Some Remarks on Congruences obtained from the L-Fuzzy Nakano Hyperoperation

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Abstract

In this paper we study relations which are congruences with respect to $\wedge$ and $\sqcup_p$, where $\sqcup_p$is the p-cut of the L-fuzzy hyperoperation $\sqcup$. The main idea is to start from an equivalence relation R_1 which is a congruence with respect to $\wedge$ and $\sqcup_1$and, for each $p \in X$, construct an equivalence relation R_p which is a congruence with respect to $\wedge$ and $\sqcup_p$. Furthermore, for each $x \in R_p$ we construct a quotient hyperoperation $\underline{\sqcup}_p$ and we show that the hyperalgebra $\left(X/R_p, \underline{\sqcup}_p\right)$ is a join space and the hyperalgebra $\left(X/R_p, \underline{\sqcup}_p, \underline{\triangle}_p\right)$ is a hyperlattice.

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1 Introduction

In a previous paper [8] we have constructed the L-fuzzy Nakano hyperoperation $\sqcup$ in terms of its p-cuts. Our construction can be summarized as follows[1]. Consider a *generalized de Morgan lattice* $(X, \leq, \vee, \wedge, ')$ and for every $x, y, p \in X$ define

$$x \sqcup_p y = \left\{ z : x \vee y \vee p' = x \vee z \vee p' = y \vee z \vee p' \right\}. \tag{1}$$

Hence, for every $x, y, p \in X$ we obtain a *crisp*[2] set $x \sqcup_p y$; i.e. $\sqcup_p$ is a *crisp hyperoperation* which maps the pair x, y to the set $x \sqcup_p y$. In [8] we have also shown how to define (for every $x, y \in X$) the *L-fuzzy set* $x \sqcup y$ in such a manner that for every $p \in X$ the p-cut $(x \sqcup y)_p$ is equal to $x \sqcup_p y$. Hence we obtain an *L-fuzzy hyperoperation* $\sqcup$ which maps the pair x, y to the L-fuzzy set $x \sqcup y$.

In the current paper we study equivalences on X which are congruences with respect to $\wedge$ and $\sqcup_p$. The work presented here can be seen as a continuation of [8] in conjunction to previous work on congruences with respect to $\sqcup_1$ [9]. The main idea is to start from an equivalence relation R_1 which is a congruence with respect to $\wedge$ and $\sqcup_1$and, for each $p \in X$, construct an equivalence relation R_p which is a congruence with respect to $\wedge$ and $\sqcup_p$. Furthermore, for each $p \in X$ we will construct a quotient hyperoperation $\underline{\sqcup}_p$ and we will show that the hyperalgebra $\left(X/R_p, \underline{\sqcup}_p\right)$ is a join space and the hyperalgebra $\left(X/R_p, \underline{\sqcup}_p, \underline{\triangle}_p\right)$ is a hyperlattice.

In addition to our already mentioned earlier work, the current paper is related to work on *join spaces* [15, 10], *hyperlattices* [1, 6, 11, 16, 17, 12], *L-fuzzy hyperoperations* [4, 7, 20] and the *Nakano hyperoperations* [14, 2]. This is only a partial list of relevant work; further references appear in [8].

[1] For details and background material, see [8].
[2] "Crisp" is used here in contradistinction to "fuzzy" [8].

2 Preliminaries

In this section we present some basic definitions, notations and propositions which will be used in the sequel. Further related material can be found in [8].

Throughout this paper we use a structure $(X, \leq, \vee, \wedge, ')$ which we assume to be a *generalized deMorgan lattice*, i.e. a structure satisfying the following.

Definition 2.1 *A* generalized deMorgan lattice *is a structure* $(X, \leq, \vee, \wedge, ')$*, where* $(X, \leq, \vee, \wedge)$ *is a complete distributive lattice with minimum element 0 and maximum element 1; the symbol* $'$ *denotes a unary operation ("complement"); and the following properties are satisfied.*

1. *For all* $x \in X$*,* $Y \subseteq X$ *we have* $x \wedge (\vee_{y \in Y} y) = \vee_{y \in Y} (x \wedge y)$*,* $x \vee (\wedge_{y \in Y} y) = \wedge_{y \in Y} (x \vee y)$*.* (Complete distributivity).
2. *For all* $x \in X$ *we have:* $(x')' = x$*.* (Negation is involutory).
3. *For all* $x, y \in X$ *we have:* $x \leq y \Rightarrow y' \leq x'$*.* (Negation is order reversing).
4. *For all* $Y \subseteq X$ *we have* $(\vee_{y \in Y} y)' = \wedge_{y \in Y} y'$, $\quad (\wedge_{y \in Y} y)' = \vee_{y \in Y} y'$ (Complete deMorgan laws).

