^2, \overline{\sigma}^3)$ and initial state $\widetilde{s}_0 = s = \overline{s}_0$.

We also define T_1 as the earliest time in which $(\sigma^1, \overline{\sigma}^2, \overline{\sigma}^3)$ (and $(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3)$) produce different states:

$$T_1 = \min\left\{t : \widetilde{s}_t \neq \overline{s}_t\right\}.$$

If $T_1 = \infty$, then $\widetilde{h} = \overline{h}$ and (18) holds with equality:

$$Q^1(s, \overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3) = Q^1(\overline{h}) = Q^1(\widetilde{h}) = Q^1(\sigma^1, \overline{\sigma}^2, \overline{\sigma}^3). \tag{19}$$

If $T_1 < \infty$, then at $t = T_1$ player 1 deviated from ϕ_1^1, the first difference in states appeared and it was detected by players 2 and 3, who switched to ϕ_1^2 and ϕ_1^3, respectively. We have

$$Q^1(s, \overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3) = Q^1(\overline{h}) = \sum_{t=0}^{T_1-2} \gamma^t q^1(\overline{s}_t) + \sum_{t=T_1-1}^{\infty} \gamma^t q^1(\overline{s}_t), \tag{20}$$

$$Q^1(s, \sigma^1, \overline{\sigma}^2, \overline{\sigma}^3) = Q^1(\widetilde{h}) = \sum_{t=0}^{T_1-2} \gamma^t q^1(\widetilde{s}_t) + \sum_{t=T_1-1}^{\infty} \gamma^t q^1(\widetilde{s}_t). \tag{21}$$

Since $\widetilde{s}_t = \overline{s}_t$ for every $t < T_1$, it suffices to compare the second sums of (20) and (21). In what follows we let $s^* = \overline{s}_{T_1-1} = \widetilde{s}_{T_1-1}$.

1. Consider first $\overline{h} = (\overline{s}_0, \overline{s}_1, \overline{s}_2, ...)$. It is produced by $\overline{\sigma} = (\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3)$ (and $\overline{s}_0$) which means that the entire $\overline{h}$ is actually produced by $\left(\phi_1^1, \phi_2^2, \phi_3^3\right)$ (and $\overline{s}_0$). Since every ϕ_m^m is positional, the history $(\overline{s}_0, \overline{s}_1, ..., \overline{s}_{T_1-2})$ does not influence the moves produced at times $T_1, T_1 + 1, ...$ Hence we have

$$\sum_{t=T_1-1}^{\infty} \gamma^t q^1(\overline{s}_t) = \gamma^{T_1-1} \sum_{t=0}^{\infty} \gamma^t q^1(\overline{s}_{T_1-1+t}) = \gamma^{T_1-1} Q^1(s^*, \phi_1^1, \phi_2^2, \phi_3^3). \tag{22}$$

   In other words, the sum in (22) is proportional to the payoff of player 1 in $\Gamma_3(G|s^*)$ when each player $n \in \{1, 2, 3\}$ uses strategy ϕ_n^n. But $Q^1(s^*, \phi_1^1, \phi_2^2, \phi_3^3)$ is also the payoff of P_1 in $\Gamma_3^1(G|s^*)$ (which starts at s^*) with P_1 playing ϕ_1^1 and P_{-1} playing (ϕ_2^2, ϕ_3^3). However, in $\Gamma_3^1(G|s^*)$ the optimal strategy of P_{-1} is $\left(\phi_1^2, \phi_1^3\right)$; hence we have the following

$$\gamma^{T_1-1} Q^1(s^*, \phi_1^1, \phi_2^2, \phi_3^3) \geq \gamma^{T_1-1} Q^1(s^*, \phi_1^1, \phi_1^2, \phi_1^3). \tag{23}$$

2. Next consider $\widetilde{h} = (\widetilde{s}_0, \widetilde{s}_1, \widetilde{s}_2, ...)$. It is produced by $(\sigma^1, \overline{\sigma}^2, \overline{\sigma}^3)$ (and $\widetilde{s}_0 = \overline{s}_0$) and, since σ^1 is not necessarily positional, $\widetilde{s}_{T_1}, \widetilde{s}_{T_1+1}, \widetilde{s}_{T_1+2} ...$ could depend on $(\widetilde{s}_0, \widetilde{s}_1, ..., \widetilde{s}_{T_1-2})$. However, we can introduce the strategy ρ^1 *induced* by σ^1 on the game starting at s^*, which will produce the same history $(\widetilde{s}_{T_1}, \widetilde{s}_{T_1+1}, \widetilde{s}_{T_1+2}, ...)$ as σ^1.[10] Then, from the optimality of ϕ_1^1 as a response to $\left(\phi_1^2, \phi_1^3\right)$ in $\Gamma_3^1(G|s^*)$, we have

[10] We define ρ^1 such that, when combined with $\widetilde{s}_{T_1-1}, \phi_1^2, \phi_1^3$, will produce the same history $(\widetilde{s}_{T_1}, \widetilde{s}_{T_1+1}, \widetilde{s}_{T_1+2}, ...)$ as σ^1. Note that ρ^1 will in general depend (in an indirect way) on $(\widetilde{s}_0, \widetilde{s}_1, ..., \widetilde{s}_{T_1-2})$.

$$\sum_{t=T_1-1}^{\infty} q^1(\widetilde{s}_t) = \gamma^{T_1-1} Q^1(s^*, \rho^1, \phi_1^2, \phi_1^3) \leq \gamma^{T_1-1} Q^1(s^*, \phi_1^1, \phi_1^2, \phi_1^3). \tag{24}$$

Combining (20)-(24) we have:

$$\begin{aligned} Q^1(s, \sigma^1, \overline{\sigma}^2, \overline{\sigma}^3) &= \sum_{t=0}^{T_1-2} \gamma^t q^1(\widetilde{s}_t) + \gamma^{T_1-1} Q^1(s^*, \rho^1, \phi_1^2, \phi_1^3) \leq \sum_{t=0}^{T_1-2} \gamma^t q^1(\widetilde{s}_t) + \gamma^{T_1-1} Q^1(s^*, \phi_1^1, \phi_1^2, \phi_1^3) \\ &\leq \sum_{t=0}^{T_1-2} \gamma^t q^1(\overline{s}_t) + \gamma^{T_1-1} Q^1(s^*, \phi_1^1, \phi_2^2, \phi_3^3) = Q^1(s, \overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3) \end{aligned}$$

and we have proved (18), which is (17) for $n = 1$. The proof for the cases $n = 2$ and $n = 3$ is similar and hence omitted. □

In general, the profile $\left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)$ is non-positional, since the action of a player at time t may be influenced by the action (deviation) performed by another player at time $t-2$. However, it is possible that $\forall m, n \in \{1, 2, 3\}$ we have $\phi_n^n = \phi_m^n$ in which case the profile $\left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)$ is positional.

