ARTICLE IN PRESS

Information Sciences xxx (2010) xxx–xxx

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Some remarks on the lattice of fuzzy intervals

Ath. Kehagias

Division of Mathematics, Department of Mathematics, Physics and Computer Sciences, Faculty of Engineering, Box 464, Aristotle University of Thessaloniki, Thessaloniki, GR 54124, Greece

A R T I C L E I N F O

Article history:
Received 15 July 2009
Received in revised form 17 March 2010
Accepted 1 May 2010
Available online xxxx

Keywords:
Fuzzy sets
Lattices
Inclusion measures

A B S T R A C T

In this paper we study the connections between three related concepts which have appeared in the fuzzy literature: *fuzzy intervals*, *fuzzy numbers* and *fuzzy interval numbers* (FIN's). We show that these three concepts are very closely related. We propose a new definition which encompasses the three previous ones and proceeds to study the properties ensuing from this definition. Given a *reference* lattice $(X, \sqsubseteq)$, we define *fuzzy intervals* to be the fuzzy sets such that their p-cuts are closed intervals of $(X, \sqsubseteq)$. We show that, given a complete lattice $(X, \sqsubseteq)$, the collection of its fuzzy intervals is a complete lattice. Furthermore we show that, if $(X, \sqsubseteq)$ is completely distributive, then the lattice of its fuzzy intervals is distributive. Finally we introduce a new *inclusion measure*, which can be used to quantify the degree in which a fuzzy interval is contained in another, an approach which is particularly valuable in engineering applications.

1. Introduction

In this paper we study the connections between three related concepts which have appeared in the fuzzy literature: *fuzzy intervals*, *fuzzy numbers* and *fuzzy interval numbers* (FIN's). We show that these three concepts are very closely related, we propose a new definition which encompasses the three previous ones and proceed to study the properties ensuing from this definition.

While fuzzy intervals [30, p.58] and fuzzy numbers [29, p.97] are well established concepts in the fuzzy literature, FIN's have been introduced relatively recently by Kaburlasos [21–23,31] as a new computational intelligence tool. From the applications point of view, the usefulness of FIN's has been demonstrated in several engineering problems [22–27,32,33]. Some theoretical work on FIN's has also been done, especially the introduction of inclusion measures and metrics [22,23,33]. But, in our opinion, much more remains to be done regarding the theoretical foundation of FIN's and the current paper takes a step in this direction.

In this paper we study *positive FIN's* in the context of fuzzy lattices. In other words, we show that, under a suitable definition, a positive FIN is a fuzzy interval. Here by "fuzzy interval" we mean a fuzzy set M satisfying the following requirements: (i) M has underlying reference set X. (ii) M takes membership values in a complete lattice $(L, \leqslant)$ with minimum and maximum elements (0 and 1).[1] (iii) The cuts of M are *closed intervals* of a *reference lattice* $(X, \sqsubseteq)$. It appears that fuzzy intervals have previously been studied mainly in the case that the underlying X is the set of real numbers $\mathbb{R}$; Kaburlasos, on the other hand, has used FIN's in the context of more general X (he has used X to represent, among other possibilities, vectors, Boolean statements and graphs – for details see [22,24,25,27,33]). To achieve compatibility with Kaburlasos' general point of view we will study FIN's (and fuzzy intervals and fuzzy numbers) on a *general* reference lattice $(X, \sqsubseteq)$ (the only requirement being that $(X, \sqsubseteq)$ is complete).

E-mail addresses: kehagiat@auth.gr, kehagiat@gmail.com

[1] Note that we do *not* restrict ourselves to the *real* interval $[0, 1] \subseteq \mathbb{R}$.

0020-0255/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.ins.2010.05.007

Our treatment is algebraic and is connected to previous work on *fuzzy algebras*. Rosenfeld wrote the first paper on *fuzzy groups* [34]; *fuzzy rings* and *fuzzy ideals of rings* are studied in [5,18,46,47]. Seselja, Tepavcevska and others have presented a far reaching framework of *L*-fuzzy and *P*-fuzzy algebras [35–37]. Additional important work on fuzzy algebras appears in, among other places, [17,40,45,4]. *Fuzzy lattices* are a particular type of fuzzy algebras. A fuzzy lattice is a fuzzy set such that its cuts are sublattices of a "reference lattice" $(X, \sqsubseteq)$. Relatively little has been published on fuzzy lattices; Yuan and Wu introduced the concept [42] and Ajmal has studied it in greater detail [1–3]; Swamy and Raju [39] and, more recently, Tepavcevska and Trajkovski [41] have studied *L-fuzzy lattices*. Also fuzzy *hyperalgebras* have been studied in the past [10–12,14,15], especially *fuzzy hypergroups* [13,16,48,49], *fuzzy hyperrings* [47] etc.

The paper is organized as follows. Preliminary concepts are presented in Section 2; the connections between FIN's, fuzzy intervals and fuzzy numbers are discussed and then the lattice of fuzzy intervals is constructed in Section 3; its distributivity is proved in Section 4; inclusion measures in the FIN lattice are studied in Section 5; concluding remarks appear in Section 6.

2. Preliminaries

In the rest of the paper we will consider fuzzy subsets of a *reference set* X. We will assume that X is endowed with an order $\sqsubseteq$ and the structure $(X, \sqsubseteq, \sqcup, \sqcap)$ is a *complete lattice* (the "reference lattice") with $\sqcup$, $\sqcap$ denoting the join and meet operations respectively. In applications, X could be $\mathbb{R}, \mathbb{R}^N$ (with $N \geqslant 2$ and an appropriate order) or more general (e.g., Boolean valued) sets; examples of the application of lattice theoretic concepts to real-world problems can be found in [22–27,32,33] and elsewhere.

Since $(X, \sqsubseteq, \sqcup, \sqcap)$ is *complete*, for every $Y \subseteq X$ the elements $\sqcap Y, \sqcup Y$ exist; in particular, there exist $\sqcap X$ (the minimum element of X) and $\sqcup X$ (the maximum element of X), hence we can write $X = [\sqcap X, \sqcup X]$.

We will also need a *target lattice* $(L, \leqslant, \vee, \wedge)$, the lattice in which fuzzy subsets of X take membership values. Initially we will only assume that $(L, \leqslant, \vee, \wedge)$ is a complete lattice with a minimum element (denoted by 0) and a maximum element (denoted by 1).[2] Given a set $P \subseteq L, \vee P$ (resp. $\wedge P$) denotes the supremum (resp. the infimum) of P (these always exist, since $(L, \leqslant, \vee, \wedge)$ is assumed to be complete).

2.1. Intervals of a lattice

Closed intervals of $(X, \sqsubseteq, \sqcup, \sqcap)$ will be of special interest to us.

Definition 2.1. Given x_1, $x_2 \in X$, with $x_1 \sqsubseteq x_2$, the *closed interval* $[x_1, x_2]$ is defined by $[x_1, x_2] \doteq \{z : x_1 \sqsubseteq z \sqsubseteq x_2\}$.

We consider the empty set $\emptyset$ to be a closed interval, the so called *empty interval*. This can also be denoted as $[x_1, x_2]$ with any x_1, x_2 such that $x_2 \sqsubseteq x_1$. We will denote by $\mathbf{I}(X)$ (or simply by $\mathbf{I}$) the collection of closed intervals of X (including the empty interval). The structure $(\mathbf{I}, \subseteq)$ is an ordered set. In fact it is a lattice, as the following propositions show (proofs are omitted for brevity; they follow from the fact that being a closed interval is a *closure property* on $(\mathbf{I}, \subseteq)$ [6]).

Proposition 2.2. *Given any nonempty interval* $A = [a_1, a_2] \subseteq X$, *we have* $a_1 = \sqcap A$, $a_2 = \sqcup A$.

Proposition 2.3. *Given any family of closed intervals* $\mathbf{J} \subseteq \mathbf{I}$ *the set* $\cap_{[a_1,a_2]\in\mathbf{J}}[a_1, a_2]$ *is a closed interval; more specifically, we have*

$$\cap_{[a_1,a_2]\in\mathbf{J}}[a_1, a_2] = [\sqcup_{[a_1,a_2]\in\mathbf{J}} a_1, \sqcap_{[a_1,a_2]\in\mathbf{J}} a_2]$$

and this is the largest closed interval contained by every member of $\mathbf{J}$.