The reader will recall that a *crisp hyperoperation* $*$ is a mapping of pairs $x, y \in X$ to *crisp sets* $x * y \subseteq X$; the set X endowed with one or more hyperoperations forms a *hyperalgebra.* [3]. The following notation is standard in the literature of hyperalgebras.

Definition 2.2 *For* $x \in X$*,* $A \subseteq X$ *and a hyperoperation* $*$*, we define* $x * A = \cup_{a \in A} x * y$*.*

We have already defined the hyperoperation $\sqcup_p$ by (1); it is the main hyperoperation of interest in this paper. Let us also note that setting $p = 1$ we obtain

$$x \sqcup_1 y = \{z : x \vee y = x \vee z = y \vee z\} ;$$

this is the classical Nakano hyperoperation which has been the object of much study [9, 2, 14]

The notion of an equivalence relationship R on X is well-known. Let us recall the following definition of the classes of an equivalence.

Definition 2.3 *Given an equivalence* R *on* X *and an element* $x \in X$*, we denote the* class *of* x *(with respect to* R*) by* $R(x)$ *and we define it by*

$$R(x) = \{y : (x, y) \in R\} ;$$

The quotient of X with respect to R *is denoted by* X/R *and defined by*

$$X/R = \{R(x)\}_{x \in X} ;$$

finally, for $A \subseteq X$ *we define*

$$R(A) = \{R(x)\}_{x \in A} .$$

[3] Clearly, a hyperoperation is a generalization of the concept of an operation, since an operation $\cdot$ maps a pair of elements $x, y \in X$ to an *element* $x \cdot y \in X$. A further generalization is that of an *L-fuzzy hyperoperation*, which maps pairs $x, y \in X$ to *L-fuzzy sets* $x * y \subseteq X$. For details on hyperoperations and hyperalgebras see the books [3, 5]; for details on *L-fuzzy hyperoperations* see [8, **?**].

The following proposition is simply a restatement of some set theoretic properties following from Definition 2.3.

Proposition 2.4 *Given an equivalence R on X, for every $B \subseteq X$ and every family $\{A_u\}$ with $A_u \subseteq X$ (for every $u \in B$) we have: $R(\cup_{u\in B} A_u) = \cup_{u\in B} R(A_u)$.*

Proof. First, $Q \in R(\cup_{u\in B} A_u) \Leftrightarrow (\exists x : x \in \cup_{u\in B} A_u,\ R(x)= Q) \Leftrightarrow (\exists u, x : u \in B,\ x \in A_u,\ R(x)= Q)$. Second, $Q \in \cup_{u\in B} R(A_u) \Leftrightarrow (\exists u : u \in B,\ Q \in R(A_u)) \Leftrightarrow (\exists u,\ x : u \in B,\ x \in A_u,\ R(x)= Q) \Leftrightarrow (\exists u, x : u \in B,\ x \in A_u,\ R(x) = Q)$. Hence $Q \in R(\cup_{u\in B} A_u) \Leftrightarrow Q \in \cup_{u\in B} R_p(A_u)$. ■

We can extend the well known definition of "congruence with respect to an operation" to "congruence with respect to an hyperoperation" as follows.