Just like Theorem 4.1, Theorem 4.4 actually holds for any bounded q^n's, not just for the specific form of the SCAR game. Moreover, the theorem depends essentially on the sequential moving of the players (so that deviations can be detected).

Finally, it is worth noting that the NE of Theorem 4.4 can be computed in *polynomial* time, since the component strategies ϕ_m^n concern the *two*-player games $\Gamma_3^n(G|s)$, for which polynomial algorithms are available [3,12].

4.3. Cop number, capturing and non-capturing NE

In this section we examine the connection of $c(G)$ to the existence of capturing and non-capturing NE in $\Gamma_3(G|s_0, \gamma, \varepsilon)$.

Theorem 4.5. *For any G with $c(G) = 1$ the following holds:*

$$\forall (\gamma, \varepsilon) \in \Omega^{(3)}, \forall s_0 \in S : \textit{every NE of } \Gamma_3(G|s_0, \gamma, \varepsilon) \textit{ is capturing.}$$

Proof. Let $G = (V, E)$ with $c(G) = 1$ and take any $s_0 \in S_{NC}$ (the case $s_0 \in S_C$ is trivial). To reach a contradiction assume that there exists a non-capturing NE $\left(\sigma^1, \sigma^2, \sigma^3\right)$ of $\Gamma_3(G|s_0)$. Then we have

$$\forall \rho^1 : 0 = Q^1(s_0, \sigma^1, \sigma^2, \sigma^3) \geq Q^1(s_0, \rho^1, \sigma^2, \sigma^3). \tag{25}$$

Now let $\widehat{\sigma}^1$ be a strategy of C_1 in $\Gamma_3(G|s_0)$ which *imitates an optimal cop strategy, in the respective CR game* with one cop. Formally, define $\widehat{\sigma}^1$ by

$$\forall (x^1, x^2, x^3) \in V^3 : \widehat{\sigma}^1\left(x^1, x^2, x^3, 1\right) := \sigma_*^1\left(x^1, x^3, 1\right),$$

where σ_*^1 is an optimal cop strategy in the one-cop CR played on G.[11]

In this latter game given $c(G) = 1$ and optimal cop play, capture will occur in some finite time which depends on R's strategy but is bounded above by $T_2(G)$ defined in Section 3.

Consider now game $\Gamma_3(G|s_0)$ and the case where C_1 employs $\widehat{\sigma}^1$. It is not hard to see (and can be formally shown) that for any pair of strategies of C_2, R capture will also occur in $\Gamma_3(G|s_0)$ in at most $T_2(G)$ moves, as C_2 may influence the game by capturing R no later than C_1. Hence, under profile $(\widehat{\sigma}^1, \sigma^2, \sigma^3)$ we have the following possibilities.

1. C_1 captures R at some time T_1.
2. C_2 captures R before C_1, i.e., at some time $T_2 < T_1$.
3. C_1 and C_2 capture R simultaneously at some time T_{12}.

At any rate, we will have $\max(T_1, T_2, T_{12}) \leq T_2(G) < \infty$. Hence C_1's payoff satisfies

$$Q^1(s_0, \widehat{\sigma}^1, \sigma^2, \sigma^3) \geq \min\left((1-\varepsilon)\gamma^{T_1}, \varepsilon\gamma^{T_2}, \frac{1}{2}\gamma^{T_{12}}\right) > 0. \tag{26}$$

But (26) contradicts (25). Thus, there does not exist a non-capturing NE of $\Gamma_3(G|s_0)$. We conclude that every NE of $\Gamma_3(G|s_0)$ is capturing. □

[11] We will repeatedly use, without further comment, this method to produce SCAR strategies from optimal CR strategies. Furthermore, for reasons of brevity we will simply call them *optimal in CR*.

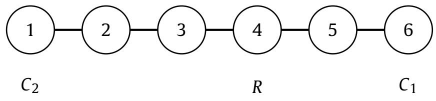

Fig. 1. An example where minimizing capture time does not yield a NE.

In light of the argument employed in the last proof, it might be assumed that in graphs G with $c(G)=1$, a NE in $\Gamma_3(G)$ can be obtained when the cops use optimal strategies from one-cop CR. But this is not true, because in certain cases a cop may unilaterally improve his payoff by delaying capture (and thus ensuring that it is effected by him) as seen in the following example.

Example 4.6. Suppose $\Gamma_3(G)$ is played on the graph G of Fig. 1 with the initial positions indicated; C_1 has the first move. Further, take $\varepsilon < \frac{1}{2}$. Let $\widehat{\sigma}^n$ $(n \in \{1,2\})$ be a one-cop CR optimal strategy of the n-th cop; in this case it consists in each cop moving towards the robber at every turn. Now suppose that for these $\widehat{\sigma}^1, \widehat{\sigma}^2$ there exists a NE $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$; it is easily seen that $\widehat{\sigma}^3$ must be an optimal robber strategy in two-cop CR. If the players use $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ the game evolves as follows:

Turn	0	1	2	3	4	5
C_1 **vertex**	6	5	5	5	4	4
C_2 **vertex**	1	1	2	2	2	3
R **vertex**	4	4	4	3	3	3

Note that the robber will move so as to be captured by C_2, because this increases capture time by 1. So the payoffs are

$$Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = \varepsilon\gamma^5,$$
$$Q^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = (1-\varepsilon)\gamma^5,$$
$$Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = -\gamma^5.$$

Now, suppose C_2, R stick to their strategies, while C_1 uses the following strategy $\widetilde{\sigma}^1$: on his first move he stays in vertex 6 and afterwards moves directly towards the robber. The game evolves as follows.

Turn	0	1	2	3	4	5	6	7
C_1 **vertex**	6	6	6	6	5	5	5	4
C_2 **vertex**	1	1	2	2	2	3	3	3
R **vertex**	4	4	4	4	4	4	4	4

So the payoffs are

$$Q^1\left(s_0, \widetilde{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = (1-\varepsilon)\gamma^7,$$
$$Q^2\left(s_0, \widetilde{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = \varepsilon\gamma^7,$$
$$Q^3\left(s_0, \widetilde{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = -\gamma^7.$$

It is easy to see that

$$\gamma^2 > \frac{\varepsilon}{1-\varepsilon} \Rightarrow (1-\varepsilon)\gamma^7 > \varepsilon\gamma^5 \Rightarrow Q^1\left(s_0, \widetilde{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) > Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right).$$

Thus, if $\gamma^2 > \frac{\varepsilon}{1-\varepsilon}$, C_1 can unilaterally improve his payoff; hence $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ is not a NE of $\Gamma_3(G|s_0, \gamma, \varepsilon)$.