Definition 2.4. Given A, $B \in \mathbf{I}$, define $\mathbf{S}(A,B) \doteq \{C : C \in \mathbf{I}, A \subseteq C, B \subseteq C\}$. Then we define the operation $\dot\cup$ as follows

$$A \dot\cup B \doteq \cap_{C\in\mathbf{S}(A,B)} C.$$

Proposition 2.5. *The structure* $(\mathbf{I}, \subseteq, \dot\cup, \cap)$ *is a lattice with respect to the* $\subseteq$ *order (i.e. set theoretic inclusion). Given any intervals* $A = [a_1, a_2] \in \mathbf{I}$, $B = [b_1, b_2] \in \mathbf{I}$, $\sup(A, B) = A \dot\cup B = [a_1 \sqcap b_1, a_2 \sqcup b_2]$, $\inf(A, B) = A \cap B = [a_1 \sqcup b_1, a_2 \sqcap b_2]$.

Remark. In other words, given any intervals $A = [a_1, a_2]$, $B = [b_1, b_2]$ we have: (i) $[a_1 \sqcap b_1, a_2 \sqcup b_2]$ is the smallest closed interval which contains both A and B and (ii) $[a_1 \sqcup b_1, a_2 \sqcap b_2]$ is the largest closed interval contained by both A and B.

[2] In Section 4 we will introduce additional assumptions on the structure of $(L, \leqslant, \vee, \wedge)$.

2.2. Fuzzy sets

By "fuzzy set" we simply mean a function M: $X \to L$. We repeat that X is the reference set and L the set of membership values.

Definition 2.6. A *fuzzy set* is a function M: $X \to L$. The collection of all fuzzy sets (from X to L) will be denoted by $\mathbf{F}(X,L)$ or simply by $\mathbf{F}$.

In a standard manner, we introduce an order on $\mathbf{F}$ using the "pointwise" order of $(L,\leqslant,\vee,\wedge)$. The symbols $\leqslant,\vee,\wedge$ will be used between elements of $\mathbf{F}$ (as well as between elements of L) without danger of confusion.

Definition 2.7. For M, $N \in \mathbf{F}$ we write $M \leqslant N$ iff for all $x \in X$ we have: $M(x) \leqslant N(x)$.
Remark. The above definition has the following interpretation: a fuzzy set M is smaller than a fuzzy set N if every element $x \in X$ belongs to N at least as much as it does to M (in other words: $M(x) \leqslant N(x)$). Note that "$\leqslant$" is a straight generalization of the inclusion relationship "$\subseteq$" of classical sets. To understand this consider that (a) the "membership function" of a classical set M is its indicator function $\mathbf{1}_M(x)$, defined to be $\mathbf{1}_M(x) = 1$ iff $x \in M$ and 0 otherwise and (b) M is a subset of classical set N iff $\mathbf{1}_M(x) \leqslant \mathbf{1}_N(x)$, i.e. whenever $\mathbf{1}_M(x) = 1$ (i.e., $x \in M$) then $\mathbf{1}_N(x) = 1$ (i.e., $x \in N$) as well. Hence the order relationship of Definition 2.7 is a generalization of fuzzy set inclusion. However alternative, fuzzy valued extensions of fuzzy set inclusion are possible and, indeed, desirable; for a discussion see Section 5 and [28].

Definition 2.8. For M, $N \in \mathbf{F}$: we define the fuzzy set $M \vee N$ by: $(M \vee N)(x) \doteq M(x) \vee N(x)$; we define the fuzzy set $M \wedge N$ by: $(M \wedge N)(x) \doteq M(x) \wedge N(x)$.

It is well known [30] that $\leqslant$ is an order on $\mathbf{F}$ and that $(\mathbf{F},\leqslant,\vee,\wedge)$ is a complete lattice with sup $(M,N) = M \vee N$, inf $(M,N) = M \wedge N$. Also, given any set $\mathbf{A} \subseteq \mathbf{F}$, the infimum of $\mathbf{A}$, denoted by $\underline{\mathbf{A}}$, is a fuzzy set defined for every $x \in X$ by

$$\underline{\mathbf{A}}(x) = \wedge\{A(x) : A \in \mathbf{A}\}$$

and the supremum of $\mathbf{A}$, denoted by $\overline{\mathbf{A}}$, is a fuzzy set defined for every $x \in X$ by

$$\overline{\mathbf{A}}(x) = \vee\{A(x) : A \in \mathbf{A}\}$$

(these always exist, since it has been assumed that $(L,\leqslant,\vee,\wedge)$ is complete).

Definition 2.9. Given a fuzzy set M: $X \to L$, the *p-cut* of M is denoted by M_p and defined by $M_p \doteq \{x: M(x) \geqslant p\}$.

We will need some properties of p-cuts, summarized in the following propositions. Their proofs can be found in [30].

Proposition 2.10. *Take any $M \in \mathbf{F}$ with p-cuts $\{M_p\}_{p\in L}$ and $N \in \mathbf{F}$ with p-cuts $\{N_p\}_{p\in L}$. Then $M = N$ iff for all $p \in L$ we have $M_p = N_p$.*

Proposition 2.11. *Take any $M \in \mathbf{F}$ with p-cuts $\{M_p\}_{p\in L}$. Then we have the following.*

(i) *For all p, $q \in L$ we have: $p \leqslant q \Rightarrow M_q \subseteq M_p$.*
(ii) *For all $P \subseteq L$ we have: $\cap_{p\in P} M_p = M_{\vee P}$.*
(iii) $M_0 = X$.

Proposition 2.12. *Consider a family of sets $\{\widetilde{M}_p\}_{p\in L}$ which satisfy the following.*

(i) *For all p, $q \in L$ we have: $p \leqslant q \Rightarrow \widetilde{M}_q \subseteq \widetilde{M}_p$.*
(ii) *For all $P \subseteq L$ we have: $\cap_{p\in P} \widetilde{M}_p = \widetilde{M}_{\vee P}$.*
(iii) $\widetilde{M}_0 = X$.

Define the fuzzy set $M(x) = \vee\{p : x \in \widetilde{M}_p\}$. Then for all $p \in L$ we have $M_p = \widetilde{M}_p$.

Proposition 2.13. *For all $M, N \in \mathbf{F} : M \leqslant N \iff (\forall p \in L : M_p \subseteq N_p)$.*

Proposition 2.14. *For all M, $N \in \mathbf{F}$, $p \in L$: (i) $(M \vee N)_p = M_p \cup N_p$, (ii) $(M \wedge N)_p = M_p \cap N_p$.*

2.3. Fuzzy lattices

The concept of *fuzzy sublattice* was introduced by Yuan and Wu [42] and the concept of *fuzzy convex sublattice* was introduced by Ajmal and Thomas [1]. These concepts were studied in [43,50] and, especially, by Ajmal and Thomas in [1–3].

We now define "fuzzy sublattice" and "fuzzy convex sublattice" in a manner different from (but equivalent to) the standard one used in [1].

Definition 2.15. We say M: it $X \to L$ is a *fuzzy sublattice* of $(X,\sqsubseteq)$ iff $\forall p \in L$ the set M_p is a sublattice of $(X,\sqsubseteq)$.

Definition 2.16. We say M: $X \to L$ is a *fuzzy convex sublattice* of $(X,\sqsubseteq)$ iff $\forall p \in L$ the set M_p is a convex sublattice of $(X,\sqsubseteq)$; (i.e. $\forall p \in L, \forall x,y \in M_p$ we have $[x \sqcap y, x \sqcup y] \subseteq M_p$).

The next proposition shows that our Definition 2.15 of fuzzy sublattice is equivalent to the one used in [1].

Proposition 2.17. *M: $X \to L$ is a fuzzy sublattice of $(X,\sqsubseteq)$ iff*

$$\forall x, y \in X: \quad M(x \sqcap y) \wedge M(x \sqcup y) \geqslant M(x) \wedge M(y).$$

Proof. See [41]. □

Proposition 2.18. *Let M: $X \to L$ be a fuzzy sublattice of $(X,\sqsubseteq)$. It is a fuzzy convex sublattice of $(X,\sqsubseteq)$ iff*

$$\forall x, y \in X, \forall z \in [x \sqcap y, x \sqcup y]: \quad M(z) \geqslant M(x \sqcap y) \wedge M(x \sqcup y) = M(x) \wedge M(y). \tag{1}$$

Proof

(i) Assume M is a fuzzy convex sublattice. Choose any x, $y \in X$. Set $p_1 = M(x \sqcap y)$, $p_2 = M(x \sqcup y)$; then $x \sqcap y, x \sqcup y \in M_{p_1 \wedge p_2}$. Take any $z \in [x \sqcap y, x \sqcup y]$. Since M is a fuzzy convex sublattice: $z \in M_{p_1 \wedge p_2} \Rightarrow M(z) \geqslant p_1 \wedge p_2 = M(x \sqcap y) \wedge M(x \sqcup y)$. Since x, $y \in [x \sqcap y, x \sqcup y]$ we have $M(x) \geqslant M(x \sqcap y) \wedge M(x \sqcup y)$, $M(y) \geqslant M(x \sqcap y) \wedge M(x \sqcup y)$; and so $M(x) \wedge M(y) \geqslant M(x \sqcap y) \wedge M(x \sqcup y)$. On the other hand, since M is a fuzzy sublattice, from Proposition 2.17 we have $M(x \sqcap y) \wedge M(x \sqcup y) \geqslant M(x) \wedge M(y)$. Hence $M(x \sqcap y) \wedge M(x \sqcup y) = M(x) \wedge M(y)$.