Definition 2.5 *Let R be an equivalence on X, let $\cdot$ be an operation and $*$ an hyperoperation.*

1. *We say that R is a* congruence with respect to $\cdot$ *iff the following holds for every $x, y, z \in X$:*

$$R(x) = R(y) \Rightarrow R(x \cdot z) = R(y \cdot z). \tag{2}$$

2. *We say that R is a* congruence with respect to $*$ *iff the following holds for every $x, y, z \in X$:*

$$R(x) = R(y) \Rightarrow R(x * z) = R(y * z). \tag{3}$$

Note that $R(x \cdot z)$ and $R(y \cdot z)$ in (2) are sets, while $R(x * z)$ and $R(y * z)$ in (3) are *families of sets.* Hence $R(x * z) = R(y * z)$ is equivalent to

$$\begin{array}{ll} \forall u \in x * z & \exists w \in y * z : R(u) = R(w) \\ \forall w \in y * z & \exists u \in x * z : R(u) = R(w). \end{array}$$

Regarding (2), let us also remark that in the context of "classical" lattice theory we simply say that "R is a congruence" meaning that it is a congruence with respect to $\vee$ and $\wedge$, i.e. that (2) specializes to

$$R(x) = R(y) \Rightarrow (R(x \vee z) = R(y \vee z) \text{ and } R(x \wedge z) = R(y \wedge z)).$$

3 The Family of Nakano Congruences

3.1 The "Generating" Congruence R_1

We start with an arbitrary crisp equivalence R_1. In the rest of the paper we assume that R_1 is a congruence with respect to $\wedge$ and $\sqcup_1$. The following propositions describe well-known (classical) properties of R_1 which can be obtained using only congruence with respect to $\wedge$.

Proposition 3.1 *The classes of R_1 are convex.*

Definition 3.2 *We write $R_1(x) \preceq_1 R_1(y)$ iff $R_1(x \wedge y) = R_1(x)$.*

Proposition 3.3 *$\preceq_1$ is an order on X/R_1 and for all $x, y \in X$ we have: $x \leq y \Rightarrow R_1(x) \preceq_1 R_1(y)$.*

If we also use the fact that R_1 is a congruence with respect to $\sqcup_1$we can show that R_1 is also a congruence with respect to $\vee$ (hence R_1 is a congruence in the "classical" sense).

Proposition 3.4 *R_1 is a congruence with respect to $\vee$.*

Proof. Choose any $x, y, z \in X$ such that $R_1(x) = R_1(y)$. Then we also have $R_1(x \sqcup_1 z) = R_1(y \sqcup_1 z)$. Since $x \vee z \in x \sqcup_1 z$ there exists some u such that $u \in y \sqcup_1 z$ and $R_1(u) = R_1(x \vee z)$. Now $u \in y \sqcup_1 z \Rightarrow u \leq u \vee z = y \vee z \Rightarrow R_1(u) \preceq_1 R_1(y \vee z) \Rightarrow R_1(x \vee z) \preceq_1 R_1(y \vee z)$. Similarly we show $R_1(y \vee z) \preceq_1 R_1(x \vee z)$ and so we conclude $R(y \vee z) = R(x \vee z)$. ■

Since R_1 is a congruence with respect to $\vee, \wedge$, the following propositions (well known properties of classical congruences) also hold.

Proposition 3.5 *For every $x, y \in X$: $R_1(x) \preceq_1 R_1(y) \Leftrightarrow R_1(y) = R_1(x \vee y)$.*

Proposition 3.6 *The classes of R_1 are convex sublattices.*

Proposition 3.7 *For all $A, B \in X/R_1$ such that $A \preceq_1 B$ we have:*

$$\begin{array}{ll} \forall a \in A & \exists b \in B \text{ such that } a \leq b, \\ \forall b \in B & \exists a \in A \text{ such that } a \leq b. \end{array}$$

The next proposition will prove quite useful in the sequel. Its proof makes essential use of the fact that R_1 is a congruence with respect to $\sqcup_1$.

Proposition 3.8 *Let $A \in X/R_1$ and $x, y \in A$ with $x < y$. Then there exists no (nonempty) $B \in X/R_1$ such that $\forall z \in B$ we have $z \leq x$.*

Proof. Suppose there exists some (nonempty) $B \in X/R_1$ such that $\forall z \in B$ we have $z \leq x$. Then $z \in B \Rightarrow\ z \vee x = x \vee x \Rightarrow\ z \in x \sqcup_1 x \Rightarrow\ B = R_1(z) \in R_1(x \sqcup_1 x)$. In short we have shown

$$z \in B \Rightarrow\ B \in R_1(x \sqcup_1 x) = R_1(x \sqcup_1 y). \tag{4}$$

On the other hand

$$z \in B \Rightarrow \begin{cases} x \vee z = x \\ y \vee z = y \neq x \\ y \vee x = y \end{cases}$$