We now move to graphs with cop number greater than one.

Theorem 4.7. *For any G with $c(G)=2$ the following holds:*

$$\forall (\gamma, \varepsilon) \in \Omega^{(3)}, \forall s_0 \in S: \textit{there exists a capturing NE of } \Gamma_3(G|s_0, \gamma, \varepsilon).$$

Proof. Take any G with $c(G)=2$, any $(\gamma,\varepsilon) \in \Omega^{(3)}$ and any $s_0 \in S_{NC}$ (the case $s_0 \in S_C$ is trivial) and fix them for the rest of the proof. Now take any threat strategy profile $\overline{\sigma} = \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)$; according to Theorem 4.4, $\overline{\sigma}$ is a NE and it can be either capturing or non-capturing. If it is capturing we are done; let us then suppose that $\overline{\sigma}$ is non-capturing. Recall that, for all $n \in \{1,2,3\}$: $\overline{\sigma}^n$ has the form of (16), where, in the game $\Gamma_3^n(G|s_0, \gamma, \varepsilon)$:

1. ϕ_n^n is an optimal strategy of P_n against P_{-n};
2. ϕ_n^m (for $m \neq n$) is an optimal strategy used (for the m-th token) by P_{-n} against P_n.

As mentioned, when $\overline{\sigma}$ is used in $\Gamma_3(G|s_0,\gamma,\varepsilon)$ the n-th player (for $n \in \{1,2,3\}$) will follow strategy ϕ_n^n for the entire game. We will now construct a new profile $\widetilde{\sigma} = \left(\widetilde{\sigma}^1, \widetilde{\sigma}^2, \widetilde{\sigma}^3\right)$ which *will* be a capturing NE of $\Gamma_3(G|s_0,\gamma,\varepsilon)$. To this end we first select an optimal strategy profile $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ in the two-cops CR; since $c(G) = 2$, $\widehat{\sigma}$ will be capturing for every s_0. Then, for each $n \in \{1,2,3\}$, we let

$$\widetilde{\sigma}^n = \begin{cases} \widehat{\sigma}^n & \text{as long as every player } m \in \{1,2,3\} \backslash n \text{ follows } \widehat{\sigma}^m; \\ \phi_m^n & \text{as soon as some player } m \in \{1,2,3\} \backslash n \text{ deviates from } \widehat{\sigma}^m. \end{cases}$$

The ϕ_m^n's above are the same as in $\overline{\sigma}$. Hence, to show that $\widetilde{\sigma}$ is of the form prescribed by Theorem 4.4, we have to show that each $\widehat{\sigma}^n$ is optimal in the corresponding $\Gamma_3^n(G|s_0,\gamma,\varepsilon)$. In what follows, note that $\Gamma_3(G|s_0,\gamma,\varepsilon)$ is (for all $n \in \{1,2,3\}$) path-equivalent[12] to $\Gamma_3^n(G|s_0,\gamma,\varepsilon)$.

1. Consider C_1 playing ϕ_1^1 in $\Gamma_3(G|s_0,\gamma,\varepsilon)$; by assumption $\overline{\sigma}$ is non-capturing in $\Gamma_3(G|s_0,\gamma,\varepsilon)$, hence also in $\Gamma_3^1(G|s_0,\gamma,\varepsilon)$. Therefore, C_1 playing ϕ_1^1 against ϕ_2^2 and ϕ_3^3 receives a payoff of zero, in both $\Gamma_3(G|s_0,\gamma,\varepsilon)$ and $\Gamma_3^1(G|s_0,\gamma,\varepsilon)$. But then, by the optimality of (ϕ_1^2, ϕ_1^3) against ϕ_1^1 in $\Gamma_3^1(G|s_0,\gamma,\varepsilon)$ the same holds when C_1 plays ϕ_1^1 against (ϕ_1^2, ϕ_1^3). So C_1's optimal payoff in $\Gamma_3^1(G|s_0,\gamma,\varepsilon)$ is zero and hence *any* strategy is optimal for him in $\Gamma_3^1(G|s_0,\gamma,\varepsilon)$; so is, in particular $\widehat{\sigma}^1$.
2. By a similar argument, any strategy, and in particular $\widehat{\sigma}^2$, will be optimal for C_2 in $\Gamma_3^2(G|s_0,\gamma,\varepsilon)$.
3. Finally, the game $\Gamma_3^3(G|s_0,\gamma,\varepsilon)$ is the two-cop CR, and hence $\widehat{\sigma}^3$ will be optimal for R in $\Gamma_3^3(G|s_0,\gamma,\varepsilon)$.

By the above observations we see that, according to Theorem 4.4, $\widetilde{\sigma}$ is a NE (in threat strategies) of $\Gamma_3(G|s_0,\gamma,\varepsilon)$. Furthermore, playing $\widetilde{\sigma}$ at equilibrium is equivalent to playing $\widehat{\sigma}$, which is a capturing profile in both two-cop CR (i.e., $\Gamma_3^3(G|s_0,\gamma,\varepsilon)$) and $\Gamma_3(G|s_0,\gamma,\varepsilon)$. So $\widetilde{\sigma}$ is a capturing NE of $\Gamma_3(G|s_0,\gamma,\varepsilon)$. $\square$

Remark 4.8. Note that the NE of the above theorem is non-positional (unless, for all m, n, we have $\widehat{\sigma}^n = \phi_m^n$).