(ii) Conversely, assume (1) holds. Take any $p \in L$. If M_p is empty, then it is a convex sublattice. If M_p is not empty, take any x, $y \in M_p$. Set $p_1 = M(x)$, $p_2 = M(y)$. We have $x \in M_p \Rightarrow p_1 = M(x) \geqslant p$, $y \in M_p \Rightarrow p_2 = M(y) \geqslant p$. From (1) we have $M(x \sqcap y) \geqslant M(x) \wedge M(y) = p_1 \wedge p_2 \geqslant p \Rightarrow x \sqcap y \in M_p$. Similarly $x \sqcup y \in M_p$ and so M_p is a sublattice. Set $q_1 = M(x \sqcap y)$, $q_2 = M(x \sqcup y)$. Now take any $z \in [x \sqcap y, x \sqcup y]$. From (1) we have $M(z) \geqslant q_1 \wedge q_2 = p_1 \wedge p_2 \geqslant p \Rightarrow z \in M_p$. Hence M_p is a convex sublattice for all $p \in L$, i.e. M is a fuzzy convex sublattice. □

3. The fuzzy intervals lattice

The concepts of *fuzzy number* and *fuzzy interval* have appeared in the literature.[3] For example, Klir and Yuan [29] define a *fuzzy number* to be a fuzzy set M which

1. is normal (i.e., $\exists x$: $M(x) = 1$);
2. has finite support (i.e., $\exists x_1, x_2$: $\forall x \notin [x_1, x_2]$: $M(x) = 0$);
3. for all $p \in \mathbb{R}, M_p$ is a closed interval.

On the other hand, Nguyen and Walker [30] define a *fuzzy interval* to be a fuzzy set M which

1. is normal;
2. has finite support;
3. for all $p \in \mathbb{R}, M_p$ is a closed interval.

It is clear that "fuzzy number" as defined by Klir and Yuan and "fuzzy interval" as defined by Nguyen and Walker are identical. On the other hand, Kaburlasos, [33] gives the following more general definition, which combines aspects of both fuzzy interval and fuzzy number.

Definition 3.1. A *positive fuzzy interval number* (positive FIN) is a function M: $(0,1] \to \mathbf{I}(X)$ which satisfies

$$p_1 \leqslant p_2 \Rightarrow M(p_2) \subseteq M(p_1). \tag{2}$$

We see that Kaburlasos drops the requirements of normality and finite support; but we consider these differences not very significant. We also recognize that the basic requirement for both fuzzy interval and fuzzy number appears to be that the sets M_p (or $M(p)$ in Kaburlasos' notation) are closed intervals. Kaburlasos does not explicitly state that the $M(p)$'s are the p-cuts of a fuzzy set, but in [33] and elsewhere corresponds FIN's to fuzzy sets (as one possible interpretation of FIN's). In-

[3] L-fuzzy numbers have also been studied; for example, see [20].

Please cite this article in press as: A. Kehagias, Some remarks on the lattice of fuzzy intervals, Inform. Sci. (2010), doi:10.1016/j.ins.2010.05.007

deed the requirement (2) makes it quite obvious that $M(p)$ can be interpeted as a p-cut of a fuzzy set.[4] Kaburlasos' main contribution is that he allows the M_p's to be closed intervals of a *general* lattice X (note the use of $\mathbf{I}(X)$ in Definition 3.1) rather than intervals of $\mathbb{R}$. This allows the use of fuzzy intervals (or FIN's) in a much wider variety of applications. In our opinion, Kaburlasos' FIN's are the natural generalization of fuzzy intervals (and fuzzy numbers) which many fuzzy researchers have previously used in the context of the real number system.

Based on Kaburlasos' generalization, we will now proceed to define fuzzy intervals in a rigorous manner and derive some of their properties. We emphasize that all the results derived in the remainder of the paper, although phrased in terms of fuzzy intervals, can be equally well applied to FIN's. The following (new) definition of *fuzzy intervals* is the one that will be used in the rest of the paper.

Definition 3.2. We say $M: X \to L$ *is* a fuzzy interval of $(X, \sqsubseteq)$ *iff*

$$\forall p \in L: \ M_p \quad \text{is a closed interval of} (X, \leqslant).$$

The collection all fuzzy intervals will be denoted by $\widetilde{\mathbf{I}}(X, L)$ or simply by $\widetilde{\mathbf{I}}$.

The following proposition will be often used in the sequel. It states that an arbitrary intersection of fuzzy intervals yields a fuzzy interval.

Proposition 3.3. *For all* $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}}$ *we have:* $\wedge_{M \in \widetilde{\mathbf{J}}} M \in \widetilde{\mathbf{I}}$

Proof. Choose any $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}} \subseteq \mathbf{F}$. The fuzzy set $\wedge_{M \in \widetilde{\mathbf{J}}} M$ is well defined, in view of the fact that $(\mathbf{F}, \leqslant, \vee, \wedge)$ is a complete lattice. Choose any $p \in L$. It is easy to show that $(\wedge_{M \in \widetilde{\mathbf{J}}} M)_p = \cap_{M \in \widetilde{\mathbf{J}}} M_p$. Then for every $M \in \widetilde{\mathbf{J}}$, the cut M_p will be a closed interval (perhaps the empty interval). From Proposition 2.3, an arbitrary intersection of closed intervals yields a closed interval. Hence, for every $p \in L$ the set $(\wedge_{M \in \widetilde{\mathbf{J}}} M)_p$ is a closed interval, i.e. $\wedge_{M \in \widetilde{\mathbf{J}}} M$ is a fuzzy interval.

Since $\widetilde{\mathbf{I}} \subseteq \mathbf{F}$, it follows that $(\widetilde{\mathbf{I}}, \leqslant)$ is an ordered set. We now establish (using Proposition 3.3) that $(\widetilde{\mathbf{I}}, \leqslant)$ is a lattice. □

Definition 3.4. For all $M, N \in \widetilde{\mathbf{I}}$ we define $M \dot{\vee} N$ as follows. We define $\widetilde{\mathbf{S}}(M, N) \doteq \{A : A \in \widetilde{\mathbf{I}}, M \leqslant A, N \leqslant A\}$ and then define

$$M \dot{\vee} N \doteq \wedge_{A \in \widetilde{\mathbf{S}}(M,N)} A$$

Proposition 3.5. $(\widetilde{\mathbf{I}}, \leqslant, \dot{\vee}, \wedge)$ *is a complete lattice.*

Proof

(i) $M \wedge N$ is the infimum in $\mathbf{F}$ of M and N. From Proposition 3.3 we have $M \wedge N \in \widetilde{\mathbf{I}}$, hence $M \wedge N$ is also the infimum of M and N in $\widetilde{\mathbf{I}}$.

(ii) For all $A \in \widetilde{\mathbf{S}}(M, N)$ we have $M \leqslant A$ and so $M \leqslant \wedge_{A \in \widetilde{\mathbf{S}}(M,N)} A = M \dot{\vee} N$; similarly $N \leqslant M \dot{\vee} N$. Furthermore, if there is some $B \in \widetilde{\mathbf{I}}$ such that $M \leqslant B$, $N \leqslant B$, then $B \in \widetilde{\mathbf{S}}(M, N)$. Hence $M \dot{\vee} N = \wedge_{A \in \widetilde{\mathbf{S}}(M,N)} A \leqslant B$. Finally, since $\widetilde{\mathbf{S}}(M, N) \subseteq \widetilde{\mathbf{I}}$, we have $M \dot{\vee} N = \wedge_{A \in \widetilde{\mathbf{S}}(M,N)} A \in \widetilde{\mathbf{I}}$. Hence $M \dot{\vee} N$ is the supremum in $\widetilde{\mathbf{I}}$ of M and N.