hence $z \notin\ x \sqcup_1 y$. In short we have shown

$$z \in B \Rightarrow z \notin x \sqcup_1 y. \tag{5}$$

But (5) implies that $B \notin R_1(x \sqcup_1 y)$. Indeed, if $B \in R_1(x \sqcup_1 y)$ then exists some w such that $R_1(w) = B$ (i.e. $w \in B$) and $w \in x \sqcup_1 y$ and this contradicts (5) which states that $w \in B \Rightarrow w \notin x \sqcup_1 y$. Hence the assumption that a (nonempty) B exists with the property ($z \in B \Rightarrow z \leq x$) leads to both $B \in R_1(x \sqcup_1 y)$ and $B \notin R_1(x \sqcup_1 y)$ which is absurd. ■

Using Proposition 3.8 we will now show that in certain cases the classes of R_1 can be obtained from "pointwise" operations.

Proposition 3.9 *Suppose that for some $x, y \in X$ we have $R_1(x) = [x_1, x_2]$, $R_1(y) = [y_1, y_2]$ and $R_1(x \vee y) = [a, b]$. Then $R_1(x \vee y) = R_1(x) \vee R_1(y)$.*

Proof. (i) We have

$$\left.\begin{array}{l} R_1(x_1)=R_1(x) \\ R_1(y_1)=R_1(y) \end{array}\right\} \Rightarrow R_1(x_1\vee y_1)=R_1(x\vee y)=[a,b]\Rightarrow x_1\vee y_1\in[a,b]\Rightarrow a\leq x_1\vee y_1.$$

Also $x\leq x\vee y\Rightarrow R_1(x)\preceq_1 R_1(x\vee y)=[a,b]$ which implies that there exists some x_0 such that $x_0\in R_1(x)$ and $x_1\leq x_0\leq a$. Similarly $y\leq x\vee y\Rightarrow R_1(y)\preceq R_1(x\vee y)=[a,b]$ which implies that there exists some y_0 such that $y_0\in R_1(y)$ and $y_1\leq y_0\leq a$. Hence $x_1\vee y_1\leq a$. In short $x_1\vee y_1=a$.

(ii.1) If $R_1(x)=R_1(y)$ then

$$[a,b]=R_1(x\vee y)=R_1(y\vee y)=R_1(y)=[y_1,y_2]\Rightarrow b=y_2=x_2\vee y_2.$$

(ii.2) If $R_1(x)\neq R_1(y)$ then either $R_1(x)\neq R_1(x\vee y)$ or $R_1(y)\neq R_1(x\vee y)$ or both. Assume (withour loss of generality) that $R_1(x)\neq R_1(x\vee y)$. Now

$$R_1(x_2\vee y_2)=R_1(x\vee y)=[a,b]\Rightarrow x_2\vee y_2\in[a,b]\Rightarrow x_2\vee y_2\leq b.$$

If $x_2\vee y_2=b$ we are done. Assume on the other hand, that $x_2\vee y_2<b$. Also

$$\left.\begin{array}{l} x_2\in R_1(x) \\ x_2\vee y_2\in R_1(x\vee y) \\ R_1(x)\neq R_1(x\vee y) \end{array}\right\} \Rightarrow x_2\neq x_2\vee y_2\Rightarrow x_2<x_2\vee y_2<b.$$

But then we have

$$z\in R_1(x)\Rightarrow z<x_2\vee y_2<b$$

which contradicts Proposition 3.8 (if one takes B to be $R_1(x)$, x to be $x_2\vee y_2$ and y to be b). Hence we must have $x_2\vee y_2=b$.

(iii) We have concluded that $R_1(x\vee y)=[a,b]=[x_1\vee y_1,x_2\vee y_2]$. But also $R_1(x)=[x_1,x_2]$, $R_1(y)=[y_1,y_2]$ and $R_1(x)\vee R_1(y)=[x_1,x_2]\vee[y_1,y_2]=[x_1\vee y_1,x_2\vee y_2]$, since $(X,\leq)$ is a distributive lattice. Hence the proof is complete. ■

3.2 The Family of Congruences R_p

Now we will use R_1 to construct a *family* of relations R_p, one for every $p\in X$. These will be constructed in such a manner that, for every $p\in X$, R_p will be a congruence with respect to $\wedge,\vee,\sqcup_p$.