The next theorem holds on a restricted set of (γ,ε) values:

$$\widetilde{\Omega}^{(3)} = \left\{(\gamma,\varepsilon) : \gamma \in (0,1), \varepsilon \in \left[0, \frac{1}{2}\right], \gamma < \frac{\varepsilon}{1-\varepsilon}\right\}.$$

Theorem 4.9. *For any G with $c(G) = 2$, let $\widehat{\sigma} = \left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ be an optimal strategy profile in the two-cop CR game. Then the following holds:*

$$\forall (\gamma,\varepsilon) \in \widetilde{\Omega}^{(3)}, \forall s_0 \in S : \widehat{\sigma} \textit{ is a capturing NE of } \Gamma_3(G|s_0,\gamma,\varepsilon).$$

Proof. Take any G with $c(G) = 2$, any $(\gamma,\varepsilon) \in \widetilde{\Omega}^{(3)}$ and any $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ which is optimal in the two-cop CR game; we fix these for the rest of the proof. Obviously $\widehat{\sigma}$ is a capturing profile, since it is optimal in CR and $c(G) = 2$. So we need to show that it is also a NE of $\Gamma_3(G|s_0,\gamma,\varepsilon)$. This will obviously be true when $s_0 \in S_C$, so let us consider any $s_0 \in S_{NC}$. Let T_1 be the capture time (the same in the path-equivalent games CR and $\Gamma_3(G|s_0,\gamma,\varepsilon)$) corresponding to $\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$.

Assume for the time being that the capturing cop is C_1; then the payoffs are

$$Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = (1-\varepsilon)\gamma^{T_1},$$
$$Q^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = \varepsilon\gamma^{T_1},$$
$$Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) = -\gamma^{T_1}.$$

We will show that no player can improve his payoff by unilaterally changing his strategy.

1. Suppose R uses some strategy σ^3 and the capture time of $\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \sigma^3\right)$ is T_2. By the optimality (in CR) of $\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3$, we have $T_2 \leq T_1$ and so

$$Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \sigma^3\right) = -\gamma^{T_2} \leq -\gamma^{T_1} = Q^3\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right).$$

[12] In the sense of Section 3.

So R has no motive to deviate from $\widehat{\sigma}^3$.

2. Similarly, suppose C_1 uses some strategy σ^1 and the capture time of $\left(s_0, \sigma^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ is T_2; if $T_2 = \infty$ we have no capture; otherwise capture can be effected by either C_1 or C_2. At any rate, by the optimality of $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$, we have $T_2 \geq T_1$ and the maximum possible payoff to C_1 is $(1-\varepsilon)\gamma^{T_2}$. Since
$$Q^1\left(s_0, \sigma^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) \leq (1-\varepsilon)\gamma^{T_2} \leq (1-\varepsilon)\gamma^{T_1} = Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right),$$
C_1 has no motive to deviate from $\widehat{\sigma}^1$.
3. Finally, suppose C_2 uses some strategy σ^2 and the capture time of $\left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right)$ is T_2. If $T_2 = \infty$ we have no capture; otherwise capture can be effected by either C_1 or C_2. If we have no capture then
$$Q^2\left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right) = 0 < \varepsilon\gamma^{T_1} = Q^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right).$$
If capture is effected by C_1, we have $T_2 \geq T_1$ and
$$Q^2\left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right) = \varepsilon\gamma^{T_2} \leq \varepsilon\gamma^{T1} = Q^2\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right).$$
Finally, if capture is effected by C_2, we have $T_2 \geq T_1 + 1$ (if C_2 could capture before C_1 this would be achieved by $\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$) and, since $(\gamma, \varepsilon) \in \widetilde{\Omega^{(3)}}$ implies $\gamma < \frac{\varepsilon}{1-\varepsilon}$, we have
$$Q^2\left(s_0, \widehat{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right) = (1-\varepsilon)\gamma^{T_2} \leq (1-\varepsilon)\gamma^{T_1+1} < \varepsilon\gamma^{T_1}.$$
In every case, C_2 has no motive to deviate from $\widehat{\sigma}^2$.

Having assumed that the starting position s_0 and the strategy profile$\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ result in a capture by C_1, we have shown that no player has a motive to change his strategy. By an analogous argument, the same holds when $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ results in a capture by C_2. As already mentioned, $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ is a capturing profile, hence *some* cop will capture the robber, and no player has a motive to unilaterally change his strategy. Consequently $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ is a capturing NE of $\Gamma_3(G|s_0, \gamma, \varepsilon)$. □

Remark 4.10. Note that in the above Theorem the NE $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ is positional.

We also have the following.

Theorem 4.11. *For any G with $c(G) \geq 2$, the following holds:*
$$\forall(\gamma, \varepsilon) \in \Omega^{(3)}, \exists s_0 \in S : \textit{there exists a non-capturing NE of } \Gamma_3(G|s_0, \gamma, \varepsilon).$$

Proof. Choose an $s_0 = (x, x, y, 1)$ of the following form: x can be any vertex of G and y is such that, when the one-cop CR is started from $s_0' = (x, y, 1)$, the robber can avoid capture (such an s_0 will always exist, since $c(G) \geq 2$). The strategies are chosen as follows.

1. R's strategy $\widehat{\sigma}^3$ is the following:
   (a) as long as C_1, C_2 stay in place R also stays in place;
   (b) if at some time C_1 (resp. C_2) is the first cop to move, R starts playing an optimal one-cop CR strategy with respect to C_1 (resp. C_2).
2. C_1's strategy $\widetilde{\sigma}^1$ is defined as follows:
   (a) if C_1 and C_2 are in the same vertex, C_1 stays in place;
   (b) if C_1 and C_2 are in different vertices, C_1 moves in a shortest path towards C_2.
3. C_2's strategy $\widetilde{\sigma}^2$ is the same as $\widetilde{\sigma}^1$, with the roles of C_1 and C_2 interchanged.

We now show that $\left(\widetilde{\sigma}^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right)$ is a non-capturing NE of $\Gamma_3(G|s_0, \gamma, \varepsilon)$. First, since C_1 and C_2 start at the same vertex x, by $\widetilde{\sigma}^1, \widetilde{\sigma}^2$ they will never move towards y; hence, under $\left(\widetilde{\sigma}^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right)$, R is not captured.