(iii) To establish completeness of $(\widetilde{\mathbf{I}}, \leqslant, \dot{\vee}, \wedge)$ we must show that any $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}}$ has an infimum and a supremum in $\widetilde{\mathbf{I}}$. We have already remarked (Proposition 3.3) that, for any $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{I}}$, the set $\wedge_{M \in \widetilde{\mathbf{J}}} M$ is a well defined fuzzy interval. Since $\wedge \widetilde{\mathbf{J}} = \wedge_{M \in \widetilde{\mathbf{J}}} M$ is the infimum of $\widetilde{\mathbf{J}}$ in $\mathbf{F}$, it will also be the infimum of $\widetilde{\mathbf{J}}$ in $\widetilde{\mathbf{I}} \subseteq \mathbf{F}$. Regarding the supremum, we must define appropriately $\dot{\vee} \widetilde{\mathbf{J}}$. Define a set $\widetilde{\mathbf{S}}(\widetilde{\mathbf{J}}) = \{A \in \widetilde{\mathbf{I}} : \forall M \in \widetilde{\mathbf{J}} \text{ we have } M \leqslant A\}$. Define $\dot{\vee} \widetilde{\mathbf{J}} \doteq \wedge_{A \in \widetilde{\mathbf{S}}(\widetilde{\mathbf{J}})} A$. Then $\dot{\vee} \widetilde{\mathbf{J}} \in \widetilde{\mathbf{I}}$ (as an intersection of fuzzy intervals), and it is easy to show that: $\forall M \in \widetilde{\mathbf{J}}$ we have $M \leqslant \dot{\vee} \widetilde{\mathbf{J}}$, $\forall A \in \widetilde{\mathbf{S}}(\widetilde{\mathbf{J}})$ we have $\dot{\vee} \widetilde{\mathbf{J}} \leqslant A$. Hence $\dot{\vee} \widetilde{\mathbf{J}}$ is the supremum of $\widetilde{\mathbf{J}}$ and completeness has been established. □

The following propositions establish some properties of fuzzy intervals.

Definition 3.6. For every fuzzy set M we define $L_M \doteq \{p: M_p \neq \emptyset\}$.

Proposition 3.7. *Let* $M: X \to L$ *be a fuzzy set. Then* M *is a fuzzy interval of* X *if and only if* M *is a fuzzy convex sublattice of* X *and satisfies the condition*

[4] The technical condition $\cap_{p \in P} M_p = M_{\vee P}$ is not required in [33]; however it is a necessary condition for a collection $\{M_p\}_{p \in L}$ to be the cuts of a fuzzy set.

Please cite this article in press as: A. Kehagias, Some remarks on the lattice of fuzzy intervals, Inform. Sci. (2010), doi:10.1016/j.ins.2010.05.007

$$\forall p \in L_M : M(\sqcap M_p) \geqslant \wedge_{x \in M_p} M(x), \quad M(\sqcup M_p) \geqslant \wedge_{x \in M_p} M(x). \tag{3}$$

Proof

(i) Assume M is a fuzzy convex sublattice of X and satisfies (3). Choose any $p \in L_M$. Now, by completeness of $(X, \sqsubseteq)$, $\sqcap M_p$ and $\sqcup M_p$ exist. Clearly $M_p \subseteq [\sqcap M_p, \sqcup M_p]$. On the other hand, from (3), $M(\sqcap M_p) \geqslant \wedge_{x \in M_p} M(x) \geqslant p \Rightarrow \sqcap M_p \in M_p$, i.e. M_p contains its infimum. Similarly $M(\sqcup M_p) \geqslant \wedge_{x \in M_p} M(x) \geqslant p \Rightarrow \sqcup M_p \in M_p$. Since M_p is a convex sublattice and $\sqcap M_p$, $\sqcup M_p \in M_p$, it follows that $[\sqcap M_p, \sqcup M_p] \subseteq M_p$. Hence for all $p \in L_M$ we have that $M_p = [\sqcap M_p, \sqcup M_p]$. Further, for all $p \in L - L_M$, M_p is the empty set, which is considered a closed interval. Hence for all $p \in L$ the set M_p is a closed interval, i.e. M is a fuzzy interval.

(ii) Conversely, assume M is a fuzzy interval. Then for all $p \in L_M$ we have $M_p = [\sqcap M_p, \sqcup M_p]$, which is a closed interval and *a fortiori* a convex sublattice. Hence M is a fuzzy convex sublattice. Furthermore, $M_p = [\sqcap M_p, \sqcup M_p] \Rightarrow \sqcap M_p \in M_p \Rightarrow M(\sqcap M_p) \geqslant \wedge_{x \in M_p} M(x)$. Similarly, $\sqcup M_p \in M_p \Rightarrow M(\sqcup M_p) \geqslant \wedge_{x \in M_p} M(x)$. □

Corollary 3.8. *If M is a fuzzy interval, then $\forall p \in L_M$ we have $M(\sqcap M_p) \wedge M(\sqcup M_p) = \wedge_{x \in M_p} M(x)$.*

Corollary 3.9. *Let X be finite. Then every fuzzy convex sublattice is a fuzzy interval and conversely.*

Proposition 3.10. *If M is a fuzzy interval, then $\forall p \in L_M$ we have $M_p = M_{p_1 \wedge p_2}$, where $p_1 = M(\sqcap M_p)$, $p_2 = M(\sqcup M_p)$.*

Proof. Choose any $p \in L_M$. Since M is a fuzzy interval, we have $M_p = [\sqcap M_p, \sqcup M_p]$. Set $p_1 = M(\sqcap M_p) \geqslant p$, $p_2 = M(\sqcup M_p) \geqslant p$. Then $M(\sqcap M_p) = p_1 \geqslant p_1 \wedge p_2$ and so $\sqcap M_p \in M_{p_1 \wedge p_2}$. Similarly $\sqcup M_p \in M_{p_1 \wedge p_2}$. Since M is a fuzzy interval (and so a fuzzy convex sublattice) it follows that $[\sqcap M_p, \sqcup M_p] \subseteq M_{p_1 \wedge p_2}$. On the other hand $p_1 \wedge p_2 \geqslant p \Rightarrow M_{p_1 \wedge p_2} \subseteq M_p = [\sqcap M_p, \sqcup M_p]$. Hence $M_{p_1 \wedge p_2} = M_p$. □

4. Distributivity of the fuzzy intervals lattice

In all of this section we assume $(X, \sqsubseteq, \sqcup, \sqcap)$ to be *completely distributive* according to the following definition.

Definition 4.1. The lattice $(X, \sqsubseteq, \sqcup, \sqcap)$ is said to be *completely distributive*, iff for every set $Y \subseteq X$ we have $x \sqcup (\sqcap_{y \in Y} y) = \sqcap_{y \in Y}(x \sqcup y)$, $x \sqcap (\sqcup_{y \in Y} y) = \sqcup_{y \in Y}(x \sqcap y)$.

In addition, we will assume $(L, \leqslant, \vee, \wedge)$ to be completely distributive and a *chain* (i.e., L is totally ordered with respect to $\leqslant$). We also retain the assumptions that $(L, \leqslant, \vee, \wedge)$ is a complete lattice, with minimum element 0 and maximum element 1.

Let M, N be fuzzy intervals. Our first task is to establish some properties of the cuts $(M \wedge N)_p$ and $(M \dot{\vee} N)_p$. From Proposition 3.5 we see that $M \wedge N$ and $M \dot{\vee} N$ are fuzzy intervals; hence $\forall p \in L$ the cuts $(M \wedge N)_p$ and $(M \dot{\vee} N)_p$ are closed intervals.

Definition 4.2. For all $M, N \in \widetilde{\mathbf{I}}$ and for all $p \in L$ we define $C_p(M, N) = M_p \cap N_p$.

We recall (Proposition 2.14) the following.

Proposition 4.3. *For all $M, N \in \widetilde{\mathbf{I}}$ and for all $p \in L$ we have: $(M \wedge N)_p = C_p(M, N)$.*

Proposition 4.4. *Take any $M, N \in \widetilde{\mathbf{I}}$. We have:*

(i) $\forall p, q \in L : p \leqslant q \Rightarrow C_q(M, N) \subseteq C_p(M, N)$,
(ii) $\forall P \subseteq L : \cap_{p \in P} C_p(M, N) = C_{\vee P}(M, N)$.
(iii) $C_0(M, N) = X$.

Proof. These properties follow from the fact that for all $p \in L$ we have $C_p(M, N) = (M \wedge N)_p$, i.e. the family $\{C_p(M, N)\}_{p \in L}$ is a family of cuts. □

Hence we have characterized the cuts of $M \wedge N$ in terms of the cuts of M and N. We will now do the same for the cuts of $M \dot{\vee} N$. However, before proceeding we need some auxiliary definitions and propositions.