Definition 3.10 *For all $p\in X$ we define the relation R_p by:*

$$(x,y)\in R_p \textit{ iff } \left(x\vee p',y\vee p'\right)\in R_1.$$

It is clear that if we set $p=1$ in the above definition, then R_p becomes the original R_1. Furthermore, from congruence with respect to $\vee$ the following proposition is obvious.

Proposition 3.11 *For all $p\in X$, R_p is an equivalence and $R_p\supseteq R_1$.*

The following properties of R_p classes are immediate consequences of Definition 3.10 (hence their proofs are omitted).

Proposition 3.12 *For all $x,y,p\in X$ we have:*

1. $R_1(x)=R_1(y)\Rightarrow R_p(x)=R_p(y)$.

2. $R_p(x) = R_p(y) \Leftrightarrow R_1(x \vee p') = R_1(y \vee p')$.

3. $R_1(x) \subseteq R_p(x)$.

4. $R_1(x \vee p') \subseteq R_p(x) = R_p(x \vee p')$.

Now we can prove that R_p is a congruence with respect to $\vee, \wedge, \sqcup_p$ (for every $p \in X$).

Proposition 3.13 *For all $x, y, z, p \in X$ we have:*

$$R_p(x) = R_p(y) \Rightarrow \begin{cases} R_p(x \vee z) = R_p(y \vee z) \\ R_p(x \wedge z) = R_p(y \wedge z) \\ R_p(x \sqcup_p z) = R_p(y \sqcup_p z) \end{cases}.$$

Proof. Since R_1 is a congruence with respect to $\vee$ we have $R_p(x) = R_p(y) \Rightarrow R_1(x \vee p') = R_1(y \vee p') \Rightarrow R_1(x \vee p' \vee z) = R_1(y \vee p' \vee z) \Rightarrow R_p(x \vee z) = R_p(y \vee z)$. Similarly we can show $R_p(x) = R_p(y) \Rightarrow R_p(x \wedge z) = R_p(y \wedge z)$.

To show the last implication, take any $A \in R_p(x \sqcup_p z)$. Then there exists a such that $a \in x \sqcup_p z$ and $R_p(a) = A$. Hence

$$\begin{gathered} x \vee z \vee p' = a \vee x \vee p' = a \vee z \vee p' \Rightarrow \\ a \vee p' \in \left(x \vee p'\right) \sqcup_1 \left(z \vee p'\right) \Rightarrow \\ R_1(a \vee p') \in R_1(\left(x \vee p'\right) \sqcup_1 \left(z \vee p'\right)). \end{gathered}$$

Also $R_1(x) = R_1(y) \Rightarrow R_1(x \vee p') = R_1(y \vee p')$, hence by congruence of R_1 with respect to $\sqcup_1$ we get

$$R_1(a \vee p') \in R_1(\left(y \vee p'\right) \sqcup_1 \left(z \vee p'\right))$$

and so there exists some b such that $b \in (y \vee p') \sqcup_1 (z \vee p')$ and $R_1(b) = R_1(a \vee p')$. But

$$R_1(a \vee p') = R_1(b) \Rightarrow R_1(a \vee p' \vee p') = R_1(b \vee p') \Rightarrow R_1(a \vee p') = R_1(b \vee p') \Rightarrow R_p(a) = R_p(b).$$

Since $b \in y \sqcup_p z$ and $R_p(b) = A$ it follows that $A \in R_p(y \sqcup_p z)$. Hence $R_p(x \sqcup_p z) \subseteq R_p(y \sqcup_p z)$. In similar manner we show $R_p(y \sqcup_p z) \subseteq R_p(x \sqcup_p z)$ and conclude that $R_p(x \sqcup_p z) = R_p(y \sqcup_p z)$. ■

Since R_p is a congruence with respect to $\vee, \wedge$ the following propositions are immediate.

Proposition 3.14 *For all $p \in X$ the classes of R_p are convex sublattices.*

Definition 3.15 *For every $p \in X$ we write $R_p(x) \preceq_p R_p(y)$ iff $R_p(x) = R_p(x \wedge y)$.*

Proposition 3.16 *For every $p \in X$ the relation $\preceq_p$ is an order on X/R_p.*

Proposition 3.17 *For all $p, x, y \in X$ we have:*

1. $x \vee p' \leq y \vee p' \Rightarrow R_p(x) \preceq_p R_p(y)$.