1. Hence $Q^3\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right) = 0$ and, clearly, R cannot improve his payoff, i.e.,
$$\forall\sigma^3 : 0 = Q^3\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right) \geq Q^3\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \sigma^3\right). \tag{27}$$

2. Now suppose C_1 uses some $\sigma^1 \neq \widetilde{\sigma}^1$ by which, at the start of the game, he moves to some x' neighbor of x. However, immediately afterwards C_2 moves by $\widetilde{\sigma}^2$ to the same x'. In other words, starting from a state where R can avoid capture in one-cop CR, C_1 and C_2 essentially move as one cop, specifically, as cop C_1. Given (by construction of $\widehat{\sigma}^3$) R plays optimally w.r.t C_1, capture will never occur. Hence

$$\forall \sigma^1 : 0 = Q^1\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right) \geq Q^1\left(s_0, \sigma^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right) = 0. \tag{28}$$

3. The case of C_2 is similar, but attention must be paid to some details. Suppose C_2 uses some $\sigma^2 \neq \widetilde{\sigma}^2$ by which his first nontrivial move is to some x' neighbor of x. Right after R starts playing optimally w.r.t C_2 and let y' be his next move. Obviously y' will be such that $y' \neq x$, i.e., R does not run into C_1 (or otherwise C_2 could capture R in the next move). Next it is C_1's turn to move and by $\widetilde{\sigma}^1$ he moves to the same vertex x' as C_2 (and of course capture does not occur). In other words, starting from a state where R can avoid capture in one-cop CR, C_1 and C_2 essentially move as one cop, specifically, as cop C_2. Given R plays optimally w.r.t C_2, and in doing so, he does not run into C_1, capture will never occur. So

$$\forall \sigma^2 : 0 = Q^2\left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right) \geq Q^2\left(s_0, \widetilde{\sigma}^1, \sigma^2, \widehat{\sigma}^3\right) = 0. \tag{29}$$

Combining (27)-(29) we see that $\widehat{\sigma} = \left(s_0, \widetilde{\sigma}^1, \widetilde{\sigma}^2, \widehat{\sigma}^3\right)$ is a non-capturing NE of $\Gamma_3(G|s_0)$. $\square$

The above result is rather surprising when G has $c(G) = 2$: while *in CR* played on G two optimally playing (and cooperating) cops always capture the robber, *in SCAR* played on the same graph there exist non-capturing NE (even when $\varepsilon = \frac{1}{2}$, the cops' interests coincide and they have the motive to cooperate fully).

On the other hand, the result is *not* surprising when applied to G's with $c(G) \geq 3$. In fact, in this case Theorem 4.11 can be strengthened significantly: there will always exist some state with *only* non-capturing NE.[13]

Theorem 4.12. *For any G with $c(G) \geq 3$ the following holds:*

$$\forall (\gamma, \varepsilon) \in \Omega^{(3)}, \exists s_0 \in S : \textit{every NE of } \Gamma_3(G|s_0, \gamma, \varepsilon) \textit{ is non-capturing.}$$

Proof. Choose an $s_0 = (x, y, z, 1)$ such that in the two-cop CR started from s_0 the robber can avoid capture; this can always be achieved, since $c(G) \geq 3$, provided R uses an optimal (in two-cop CR) strategy $\widehat{\sigma}^3$. Also take any cop strategies σ^1, σ^2. Then the profile $\left(\sigma^1, \sigma^2, \widehat{\sigma}^3\right)$ will not result in capture, in either two-cop CR or in $\Gamma_3(G|s_0)$. The $\Gamma_3(G|s_0)$ payoffs will be

$$\forall \sigma^1, \sigma^2, \quad \forall n \in \{1, 2, 3\} : Q^n\left(s_0, \sigma^1, \sigma^2, \widehat{\sigma}^3\right) = 0.$$

Clearly, no player can improve his payoff by unilaterally changing his strategy. Hence, for every σ^1, σ^2, $\left(\sigma^1, \sigma^2, \widehat{\sigma}^3\right)$ is a non-capturing NE in $\Gamma_3(G|s_0)$. On the other hand, take *any* NE $\left(\sigma^1, \sigma^2, \sigma^3\right)$ of $\Gamma_3(G|s_0)$; then we must have

$$Q^3\left(s_0, \sigma^1, \sigma^2, \sigma^3\right) = 0$$

because otherwise R could use $\widehat{\sigma}^3$ and unilaterally improve his payoff. Hence *every* NE $\left(\sigma^1, \sigma^2, \sigma^3\right)$ of $\Gamma_3(G|s_0)$ is non-capturing. $\square$

The following corollary illuminates the connection of capturing and non-capturing NE of $\Gamma_3(G|s_0, \gamma, \varepsilon)$ to the classic cop number. The first part of the corollary is obtained from Theorem 4.11; the second from Theorems 4.5 and 4.7.

Corollary 4.13. *Given a graph G:*

1. *suppose that for all $(\gamma, \varepsilon) \in \Omega^{(3)}$ and $s_0 \in S$, every NE of $\Gamma_3(G|s_0, \gamma, \varepsilon)$ is capturing; then $c(G) = 1$.*
2. *suppose that for all $(\gamma, \varepsilon) \in \Omega^{(3)}$ there exists some $s_0 \in S$ such that every NE of $\Gamma_3(G|s_0, \gamma, \varepsilon)$ is non-capturing; then $c(G) \geq 3$.*

Finally, combining Theorem 4.5 and the first part of Corollary 4.13 we get the following.

Corollary 4.14. *G is cop-win iff : for all $(\gamma, \varepsilon) \in \Omega^{(3)}$ and $s_0 \in S$, every NE of $\Gamma_3(G|s_0, \gamma, \varepsilon)$ is capturing.*

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[13] However, we still have initial positions with capturing NE; e.g., when all players start at the same vertex.

5. *N*-player SCAR

5.1. Preliminaries

The generalization of $\Gamma_3(G)$ to $\Gamma_N(G)$, i.e., the N-player SCAR game is straightforward. For any $N \geq 2$, $\Gamma_N(G)$ is played by $N-1$ cops (denoted by $C_1, ..., C_{N-1}$) and a robber (denoted by R) who move along the edges of G.[14] The game starts from a prescribed initial position s_0 and is played in turns, with a single player moving at every turn; the moving sequence is $... \to C_1 \to C_2 \to ... \to C_{N-1} \to R \to ....$ The following briefly presented quantities are direct generalizations of those defined in Section 2.

The player set is $I = \{1, 2, ..., N\}$ or $I = \{C_1, C_2, ..., C_{N-1}, R\}$. A game position or state has the form $s = \left(x^1, x^2, ..., x^N, p\right)$, where x^n denotes the position of the n-th player and p denotes the player who has the next move. For $n \in \{1, 2, ..., N\}$, set

$$S^n = \left\{s = \left(x^1, ..., x^N, n\right) : (x^1, ..., x^N) \in V^N \text{ and } n \in I\right\}$$

is the set of states where player n has the next move. The set S of all states of the game is

$$S = S^1 \cup S^2 \cup ... \cup S^N \cup \{\tau\},$$

where τ is as before the terminal state and the set $S' = S \backslash \{\tau\}$ is the set of non-terminal states. We also define the set S_C of capture states and the set S_{NC} of non-capture states as follows:

$$S_C := \{s = (x^1, x^2, ..., x^N, n) \in S' : \exists i \in \{1, 2, ..., N-1\} : x^i = x^N\},$$
$$S_{NC} := \{s = (x^1, x^2, ..., x^N, n) \in S' : \forall i \in \{1, 2, ..., N-1\}\ x^i \neq x^N\ \}.$$

An alternative partition of S is therefore

$$S = S_C \cup S_{NC} \cup \{\tau\}.$$

Moreover, and since we can have simultaneous captures by any subset $\{n_1, n_2, ...\}$ of $\{1, 2, ..., N-1\}$, we define sets $S_C^{n_1}$, $S_C^{n_1 n_2}$, ..., $S_C^{12...N-1}$ analogously to the sets S_C^1, S_C^2, S_C^{12}; the union of all these sets is of course S_C.