Definition 4.5. For every $M \in \widetilde{\mathbf{I}}$, we define the functions $\underline{M} : L \to X, \overline{M} : L \to X$ as follows. For $p \in L_M$, $\underline{M}(p) \doteq \sqcap M_p$, $\overline{M}(p) \doteq \sqcup M_p$; for $p \in L - L_M$, $\underline{M}(p) \doteq \sqcup X$, $\overline{M}(p) \doteq \sqcap X$.

Remark. Hence we can write $M_p = [\underline{M}(p), \overline{M}(p)]$ for *every* $p \in L$. Because: if $p \in L_M$, then $M_p = [\sqcap M_p, \sqcup M_p] = [\underline{M}(p), \overline{M}(p)]$; if $p \in L - L_M$, then $M_p = \emptyset = [\sqcup X, \sqcap X] = [\underline{M}(p), \overline{M}(p)]$.

Please cite this article in press as: A. Kehagias, Some remarks on the lattice of fuzzy intervals, Inform. Sci. (2010), doi:10.1016/j.ins.2010.05.007

Proposition 4.6. *Take any $M \in \widetilde{\mathbf{I}}$ and for all $p \in L$ set $M_p = [\underline{M}(p), \overline{M}(p)]$. Then*

(i) $\forall p, q \in L : p \leqslant q \Rightarrow (\underline{M}(p) \sqsubseteq \underline{M}(q), \overline{M}(p) \sqsupseteq \overline{M}(q))$.
(ii) $\forall P \subseteq L : \sqcup_{p \in P} \underline{M}(p) = \underline{M}(\vee P), \sqcap_{p \in P} \overline{M}(p) = \overline{M}(\vee P)$.

Proof

(i) Since $\{M_p\}_{p \in P}$ are cuts, from Proposition 2.11.(i) we have: $p \leqslant q \Rightarrow M_q \subseteq M_p \Rightarrow [\underline{M}(q), \overline{M}(q)] \subseteq [\underline{M}(p), \overline{M}(p)] \Rightarrow (\underline{M}(p) \leqslant \underline{M}(q), \overline{M}(p) \geqslant \overline{M}(q))$. Note in particular that: if $q \notin L_M$, then $\underline{M}(p) \sqsubseteq \underline{M}(q) = \sqcup X$ and $\overline{M}(p) \sqsupseteq \overline{M}(q) = \sqcap X$.
(ii) Since $\{M_p\}_{p \in P}$ are cuts, from Proposition 2.11.(ii) we have: $\cap_{p \in P} M_p = M_{\vee P}$. But $M_{\vee P} = [\underline{M}(\vee P), \overline{M}(\vee P)]$ and (Proposition 2.3) $\cap_{p \in P} M_p = [\sqcup_{p \in P} \underline{M}(p), \sqcap_{p \in P} \overline{M}(p)]$ which yields the required result. Note in particular that: if there exists some $q \in P$ such that $q \in L - L_M$, then $M_q = \emptyset, \cap_{p \in P} M_p = \emptyset$, and $M_{\vee P} = \emptyset = [\underline{M}(\vee P), \overline{M}(\vee P)]$ with $\underline{M}(\vee P) = \sqcup X, \overline{M}(\vee P) = \sqcap X$. Also, in this case $\underline{M}(q) = \sqcup X, \sqcup_{p \in P} \underline{M}(p) = \sqcup X, \overline{M}(q) = \sqcap X, \sqcap_{p \in P} \underline{M}(p) = \sqcap X$. □

Proposition 4.7

(i) *Take any $P \subseteq L$ and any functions F: $L \to \to X$ which satisfy*

$$p \leqslant q \Rightarrow F(p) \sqsubseteq F(q), \quad \sqcup_{p \in P} F(p) = F(\vee P),$$
$$p \leqslant q \Rightarrow G(p) \sqsubseteq G(q), \quad \sqcup_{p \in P} G(p) = G(\vee P).$$

Then $\sqcup_{p \in P}(F(p) \sqcap G(p)) = F(\vee P) \sqcap G(\vee P)$.
(ii) *Take any $P \subseteq L$ and any functions F: $L \to \to X$ which satisfy*

$$p \leqslant q \Rightarrow F(p) \sqsupseteq F(q), \qquad \sqcap_{p \in P} F(p) = F(\vee P),$$
$$p \leqslant q \Rightarrow G(p) \sqsupseteq G(q), \qquad \sqcap_{p \in P} G(p) = G(\vee P).$$

Then $\sqcap_{p \in P}(F(p) \sqcup G(p)) = F(\vee P) \sqcup G(\vee P)$.

Proof. For (i), take any $p \in P$. Then $F(p) \sqcap G(p) \sqsubseteq F(p)$. Hence $\sqcup_{p \in P}(F(p) \sqcap G(p)) \sqsubseteq \sqcup_{p \in P} F(p) = F(\vee P)$. Similarly $\sqcup_{p \in P}(F(p) \sqcap G(p)) \sqsubseteq \sqcup_{p \in P} G(p) = G(\vee P)$. It follows that

$$\sqcup_{p \in P}(F(p) \sqcap G(p)) \sqsubseteq F(\vee P) \sqcap G(\vee P). \tag{4}$$

On the other hand, using complete distributivity, we have $\sqcup_{p \in P, q \in P}(F(p) \sqcap G(q)) = \sqcup_{p \in P}(F(p) \sqcap 0(\sqcup_{q \in P} G(q))) = \sqcup_{p \in P}(F(p) \sqcap G(\vee P)) = (\sqcup_{p \in P} F(p)) \sqcap G(\vee P) = F(\vee P) \sqcap G(\vee P)$. In short

$$F(\vee P) \sqcap G(\vee P) = \sqcup_{p \in P, q \in P}(F(p) \sqcap G(q)) \tag{5}$$

Finally, since $(L, \leqslant)$ is totally ordered, P is a sublattice of $(L, \leqslant)$; so for any p, $q \in P$ we have $p \vee q \in P$. Then $(p \leqslant p \vee q, q \leqslant p \vee q) \Rightarrow F(p) \sqcap G(q) \sqsubseteq F(p \vee q) \sqcap G(p \vee q)$. So $\sqcup_{p \in P, q \in P}(F(p) \sqcap G(q)) \sqsubseteq \sqcup_{p \in P, q \in P}(F(p \vee q) \sqcap G(p \vee q)) \sqsubseteq \sqcup_{r \in P}(F(r) \sqcap G(r))$. Hence

$$\sqcup_{p \in P, q \in P}(F(p) \sqcap G(q)) \sqsubseteq \sqcup_{p \in P}(F(p) \sqcap G(p)) \tag{6}$$

From (4)–(6) it follows that $\sqcup_{p \in P}(F(p) \sqcap G(p)) = F(\vee P) \sqcap G(\vee P)$ and (i) has been proved; (ii) is proved dually. □

Now we return to the cuts of $M \dot{\vee} N$.

Definition 4.8. For all $M, N \in \widetilde{\mathbf{I}}$ and for all $p \in L$ we define $D_p(M, N) = M_p \dot{\cup} N_p$.

Proposition 4.9. *Take any $M, N \in \widetilde{\mathbf{I}}$. We have*

(i) $\forall p$, $q \in L$: $p \leqslant q \Rightarrow D_q(M, N) \subseteq D_p(M, N)$,
(ii) $\forall P \subseteq L$: $\cap_{p \in P} D_p(M, N) = D_{\vee P}(M, N)$.
(iii) $D_0(M, N) = X$.

Proof

(i) Assume $p \leqslant q$. Then $(M_q \subseteq M_p, N_q \subseteq N_p) \Rightarrow M_q \dot{\cup} N_q \subseteq M_p \dot{\cup} N_p \Rightarrow D_q(M, N) \subseteq D_p(M, N)$.
(ii) Take any $P \subseteq L$ and any $p \in P$. We have $D_p(M, N) = [\underline{M}(p) \sqcap \underline{N}(p), \overline{M}(p) \sqcup \overline{N}(p)]$, hence

$$\cap_{p \in P} D_p(M, N) = [\sqcup_{p \in P}(\underline{M}(p) \sqcap \underline{N}(p)), \sqcap_{p \in P}(\overline{M}(p) \sqcup \overline{N}(p))]. \tag{7}$$

Also

$$D_{\vee P}(M,N) = \left[\underline{M}(\vee P) \sqcap \underline{N}(\vee P), \overline{M}(\vee P) \sqcup \overline{N}(\vee P)\right]. \tag{8}$$

Use Proposition 4.7.(i) with $F(p) = \underline{M}(p)$ and $G(p) = \underline{N}(p)$. Then

$$\sqcup_{p \in P}(\underline{M}(p) \sqcap \underline{N}(p)) = \underline{M}(\vee P) \sqcap \underline{N}(\vee P). \tag{9}$$

Use Proposition 4.7.(ii) with $F(p) = \overline{M}(p)$ and $G(p) = \overline{N}(p)$. Then

$$\sqcap_{p \in P}\left(\overline{M}(p) \sqcup \overline{N}(p)\right) = \overline{M}(\vee P) \sqcup \overline{N}(\vee P). \tag{10}$$

Eqs. (7)–(10) yield the required result.