2. $R_p(x) \preceq_p R_p(y) \Leftrightarrow R_p(y) = R_p(x \vee y)$.

3. $R_1(x) \preceq_1 R_1(y) \Leftrightarrow R_p(x) \preceq_p R_p(y)$.

Proposition 3.18 *For every p and all $A, B \in X/R_p$ such that $A \preceq_p B$ we have:*

$$\begin{aligned} &\forall a \in A \quad \exists b \in B \text{ such that } a \leq b, \\ &\forall b \in B \quad \exists a \in A \text{ such that } a \leq b. \end{aligned}$$

and

$$(a \in A, b \in B) \Rightarrow (a \wedge b \in A, a \vee b \in B).$$

Proposition 3.19 *Suppose that for some $p, x \in X$ we have $R_p(x) = [x_1, x_2]$. Then $p' \leq x_2$.*

Proof. $x_2 \in R_p(x) = [x_1, x_2] = R_p(x_2) = R_p(x_2 \vee p') \ni x_2 \vee p'$. Hence $x_2 \vee p' \leq x_2 \Rightarrow p' \leq x_2$. ■
Next we prove the generalizations of Propositions 3.8, 3.9 for arbitrary p.

Proposition 3.20 *Let $A \in X/R_p$ and $x, y \in A$ with $x \vee p' < y \vee p'$. Then there exists no $B \in X/R_p$ such that $\forall z \in B$ we have $z \leq x$.*

Proof. Suppose there exists some $B \in X/R_p$ such that $\forall z \in B$ we have $z \leq x$. Then $z \in B \Rightarrow z \vee x \vee p' = x \vee x \vee p' \Rightarrow \ z \in x \sqcup_p x \Rightarrow \ B = R_p(z) \in R_p(x \sqcup_p x)$. In short we have shown

$$z \in B \Rightarrow \ B \in R_p(x \sqcup_p x) = R_p(x \sqcup_p y). \tag{6}$$

However

$$z \in B \Rightarrow \begin{cases} x \vee z \vee p' = x \vee p' \\ y \vee z \vee p' = y \vee p' \neq x \vee p' \\ y \vee x \vee p' = y \vee p' \end{cases}$$

hence $z \notin \ x \sqcup_p y$. In short we have shown

$$z \in B \Rightarrow z \notin x \sqcup_p y. \tag{7}$$

But (7) implies that $B \notin R_p(x \sqcup_p y)$. Indeed

$$B \in R_p(x \sqcup_p y) \Rightarrow \left\{ \exists z : \begin{array}{l} z \in x \sqcup_p y \\ z \in B \end{array} \right\} \Rightarrow \left\{ \exists z : \begin{array}{l} z \in x \sqcup_p y \\ z \notin x \sqcup_p y. \end{array} \right\}$$

hence we have a contradiction. So we have

$$z \in B \Rightarrow B \notin R_p(x \sqcup_p y). \tag{8}$$

But (8) contradicts (6) so we have wrongly assumed that there exists some $B \in H/R_p$ such that $\forall z \in B$ we have $z \leq x$. ■

Proposition 3.21 *Suppose that for some p, x, y we have $R_p(x) = [x_1, x_2]$, $R_p(y) = [y_1, y_2]$ and $R_p(x \vee y) = [a, b]$. Then $R_p(x \vee y) = R_p(x) \vee R_p(y)$.*

Proof. (i) We show that $a = x_1 \vee y_1$ in exactly the same maner as in Proposition 3.9.
(ii.1) If $R_p(x) = R_p(y)$ we show that $b = x_2 \vee y_2$ in exactly the same manner as in Proposition 3.9.
(ii.2) Now suppose $R_p(x) \neq R_p(y)$. Then we can assume (without loss of generality) that $R_p(x) \neq R_p(x \vee y)$. Also

$$R_p(x_2 \vee y_2) = R_p(x \vee y) = [a, b] \Rightarrow x_2 \vee y_2 \in [a, b] \Rightarrow x_2 \vee y_2 \leq b \Rightarrow x_2 \vee y_2 \vee p' \leq b \vee p'.$$