Action sets $A^n(s)$ and the transition function $\mathbf{T}(s, a)$ are defined in a similar fashion as in Section 2. In case at time t the state is $s_t = \left(x_t^1, x_t^2, ..., x_t^N, n\right) \in S^n$ and at time $t+1$ the move by player n is $a_{t+1}^n \in A^n(s_t)$, then we use again the shorthand

$$s_{t+1} = \mathbf{T}\left(s_t, a_{t+1}^n\right).$$

Capture time T_C, histories and strategies are also defined analogously to Section 2. The same is true for payoffs. Specifically, at every non-capture state $s_t \in S_{NC} \cup \{\tau\}$, the immediate reward to each player $q^n(s_t)$ is zero; at every state $s_t \in S_C^{n_1 ... n_{N_1}}$ (i.e., when capture is effected by N_1 cops) the robber's loss is 1 and this is distributed between the $N-1$ cops as follows.

1. When $1 \leq N_1 \leq N-2$: each capturing (resp. non-capturing) cop receives an immediate reward of $\frac{1-\varepsilon}{N_1}$ (resp. $\frac{\varepsilon}{N-N_1-1}$).
2. When $N_1 = N-1$: all cops are capturing and each receives an immediate reward of $\frac{1}{N-1}$.

The *total* payoff of player n is $Q^n(s_0, s_1, s_2, ...) = \sum_{t=0}^{\infty} \gamma^t q^n(s_t)$. The (γ, ε) sets now are

$$\Omega^{(N)} = \left\{(\gamma, \varepsilon) : \gamma \in (0, 1), \varepsilon \in \left[0, \frac{1}{N-1}\right]\right\} = (0, 1) \times \left[0, \frac{1}{N-1}\right],$$
$$\widetilde{\Omega}^{(N)} = \left\{(\gamma, \varepsilon) : \gamma \in (0, 1), \varepsilon \in \left[0, \frac{1}{N-1}\right], \gamma < \frac{\varepsilon}{1-\varepsilon}\right\}.$$

The choice $\varepsilon \in \left[0, \frac{1}{N-1}\right]$ ensures satisfaction of the intuitive requirement that capturing cops should get at least as much as non-capturing ones:

$$\begin{aligned}
\varepsilon \leq \frac{1}{N-1} &\Rightarrow \left(\forall N_1 \in \{1, 2, ..., N-2\} : \varepsilon \leq \frac{N-1-(N-2)}{N-1} \leq \frac{N-1-N_1}{N-1}\right) \\
&\Rightarrow (\forall N_1 \in \{1, 2, ..., N-2\} : \varepsilon (N-1) \leq N-1-N_1) \\
&\Rightarrow (\forall N_1 \in \{1, 2, ..., N-2\} : \varepsilon N_1 \leq (1-\varepsilon)(N-1-N_1)) \\
&\Rightarrow \left(\forall N_1 \in \{1, 2, ..., N-2\} : \frac{\varepsilon}{N-1-N_1} \leq \frac{1-\varepsilon}{N_1}\right).
\end{aligned}$$

[14] Note that the case $N = 2$, i.e., one cop vs. one robber, is also included in the formulation.

In addition, again agreeing with our intuition, each capturing cop's reward is a decreasing function of N_1. Indeed, when $1 \le N_1 \le N-2$, their reward is $\frac{1-\varepsilon}{N_1}$ which is decreasing in N_1, with minimum value achieved at $N_1 = N-2$ and equal to $\frac{1-\varepsilon}{N-2}$; and when $N_1 = N-1$ (all cops are capturing) we have:

$$\varepsilon \le \frac{1}{N-1} \Rightarrow \frac{1-\varepsilon}{N-2} \ge \frac{1-\frac{1}{N-1}}{N-2} = \frac{1}{N-1}.$$

In short, the fewer capturing cops we have, the more is each of them rewarded.

Before proceeding, we note that (similarly to $\Gamma_3(G)$, see Section 3) $\Gamma_N(G)$ is path-equivalent to the modified $N-1$ cops CR.

5.2. Existence of NE

The next theorem shows the existence of positional NE for every $\Gamma_N(G|s_0)$. It generalizes Theorem 4.1 and is proved very similarly; hence the proof is omitted.

Theorem 5.1. *For every graph G and for every $s_0 \in S$, $(\gamma, \varepsilon) \in \Omega^{(N)}$ the game $\Gamma_N(G|s_0, \gamma, \varepsilon)$ has deterministic positional NE. Specifically, there exists a deterministic positional profile $\sigma_* = (\sigma_*^1, \sigma_*^2, ..., \sigma_*^N)$ such that*

$$\forall n, \forall s_0, \forall \sigma^n : Q^n\left(s_0, \sigma_*^n, \sigma_*^{-n}\right) \ge Q^n\left(s_0, \sigma^n, \sigma_*^{-n}\right). \tag{30}$$

For every s and n, let $u^n(s) = Q^n(s, \sigma_*)$. Then the following equations are satisfied

$$\forall n, \forall s \in S^n : \sigma_*^n(s) = \arg\max_{a^n \in A^n(s)} \left[q^n(s) + \gamma u^n\left(\mathbf{T}\left(s, a^n\right)\right)\right], \tag{31}$$

$$\forall n, m, \forall s \in S^n : u^m(s) = q^m(s) + \gamma u^m\left(\mathbf{T}\left(s, \sigma_*^n(s)\right)\right). \tag{32}$$