(iii) $D_0(M,N) = M_0 \dot{\cup} N_0 = X \dot{\cup} X = X$. $\square$

Proposition 4.10. *For all $M, N \in \widetilde{\mathbf{I}}$ and for all $p \in L$ we have: $(M \dot{\vee} N)_p = D_p(M,N)$.*

Proof. From Proposition 4.9 it follows that $\{D_p(M,N)\}_{p \in L}$ is a family of cuts. Hence, if we define a fuzzy set $(M \veebar N)$ by setting

$$\forall x \in X : (M \veebar N)(x) \doteq \vee \{p : x \in D_p(M,N)\}$$

then $\forall p \in L$ we will have $(M \veebar N)_p = D_p(M,N)$ (Proposition 2.12). From this also follows that $(M \veebar N)$ is a fuzzy interval (since $\forall p \in L$ we have $(M \veebar N)_p = D_p(M,N) = M_p \dot{\cup} N_p$). Now choose any $p \in L$; we will show that $(M \dot{\vee} N)_p = (M \veebar N)_p$.

First, $(M \dot{\vee} N)_p$ is a closed interval. Also, $x \in M_p \Rightarrow (M \dot{\vee} N)(x) \geqslant M(x) \geqslant p \Rightarrow x \in (M \dot{\vee} N)_p$. So $M_p \subseteq (M \dot{\vee} N)_p$. Similarly $N_p \subseteq (M \dot{\vee} N)_p$. Hence $(M \dot{\vee} N)_p \in \mathbf{S}(M_p, N_p)$ which implies that $(M \veebar N)_p = D_p(M,N) = M_p \dot{\cup} N_p = \cap_{A \in \mathbf{S}(M_p,N_p)} A \subseteq (M \dot{\vee} N)_p$.

Second, choose any $x \in X$ and set $p = M(x)$. Then $x \in M_p \subseteq D_p(M,N) = (M \veebar N)_p$. Hence $(M \veebar N)(x) \geqslant p = M(x)$; similarly $(M \veebar N)(x) \geqslant N(x)$. Since $M \dot{\vee} N = \sup(M,N)$, it follows that $(M \veebar N)(x) \geqslant (M \dot{\vee} N)(x)$ and so $(M \veebar N)_p \supseteq (M \dot{\vee} N)_p$.

So we have $(M \veebar N)_p = (M \dot{\vee} N)_p$ which (Proposition 2.10) implies $M \veebar N = M \dot{\vee} N$. $\square$

Proposition 4.11. *$(\widetilde{\mathbf{I}}, \leqslant, \dot{\vee}, \wedge)$ is a distributive lattice.*

Proof We must show that for any $A, B, C \in \widetilde{\mathbf{I}}$ we have $(A \dot{\vee} B) \wedge C = (A \wedge C) \dot{\vee} (B \wedge C)$ and $(A \wedge B) \dot{\vee} C = (A \dot{\vee} C) \wedge (B \dot{\vee} C)$. We will show this by showing equality of the p-cuts.

Indeed, choose any $p \in L$ and set $A_p = [a_1, a_2]$, $B_p = [b_1, b_2]$, $C_p = [c_1, c_2]$ (in case any of these intervals is empty, denote it by $[\sqcup X, \sqcap X]$). Now

$$\begin{aligned}
((A \dot{\vee} B) \wedge C)_p &= (A \dot{\vee} B)_p \cap C_p = \left(A_p \dot{\cup} B_p\right) \cap C_p = ([a_1, a_2] \dot{\cup} [b_1, b_2]) \cap [c_1, c_2] = [a_1 \sqcap b_1, a_2 \sqcup b_2] \cap [c_1, c_2] \\
&= [(a_1 \sqcap b_1) \sqcup c_1, (a_2 \sqcup b_2) \sqcap c_2] = [(a_1 \sqcup c_1) \sqcap (b_1 \sqcup c_1), (a_2 \sqcap c_2) \sqcup (b_2 \sqcap c_2))] \\
&= [a_1 \sqcup c_1, a_2 \sqcap c_2] \dot{\cup} [b_1 \sqcup c_1, b_2 \sqcap c_2] = ([a_1, a_2] \cap [c_1, c_2]) \dot{\cup} ([b_1, b_2] \cap [c_1, c_2]) = \left(A_p \cap C_p\right) \dot{\cup} \left(B_p \cap C_p\right) \\
&= (A \wedge C)_p \dot{\cup} (B \wedge C)_p = ((A \wedge C) \dot{\vee} (B \wedge C))_p.
\end{aligned}$$

Since for all $p \in L$ we have $((A \dot{\vee} B) \wedge C)_p = ((A \wedge C) \dot{\vee} (B \wedge C))_p$, it follows that $(A \dot{\vee} B) \wedge C = (A \wedge C) \dot{\vee} (B \wedge C)$. Dually we show that $(A \wedge B) \dot{\vee} C = (A \dot{\vee} C) \wedge (B \dot{\vee} C)$. $\square$

5. Inclusion measures in the fuzzy intervals lattice

Inclusion measures are used widely in the FIN papers by Kaburlasos et al. [21–23,31,33] and, more generally, have been studied widely in the fuzzy literature (a rather extensive survey appears in [28], see also [8,9]). In this section we present an inclusion measure which possesses many desirable properties and (as far as we know) has not been previously considered in the literature.

An inclusion meaure is a mapping σ: $\mathbf{F} \times \mathbf{F} \to L$; the value $\sigma(A,B)$ quantifies the degree to which a fuzzy set A is contained in fuzzy set B. As explained in [28], considerable effort has been expended in discovering inclusion measures which have "desirable properties". We list below such desirable properties, which we divide into two groups: "basic properties" and "additional properties". The basic properties are listed in Table 1.

We consider properties **A1–A3** basic, because they are the properties which define a *fuzzy order* [29]. We consider a fuzzy inclusion measure to be the fuzzification of the (set inclusion relation $\subseteq$ which, as is well known, is a order. Unfortunately, especially the transitivity property **A3** appears to be particularly hard to obtain and is not satisfied by the "usual" inclusion measures which most often appear in the literature (for a discussion see [28]).

In addition, various authors [7,19,38,44] have considered the properties listed in Table 2 to be "appropriate" or desirable to be satisfied by an inclusion measure.

Again, many of these properties are not satisfied by the "usual" inclusion measures. For an extensive discussion see [28].

Table 1
Basic properties of an inclusion measure σ.

A1	$\forall A \in \mathbf{F}$:	$\sigma(A,A) = 1$	(Reflexivity)
A2	$\forall A,B \in \mathbf{F}$:	$\sigma(A,B) = \sigma(B,A) \Rightarrow A = B$	(Antisymmetry)
A3	$\forall A,B,C \in \mathbf{F}$:	$\sigma(A,B) \wedge (B,C) \leqslant \sigma(A,C)$	(Transitivity)

Table 2
Additional properties of an inclusion measure σ.

B1	$\forall A,B \in \mathbf{F}$:	$A \leqslant B \Leftrightarrow \sigma(A,B) = 1$
B2	$\forall A \in \mathbf{F}$:	$\sigma(A,A') = 0 \Leftrightarrow A$ is set
B3	$\forall A,B \in \mathbf{F}$:	$\sigma(A,B) = \sigma(B',A')$
B4	$\forall A,B,C \in \mathbf{F}$:	$B \leqslant C \Rightarrow \sigma(A,B) \leqslant \sigma(A,C)$
B5	$\forall A,B,C \in \mathbf{F}$:	$B \leqslant C \Rightarrow \sigma(C,A) \leqslant \sigma(B,A)$
B6	$\forall A,B,C \in \mathbf{F}$:	$\sigma(A \vee B,C) = \sigma(A,C) \wedge \sigma(B,C)$
B7	$\forall A,B,C \in \mathbf{F}$:	$\sigma(A \wedge B,C) \geqslant \sigma(A,C) \vee \sigma(B,C)$
B8	$\forall A,B,C \in \mathbf{F}$:	$\sigma(A,B \vee C) \geqslant \sigma(A,B) \vee \sigma(A,C)$
B9	$\forall A,B,C \in \mathbf{F}$:	$\sigma(A,B \wedge C) = \sigma(A,B) \wedge \sigma(A,C)$

We will now introduce an inclusion measure σ (proposed, as far as we know, for the first time) which satisfies most of the above properties – and in particular it satisfies transitivity, which we consider especially important for any reasonable inclusion measure. We emphasize that the new inclusion measure applies to fuzzy sets in general, not only to fuzzy intervals; we believe its introduction and study is particularly pertinent in the context of fuzzy intervals since, as demonstrated by Kaburlasos, fuzzy intervals (and their equivalent FIN's) combined with inclusion measures yield very useful approaches to various applied problems.