If $x_2 \vee y_2 \vee p' = b \vee p'$ then we are done, because by Proposition 3.19,we also have $p' \leq x_2$, $p' \leq y_2$, $p' \leq b$; hence $x_2 \vee y_2 = x_2 \vee y_2 \vee p' = b \vee p' = b$. If, on the other hand $x_2 \vee y_2 \vee p' < b \vee p'$, then

$$\left.\begin{array}{l} x_2 \vee p' \in R_p(x) \\ x_2 \vee y_2 \vee p' \in R_p(x \vee y) \\ R_p(x) \neq R_p(x \vee y) \end{array}\right\} \Rightarrow x_2 \vee p' \neq x_2 \vee y_2 \vee p' \Rightarrow x_2 \vee p' < x_2 \vee y_2 \vee p' < b \vee p'.$$

But then we have

$$z \in R_p(x) \Rightarrow z < x_2 \vee y_2 \vee p' < b \vee p'$$

which contradicts Proposition 3.20 (if one takes B to be $R_p(x)$, x to be $x_2 \vee y_2$ and y to be b). Hence we must have $x_2 \vee y_2 \vee p' = b \vee p'$. But, by Proposition 3.19, $p' \leq x_2 \vee y_2$ and $p' \leq b$, hence finally $x_2 \vee y_2 = b$.

(iii) Hence we have concluded that $R_p(x \vee y) = [a, b] = [x_1 \vee y_1, x_2 \vee y_2]$. But also $R_p(x) = [x_1, x_2]$, $R_p(y) = [y_1, y_2]$ and $R_p(x) \vee R_p(y) = [x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$, since $(X, \leq)$ is a distributive lattice. Hence the proof is complete. ∎

Proposition 3.21 can be applied immediately in case X is finite: since the classes of X are finite convex sublattices, they are intervals. Hence we have the following corollary.

Corollary 3.22 *If X is finite, then for all $p, x, y \in X$ we have $R_p(x \vee y) = R_p(x) \vee R_p(y)$.*

In Section 3.3 we will present another application of Proposition 3.21.

Next we show that both the family of relations $\{R_p\}_{p \in X}$ and the family of classes $\{R_p(x)\}_{p \in X}$ have the *p-cut properties* [8].

Proposition 3.23 *For all $p, q \in X$, $P \subseteq X$ we have the following:*

1. $R_0 = X \times X$.

2. $p \leq q \Rightarrow R_q \subseteq R_p$.

3. $R_{p \vee q} = R_p \cap R_q$; *more generally,* $\cap_{p \in P} R_p = R_{\vee P}$.

Proof. 1 is obvious. For 2, take p and q with $p \leq q$ (hence $q' \leq p'$). Then

$$\begin{aligned} (x, y) \in R_q &\Rightarrow \left(x \vee q', y \vee q'\right) \in R_1 \\ &\Rightarrow R_1(x \vee q') = R_1(y \vee q') \\ &\Rightarrow R_1(x \vee q' \vee p') = R_1(y \vee q' \vee p') \\ &\Rightarrow R_1(x \vee p') = R_1(y \vee p') \\ &\Rightarrow R_p(x) = R_p(y) \\ &\Rightarrow (x, y) \in R_p. \end{aligned}$$

For 3, we will prove directly the second part which is more general. Take some $P \subseteq X$. Set $s = \vee P$. For every $p \in P$ we have $p \leq s \Rightarrow R_s \subseteq R_p$. Since this is true for *every* $p \in P$, we have $R_s \subseteq \cap_{p \in P} R_p$.On the other hand, take any $p_0 \in P$. We have $p_0 \leq s \Rightarrow p_0' \geq s'$. Also

$$\begin{aligned}
&(x, y) \in R_{p_0} \\
&\qquad \Rightarrow R_{p_0}(x) = R_{p_0}(y) \\
&\qquad \Rightarrow R_1(x \vee p_0') = R_1(y \vee p_0') \\
&\qquad \Rightarrow R_1\big((x \vee p_0') \wedge (x \vee s')\big) = R_1\big((y \vee p_0') \wedge (x \vee s')\big) \\
&\qquad \Rightarrow R_1\big(x \vee (p_0' \wedge s')\big) = R_1\big(y \vee (p_0' \wedge s')\big) \\
&\qquad \Rightarrow R_1(x \vee s') = R_1(y \vee s') \\
&\qquad \Rightarrow R_s(x) = R_s(y) \\
&\qquad \Rightarrow (x, y) \in R_s.
\end{aligned}$$