The next theorem generalizes Theorem 4.4 and shows that every $\Gamma_N(G|s_0)$ has NE which are, in general, non-positional. The proof (which is similar to that of Theorem 4.4 and hence will be omitted) depends on auxiliary two-player zero-sum games $\Gamma_N^1(G|s_0)$, ..., $\Gamma_N^N(G|s_0)$, where $\Gamma_N^n(G|s_0)$ is the two-player game with initial state s_0 in which P_n (who has payoff Q^n) plays against P_{-n} (who has payoff $-Q^n$ and controls $\{1, 2, ..., N\} \setminus \{n\}$). Similarly to the 3-player case, for each $s \in S$ and $n \in \{1, 2, ..., N\}$, the game $\Gamma_N^n(G|s_0)$ has a value and the players have optimal deterministic positional strategies. Strategies ϕ_n^n and ϕ_n^m are as in Section 4. The threat strategy of the n-th player in the N-player game $\Gamma_N(G|s_0)$ is $\overline{\sigma}^n$, defined (exactly as in Section 4) as follows:

$$\overline{\sigma}^n = \begin{cases} \phi_n^n & \text{as long as every player } m \in \{1, 2, ..., N\} \setminus n \text{ follows } \phi_m^m; \\ \phi_m^n & \text{as soon as some player } m \in \{1, 2, ..., N\} \setminus n \text{ "deviates" from } \phi_m^m. \end{cases} \tag{33}$$

Keeping the above in mind, we can prove the following.

Theorem 5.2. *For every graph G and for every $N \ge 3$, $s_0 \in S$, $(\gamma, \varepsilon) \in \Omega^{(N)}$, in the game $\Gamma_N(G|s_0, \gamma, \varepsilon)$ we have*

$$\forall n \in \{1, 2, ..., N\}, \forall \sigma^n : Q^n\left(s_0, \overline{\sigma}^1, \overline{\sigma}^2, ..., \overline{\sigma}^N\right) \ge Q^n(s, \sigma^n, \overline{\sigma}^{-n}) \tag{34}$$

where $\overline{\sigma}^n$ (for $n \in \{1, 2, ..., N\}$) is a deterministic strategy of the form (33).

As in 3-player SCAR $\Gamma_3(G|s_0)$, the above hold for any $\varepsilon \in [0, 1]$, not just for $\varepsilon \in \left[0, \frac{1}{N-1}\right]$.

5.3. Cop number, capturing and non-capturing NE

The following results generalize those appearing in Section 4.3 and hold for every $N \ge 2$.

Theorem 5.3. *For any G with $c(G) = 1$ the following holds:*

$$\forall (\gamma, \varepsilon) \in \Omega^{(N)}, \forall s_0 \in S : \textit{every NE of } \Gamma_N(G|s_0, \gamma, \varepsilon) \textit{ is capturing.}$$

Theorem 5.4. *For any G with $c(G) \le N-1$ the following holds:*

$$\forall (\gamma, \varepsilon) \in \Omega^{(N)}, \forall s_0 \in S : \textit{there exists a capturing NE of } \Gamma_N(G|s_0, \gamma, \varepsilon).$$

Theorem 5.5. *For any G with $c(G) \leq N-1$, let $\widehat{\sigma} = (\widehat{\sigma}^1, \widehat{\sigma}^2, ..., \widehat{\sigma}^N)$ be a strategy profile which is optimal in the $(N-1)$-cop CR game. Then the following holds:*

$$\forall (\gamma, \varepsilon) \in \widetilde{\Omega}^{(N)}, \forall s_0 \in S : \widehat{\sigma} \text{ is a capturing NE of } \Gamma_N (G|s_0, \gamma, \varepsilon).$$

Theorem 5.6. *For any G with $c(G) \geq 2$, the following holds:*

$$\forall (\gamma, \varepsilon) \in \Omega^{(N)}, \exists s_0 \in S : \text{there exists a non-capturing NE of } \Gamma_N (G|s_0, \gamma, \varepsilon).$$

Theorem 5.7. *For any G with $c(G) \geq N$ the following holds:*

$$\forall (\gamma, \varepsilon) \in \Omega^{(N)}, \exists s_0 \in S : \text{every NE of } \Gamma_N (G|s_0, \gamma, \varepsilon) \text{ is non-capturing.}$$

Corollary 5.8. *Given a graph G:*

1. *suppose that for all $(\gamma, \varepsilon) \in \Omega^{(N)}$ and $s_0 \in S$, every NE of $\Gamma_N (G|s_0, \gamma, \varepsilon)$ is capturing; then $c(G) = 1$.*
2. *suppose that for all $(\gamma, \varepsilon) \in \Omega^{(N)}$ there exists some $s_0 \in S$ such that every NE of $\Gamma_N (G|s_0, \gamma, \varepsilon)$ is non-capturing; then $c(G) \geq N$.*

Corollary 5.9. *G is cop-win iff : for all $(\gamma, \varepsilon) \in \Omega^{(N)}$ and $s_0 \in S$, every NE of $\Gamma_N (G|s_0, \gamma, \varepsilon)$ is capturing.*

5.4. Selfish cop number

We know that the cop number $c(G)$ of graph G is the minimum number of cops required to guarantee (when the cops play optimally and for any robber strategy and starting position) capture in CR played on G. Define correspondingly the *selfish cop number* for the SCAR game.

Definition 5.10. *The* selfish cop number *of a graph G is denoted by $c_s(G)$ and defined to be the smallest K such that: for any $(\gamma, \varepsilon) \in \Omega^{(K+1)}$ and any $s_0 \in S$, there exists a capturing NE of $\Gamma_{K+1} (G|s_0, \gamma, \varepsilon)$.*

The selfish cop number equals the classic one, as demonstrated in the following.

Theorem 5.11. *For every graph G we have $c_s(G) = c(G)$.*

Proof. Take any K such that $K \geq c(G)$, then by Theorem 5.4 we have that, for every $(\gamma, \varepsilon) \in \Omega^{(K+1)}$ and every $s_0 \in S$ there exists a capturing NE of $\Gamma_{K+1} (G|s_0, \gamma, \varepsilon)$. On the other hand, take any $K \leq c(G) - 1$, then by Theorem 5.7 we have that, for every $(\gamma, \varepsilon) \in \Omega^{(K+1)}$ there exists some $s_0 \in S$, such that there exists no capturing NE of $\Gamma_{K+1} (G|s_0, \gamma, \varepsilon)$. So $c_s(G)$ (the smallest K such that for every $(\gamma, \varepsilon) \in \Omega^{(K+1)}$ and every $s_0 \in S$ there exists a capturing NE of $\Gamma_{K+1} (G|s_0, \gamma, \varepsilon)$) equals $c(G)$. □

It follows that computing $c_s(G)$ is exactly as hard as $c(G)$. Namely, given some k we can decide whether $c_s(G) \leq k$ in polynomial time [12]; but computing $c_s(G)$ is NP-hard.