Definition 5.1. For all A, $B \in \mathbf{F}$ we define

$$\sigma(A,B) = \vee\{p : \forall r \leqslant p : A_r \subseteq B_r\}.$$

We first show that the $\sigma(A,B)$ of Definition 5.1 satisfies the "basic" properties **A1** and **A3**.

Remark. In the rest of this section (and especially in the proofs of Propositions 5.2 and 5.4) we assume that fuzzy sets A, B, C are *continuous*. Actually the proofs can go through without this restriction; we assume continuity for the sake of simplicity and to make the basic ideas behind the proofs more obvious to the reader.

Proposition 5.2. *$\sigma(A,B)$ is reflexive and transitive, i.e., it satisfies properties* **A1** *and* **A3**.

Proof. The proof of **A1** is straightforward: let $p = 1$ and note that for all $r \leqslant p = 1$ we have $A_r \subseteq A_r$. Hence

$$\sigma(A,A) = \vee\{p : \forall r \leqslant p : A_r \subseteq A_r\} = 1.$$

For **A3**, take any A, B, $C \in \mathbf{F}$ and let

$$p = \sigma(A,B), \quad q = \sigma(B,C), \quad r = p \wedge q.$$

Then, for all $s \leqslant r$ we have

$$\left.\begin{array}{l} s \leqslant r = p \wedge q \leqslant p = \sigma(A,B) \Rightarrow A_s \subseteq B_s \\ s \leqslant r = p \wedge q \leqslant q = \sigma(B,C) \Rightarrow B_s \subseteq C_s \end{array}\right\} \Rightarrow (\forall s \leqslant r : A_s \subseteq C_s) \Rightarrow \sigma(A,C) \geqslant r = p \wedge q = \sigma(A,B) \wedge \sigma(B,C), \qquad \square$$

Remark. The antisymmetry property **A2** is not fully satisfied, as can be seen by the following example. Take fuzzy sets A: $[0,4] \to [0,1]$, B: $[0,4] \to [0,1]$ with

$$A(x) = \begin{cases} 1/2 & \text{iff } x \neq 2 \\ 1 & \text{iff } x = 2 \end{cases}, \quad B(x) = \begin{cases} 1/2 & \text{iff } x \neq 3 \\ 1 & \text{iff } x = 3 \end{cases}.$$

Then $\sigma(A,B) = \sigma(B,A) = \frac{1}{2}$ but $A \neq B$. However, we have the following weak form of antisymmetry

Proposition 5.3. *For all A, $B \in \mathbf{F}$ such that $\sigma(A,B) = \sigma(B,A) = 1$ we have $A = B$.*

Proof. If $\sigma(A,B) = 1$, then $\forall r \in [0,1]$ we have $A_r \subseteq B_r$. Similarly, if $\sigma(B,A) = 1$, then $\forall r \in [0,1]$ we have $B_r \subseteq A_r$. Hence $\forall r$ we have $A_r = B_r$ and so $A = B$. $\square$

Next we show that $\sigma(A,B)$ of Definition 5.1 also satisfies most of the "additional" properties.

Proposition 5.4. *$\sigma(A,B)$ satisfies properties* **B1**, **B3–B9** *but does not satisfy* **B2.**

Proof. In the following take any A, B, $C \in \mathbf{F}$. We have the following.

B1 Assume $A \leqslant B$, then (from Proposition 2.13) we have: $(\forall p\colon A_p \subseteq B_p) \Rightarrow \sigma(A,B) = 1$. Conversely, assume that $\sigma(A,B) = 1$, then $(\forall p\colon A_p \subseteq B_p) \Rightarrow A \leqslant B$.

B2 We show that it does not hold by a counterexample. Take a fuzzy set $A\colon [0,2] \to [0,1]$ with

$$A(x) = \begin{cases} x & \text{iff } x \leqslant 1 \\ 2-x & \text{iff } x \geqslant 1 \end{cases}.$$

Then $A'(x)$ is given by

$$A'(x) = \begin{cases} 1-x & \text{iff } x \leqslant 1 \\ x-1 & \text{iff } x \geqslant 1 \end{cases}$$

and we see that $\sigma(A,A') = 0$, while A is not a classical set (i.e. one with 0/1 membership).

B3 Say $\sigma(A,B) = p$. Then $(\forall r \leqslant p : A_r \subseteq B_r) \Rightarrow (\forall r \leqslant p : B'_r \subseteq A'_r) \Rightarrow q = \sigma(B',A') \geqslant p$. Starting with $\sigma(B',A') = q$ we we can show similarly that $p \geqslant q$ and so $\sigma(A,B) = p = q = \sigma(B',A')$.

B4 We have

$$B \leqslant C \Rightarrow (\forall r : B_r \subseteq C_r). \tag{11}$$

Let $\sigma(A,B) = p$, then

$$\forall r \leqslant p \Rightarrow A_r \subseteq B_r \tag{12}$$

From (11) and (12) we get

$$\forall r \leqslant p \Rightarrow A_r \subseteq C_r \tag{13}$$

from which follows $\sigma(A,C) \geqslant p = \sigma(A,B)$.

B5 This is proved similarly to **B4**.

B6 Since $A \leqslant A \vee B$ and $B \leqslant A \vee B$, from **B4** we have

$$\left.\begin{array}{l} \sigma(A,C) \geqslant \sigma(A \vee B, C) \\ \sigma(B,C) \geqslant \sigma(A \vee B, C) \end{array}\right\} \Rightarrow \sigma(A,C) \wedge \sigma(B,C) \geqslant \sigma(A \vee B, C). \tag{14}$$

On the other hand, let $p = \sigma(A,C)$, $q = \sigma(B,C)$, $r = p \wedge q$. Then

$$\left.\begin{array}{l} \forall s \leqslant r \leqslant p : A_s \subseteq C_s \\ \forall s \leqslant r \leqslant q : B_s \subseteq C_s \end{array}\right\} \Rightarrow (\forall s \leqslant r : A_s \cup B_s \subseteq C_s) \Rightarrow (\forall s \leqslant r : (A \vee B)_s \subseteq C_s) \Rightarrow \sigma(A \vee B, C) \geqslant r = p \wedge q = \sigma(A,C) \wedge \sigma(B,C). \tag{15}$$

From (14) and (15) we get $\sigma(A \vee B) = \sigma(A,C) \wedge \sigma(B,C)$.

B7 Since $A \geqslant A \wedge B$ and $B \geqslant A \wedge B$, from **B4** we have

$$\left.\begin{array}{l} \sigma(A \wedge B, C) \geqslant \sigma(A,C) \\ \sigma(A \wedge B, C) \geqslant \sigma(B,C) \end{array}\right\} \Rightarrow \sigma(A \wedge B, C) \geqslant \sigma(A,C) \vee \sigma(B,C). \tag{16}$$

B8 This is proved similarly to **B7**.

B9 This is proved similarly to **B6**. □

6. Conclusion

In this paper we have introduced a general definition of fuzzy intervals and shown its relationship to a previous more restricted definition of fuzzy intervals, as well as to fuzzy numbers and to Kaburlasos' FIN's. Furthermore we have obtained some of the basic properties of fuzzy intervals and introduced a novel inclusion measure which can be used in FIN applications.

The method we have used is rather standard in the study of fuzzy algebras – in particular we have obtained several properties of fuzzy intervals by studying their p-cuts. This method can be used to obtain further properties of fuzzy intervals.

In our analysis we have assumed that the origin lattice $(X, \sqsubseteq, \sqcup, \sqcap)$ is complete and completely distributive. These assumptions are essential. Obviously, if $(X, \sqsubseteq, \sqcup, \sqcap)$ is not complete, there is no guarantee that an infinite union of fuzzy intervals will be a fuzzy interval. Complete distributivity, on the other hand, has only been used in Section 4, but there it plays an essential role in the proof of Proposition 4.7. Let us note that in the important special case where X has finite cardinality, completeness is automatically satisfied and complete distributivity is equivalent to distributivity (which clearly is a minimum requirement for the lattice of fuzzy intervals to be distributive).