5.5. A connection between CR and SCAR

We will now show that a slightly modified version of N-player SCAR is, in a certain sense, equivalent to the CR game with $N-1$ cops. The modification consists in letting ε be a function of N and N_1, namely we use $\varepsilon(N, N_1) = \frac{N-1-N_1}{N-1}$. We will denote the modified SCAR game by $\Gamma_N (G|s_0, \gamma, \varepsilon(N, N_1))$.

The modification implies that the payoff of each capturing state does *not* depend on the number of capturing cops. The distribution of the payoff remains the same, i.e.,

1. each capturing cop receives a reward of $\frac{1-\varepsilon(N,N_1)}{N_1} = \frac{1-\frac{N-1-N_1}{N-1}}{N_1} = \frac{1}{N-1}$;
2. each non-capturing cop receives a reward of $\frac{\varepsilon(N,N_1)}{N-1-N_1} = \frac{\frac{N-1-N_1}{N-1}}{N-1-N_1} = \frac{1}{N-1}$.

In other words, for every capture each cop (whether he is capturing or non-capturing) receives the same reward.

SCAR with the above modification of ε falls under the general formulation of discounted stochastic games and all our previous results still hold. Furthermore, the $(N-1)$-cops CR game is *payoff-equivalent* to $\Gamma_N (G|s_0, \gamma, \varepsilon(N, N_1))$, by which we mean the following. Take any strategies $\sigma^1, \sigma^2, ..., \sigma^N$ and apply them

1. to $\Gamma_N (G|s_0, \gamma, \varepsilon(N, N_1))$, with σ^n being the strategy of the n-th player;

2. to the respective $(N-1)$-cops CR game starting from s_0, with σ^n (for $n \in \{1, 2, ..., N-1\}$) being the strategy the cop player uses for his n-th token, and σ^N being the strategy the robber player uses.

Then the same history $(s_0, s_1, s_2, ...)$ will be produced in the two games (they are path-equivalent) and, furthermore, the sum of the total payoffs of the cops (resp. the total payoff of the robber) in $\Gamma_N(G|s_0, \gamma, \varepsilon(N, N_1))$ will be the same as the payoff of the cop player (resp. robber player) in the $(N-1)$-cops CR game.

Since all cops in $\Gamma_N(G|s_0, \gamma, \varepsilon(N, N_1))$ receive the same payoff, their interests totally coincide: they all want *some* cop to capture the robber in the shortest possible time, just like in the $(N-1)$-cops CR game. Hence the following can be proved in a similar way as Theorems 4.9 and 5.5.

Theorem 5.12. *If the profile $((\widehat{\sigma}_1, ..., \widehat{\sigma}_{N-1}), \widehat{\sigma}_N)$ is optimal in the $(N-1)$-cops CR game, then the profile $(\widehat{\sigma}_1, ..., \widehat{\sigma}_{N-1}, \widehat{\sigma}_N)$ is a NE of $\Gamma_N(G|s_0, \gamma, \varepsilon(N, N_1))$.*

6. Conclusion

As we have already mentioned, very little work has been previously done on *multi-player* pursuit games. In this sense SCAR furnishes a novel generalization of CR and its numerous *two-player* variants. We find especially interesting the following aspects of SCAR.

1. On the "technical" side, the formulation of SCAR as a discounted game is quite advantageous. In the "natural" formulation of a pursuit game, payoff is expected capture time; since this can be unbounded, there is no obvious way to establish the existence of NE (in the multi-player case). On the other hand, in the SCAR formulation payoff is a *discounted* constant (see (5)); consequently the existence of a deterministic positional NE follows immediately from Fink's classical result. Furthermore, because SCAR is a perfect information game, its payoff can be immediately converted to capture time, thus preserving the semantics of a pursuit game.
2. On the "conceptual" side, our results indicate that (perhaps surprisingly) even when $N-1$ cops can capture the robber if they cooperate, they may settle on a non-cooperating, non-capturing Nash equilibrium. This is somewhat similar to the "lack-of-cooperation" phenomenon observed in other branches of Game Theory (e.g., in Prisoner's Dilemma and the Tragedy of the Commons).

The above facts indicate further research directions, which we intend to pursue in the future. We conclude this paper by briefly discussing some such directions.

1. *Refinement of equilibria*. The apparent paradox of non-capturing Nash equilibria may be resolved by using more refined equilibria concepts (subgame perfect equilibria, strong Nash equilibria, admissible equilibria etc.). An obvious target then is to establish the existence and nature of such equilibria in SCAR.
2. *SCAR variants*. These are obtained by changing the number and / or behaviors of the cops and robbers. Some possibilities are listed below; the methods of the current paper can be used to study the resulting variants.
   (a) One cop pursues several selfish robbers; each robber pays a penalty if he is captured and a (lower, perhaps zero) penalty if another robber is captured. In a sense this is the dual of the game we have studied in this paper.
   (b) More generally, $N-M$ cops pursue M robbers; the payoff of each player may reflect a completely or partially selfish behavior on his part.
   (c) Even more generally, *teams* of cops pursue teams of robbers; a team is a set of tokens controlled by a single player.
   (d) The robbers can be "passive": (i) they move on the graph according to predetermined, known transition functions and (ii) they do not receive any payoff (but their capture results in payoffs to the capturing and non-capturing cops). Conversely, we can have active robbers and passive cops.
3. *SCAR generalizations*. More generally, a family of *generalized multi-player pursuit/evasion games on graphs* can be obtained by varying the "capture relationship" between players. Here are two examples.
   (a) A game played by players P_1, P_2, ..., P_N, in which P_n pursues P_{n+1} (for $n \in \{1, 2, ..., N-1\}$); here we have a "linear" pursuit relationship.
   (b) The same as above but also P_N pursues P_1; here we have a "cyclic" pursuit relationship.
   Hence a player will, in general, be simultaneously pursuer and evader. Pursuit relationships are specified in terms of appropriate player payoffs,[15] e.g., the capturing (resp. captured) player receives (resp. pays) one time discounted unit. Again, the resulting games can be studied by the methods of the current paper.
4. *Non-perfect-information games*. A more drastic change (which can be used in conjunction to any of the previously mentioned variations) is to allow for *simultaneous* or *concurrent* player moves. This results in non-perfect-information games and their study will require more powerful methods than the ones presented in the current paper.

[15] A minimum requirement (to preserve the semantics of pursuit / evasion) is that total payoff is nondecreasing (resp. nonincreasing) with capture time for the evader (resp. pursuer).

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