Regarding the target lattice $(L, \leqslant, \vee, \wedge)$, we have assumed that it is a complete lattice with a minimum element 0 and a maximum element 1. These are rather weak assumptions in the context of L-fuzzy sets. In Section 4 we have further assumed that L is a completely distributive chain. This is also a crucial assumption: it does not seem obvious how to extend our results to general L-fuzzy lattices, because Proposition 4.7 requires that for every $P \subseteq L$, and for all $p, q \in P$, we have $p \vee q \in P$; for this to be true for arbitrary $P \subseteq L$, $(L, \leqslant)$ must be a chain.

We believe that the theoretical framework provided in the current paper can serve as a foundation for further mathematical study of FIN's with special emphasis placed on applications, for example on the properties of inclusion measures used in applied engineering tasks. This task, however, will be accomplished in future publications.

References

[1] N. Ajmal, K.V. Thomas, Fuzzy lattices, Information Sciences 79 (1994) 271–291.
[2] N. Ajmal, K.V. Thomas, Fuzzy lattices – I, Journal of Fuzzy Math. 10 (2002) 255–274.
[3] N. Ajmal, K.V. Thomas, Fuzzy lattices – II, Journal of Fuzzy Math. 10 (2002) 275–296.
[4] N. Ajmal, A. Jain, Some constructions of the join of fuzzy subgroups and certain lattices of fuzzy subgroups with sup property, Information Sciences 179 (2009) 4070–4082.
[5] N. Ajmal, K.V. Thomas, The lattices of fuzzy ideals of a ring, Fuzzy Sets and Systems 74 (1995) 371–379.
[6] G. Birkhoff, Lattice Theory, American Math Society, Providence, Rhode Island, 1967.
[7] P. Burillo, N. Frago, R. Fuentes, Inclusion grade and fuzzy implication operators, Fuzzy Sets and Systems 114 (2000) 417–429.
[8] H. Bustince, V. Mohedano, E. Barrenechea, M. Pagola, Definition and construction of fuzzy DI-subsethood measures, Information Sciences 176 (2006) 3190–3231.
[9] H. Bustince, M. Pagola, E. Barrenechea, Construction of fuzzy indices from fuzzy DI-subsethood measures: application to the global comparison of images, Information Sciences 177 (2007) 906–929.
[10] G. Calugareanu, V. Leoreanu, Hypergroups associated with lattices, Italian Journal of Pure and Applied Mathematics 9 (2001) 165–173.
[11] P. Corsini, Prolegomena of Hypergroup Theory, Udine, Aviani, 1993.
[12] P. Corsini, Join spaces, power sets, fuzzy sets, in: M. Stefanescu (Ed.), Algebraic Hyperstructures and Applications, Hadronic Press, Palm Harbor, 1994, pp. 45–52.
[13] P. Corsini, V. Leoreanu, Join spaces associated with fuzzy sets, Journal of Combinatorics, Information and System Sciences 20 (1995) 293–303.
[14] P. Corsini, V. Leoreanu, Fuzzy sets and join spaces associated with rough sets, Rendiconti del Circo Matematico di Palermo 51 (2002) 527–536.
[15] P. Corsini, V. Leoreanu, Applications of Hyperstructure Theory, Kluwer, 2003.
[16] P. Corsini, I. Tofan, On fuzzy hypergroups, P.U.M.A. 8 (1997) 29–37.
[17] B. Davvaz, Roughness based on fuzzy ideals, Information Sciences 176 (2006) 2417–2437.
[18] V.N. Dixit, R. Kumar, N. Ajmal, On fuzzy rings, Fuzzy Sets and Systems 49 (1992) 205–213.
[19] J. Fan, W. Xie, J. Pei, Subsethood measure: new definitions, Fuzzy Sets and Systems 106 (1999) 201–209.
[20] Han-Liang Huang, Fu-Gui Shi, L-fuzzy numbers and their properties, Information Sciences 178 (200) 1141–1151.
[21] V.G. Kaburlasos, V. Petridis, Fuzzy lattice neurocomputing (FLN) models, Neural Networks 13 (2000) 1145–1170.
[22] V.G. Kaburlasos, FINs: lattice theoretic tools for improving prediction of sugar production from populations of measurements, IEEE Transactions on Systems, Man and Cybernetics, Part B 34 (2004) 1017–1030.
[23] V.G. Kaburlasos, Ath. Kehagias, Novel fuzzy inference system (FIS) analysis and design based on lattice theory. Part I: Working principles, International Journal of General Systems 35 (2006) 45–67.
[24] V.G. Kaburlasos, I.N. Athanasiadis, P.A. Mitkas, Fuzzy lattice reasoning (FLR) classifier and its application for ambient ozone estimation, International Journal of Approximate Reasoning (2007).
[25] V.G. Kaburlasos, Improved fuzzy lattice neurocomputing (FLN) for semantic neural computing, in: Proceedings of the International Joint Conference on Neural Networks, 2003.
[26] V.G. Kaburlasos, L. Moussiades, A. Vakali, Fuzzy lattice reasoning (FLR) type neural computation for weighted graph partitioning, Neurocomputing (2008).
[27] Ath. Kehagias, V. Petridis, V.G. Kaburlasos, P. Fragkou, A comparison of word-and sense-based text categorization using several classification algorithms, Journal of Intelligent Information Systems (2003).
[28] Ath. Kehagias, M. Konstantinidou, L-fuzzy valued inclusion measure, L-fuzzy similarity and L-fuzzy distance, Fuzzy Sets and Systems 136 (2003) 313–332.
[29] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, 1995.
[30] H.T. Nguyen, E.A. Walker, A First Course on Fuzzy Logic, CRC Press, Boca Raton, 1997.
[31] V. Petridis, V.G. Kaburlasos, Fuzzy lattice neural network (FLNN): a hybrid model for learning, IEEE Transactions on Neural Networks 9 (1998) 877–890.
[32] V. Petridis, V.G. Kaburlasos, Learning in the framework of fuzzy lattices, IEEE Transactions on Fuzzy Systems 7 (1999) 422–440.
[33] V. Petridis, V.G. Kaburlasos, FINkNN: a fuzzy interval number k-nearest neighbor classifier for prediction of sugar production, Journal of Machine Learning Research 4 (2003) 17–37.
[34] A. Rosenfeld, Fuzzy groups, Journal of Mathematical Analysis and Applications 35 (1971) 512–517.
[35] B. Seselja, A. Tepavcevic, Representation of lattices by fuzzy sets, Information Sciences 79 (1994) 171–180.
[36] B. Seselja, A. Tepavcevic, On a generalization of fuzzy algebras and congruences, Fuzzy Sets and Systems 65 (1994) 85–94.
[37] B. Seselja, Lattice of partially ordered fuzzy subalgebras, Fuzzy Sets and Systems 81 (1996) 265–269.
[38] D. Sinha, E.R. Dougherty, Fuzzification of set inclusion: theory and applications, Fuzzy Sets and Systems 55 (1993) 15–42.
[39] U.M. Swamy, D.V. Raju, Fuzzy ideals and congruences of lattices, Fuzzy Sets and Systems 95 (1998) 249–253.
[40] M. Tarnauceanu, Distributivity in lattices of fuzzy subgroups, Information Sciences 179 (2009) 1163–1168.
[41] A. Tepavcevic, G. Trajkovski, L-fuzzy lattices: an introduction, Fuzzy Sets and Systems 123 (2001) 209–216.
[42] B. Yuan, W. Wu, Fuzzy ideals on a distributive lattice, Fuzzy Sets and Systems 35 (1990) 231–240.
[43] L.S. Xu, Construct fuzzy lattices from a given symmetric complete lattice, Fuzzy Sets and Systems 66 (1994) 357–362.
[44] V.R. Young, Fuzzy subsethood, Fuzzy Sets and Systems 77 (1996) 371–384.
[45] A. Weinberger, Embedding lattices of fuzzy subalgebras into lattices of crisp subalgebras, Information Sciences 108 (1998) 51–70.
[46] Z. Yue, Prime L-fuzzy ideals primary L-fuzzy ideals, Fuzzy Sets and Systems 27 (1988) 345–350.
[47] M.M. Zahedi, A characterisation of L-fuzzy prime ideals, Fuzzy Sets and Systems 44 (1991) 147–160.
[48] M.M. Zahedi, A. Hasankhani, F-Polygroups, International Journal of Fuzzy Mathematics 4 (1996) 533–548.
[49] M.M. Zahedi, A. Hasankhani, F-Polygroups (II), Information Sciences 89 (1996) 225–243.
[50] X.D. Zhao, Representation theorem of fuzzy lattices and its applications in fuzzy topology, Fuzzy Sets and Systems 25 (1988) 125–128.