Hence $\cap_{p\in P} R_p \subseteq R_{p_0} \subseteq R_s$ and so we see that $\cap_{p\in P} R_p = R_s = R_{\vee P}$. ■

Proposition 3.24 *For all $p, q, x \in X$, $P \subseteq X$ we have the following:*

1. $R_0(x) = X \times X$.
2. $p \leq q \Rightarrow R_q(x) \subseteq R_p(x)$.
3. $R_{p\vee q}(x) = R_p(x) \cap R_q(x)$*; more generally,* $\cap_{p\in P} R_p(x) = R_{\vee P}(x)$.

Proof. In fact all of the above $1-3$ are restatements of results 1–3 of Proposition 3.23, based on the equivalence $x \in R_p(a) \Leftrightarrow (x, a) \in R_p$. ■

In light of Propositions 3.23 and 3.24, both $\left(\{R_p\}_{p\in X}, \cap\right)$ and $\left(\{R_p(a)\}_{p\in X}, \cap\right)$ are closure systems. Hence the following propositions are immediate.

Proposition 3.25 *The structure $(\{R_p\}_{p\in X}, \dot\cup, \cap, \subseteq)$ is a complete lattice (where $R_p \dot\cup R_q \doteq \cap_{s:R_p\subseteq R_s, R_q\subseteq R_s} R_s$).*

Proposition 3.26 *For every $x \in X$ the structure $(\{R_p(x)\}_{p\in X}, \dot\cup, \cap, \subseteq)$ is a complete lattice (where $R_p(x) \dot\cup R_q(x) \doteq \cap_{s:R_p(x)\subseteq R_s(x), R_q(x)\subseteq R_s(x)} R_s(x)$).*

3.3 The Family of Congruences Derived from the Identity

We now turn to a special relation, namely the *identity* relation, which we denote by ρ_1.

Definition 3.27 *We define ρ_1 as follows: $(x, y) \in \rho_1$ iff $x = y$.*

It is obvious that ρ_1 is an equivalence and a congruence with respect to $\vee, \wedge$ and $\sqcup_1$. Let us define (for all $p \in X$) the family of relations ρ_p in the usual manner.

Definition 3.28 *For every $p \in X$ we define ρ_p as follows: $(x, y) \in \rho_p$ iff $(x \vee p', y \vee p') \in \rho$.*

Definition 3.29 *For every $p, x, y \in X$ we define $\preceq_p$ as follows: $\rho_p(x) \preceq_p \rho_p(y)$ iff $\rho_p(x \wedge y) = \rho_p(x)$.*

We will also write $x =_p y$ when $x \vee p' = y \vee p'$; similarly we will write $x \leq_p y$ when $x \vee p' \leq y \vee p'$ (these notations have been introduced in [8]). Obviously, for all $x, y, p \in X$ we have:

$$\rho_p(x) = \rho_p(y) \Leftrightarrow x =_p y, \qquad \rho_p(x) \preceq_p \rho_p(y) \Leftrightarrow x \leq_p y.$$

Since ρ_1 is a special case of R_1 all the results of Sections 3.2 hold for this special case as well; in particular for every $p \in X$, ρ_p is a congruence with respect to $\vee, \wedge, \sqcup_p$. Some special properties follow from the fact that ρ_1 is the identity relation.

Proposition 3.30 *For every $x, a, b, p \in X$ we have: $x \in a \sqcup_p b \Rightarrow \rho_p(x) \subseteq a \sqcup_p b$.*

Proof. Choose any $a, b, x, p \in X$. Suppose that $x \in a \sqcup_p b$. Now take any $y \in \rho_p(x)$. Then we have:

$$x \in a \sqcup_p b \Rightarrow a \vee b \vee p' = a \vee x \vee p' = b \vee x \vee p'$$
$$\rho_p(x) = \rho_p(y) \Rightarrow x \vee p' = y \vee p'$$

hence $a \vee b \vee p' = a \vee y \vee p' = b \vee y \vee p'$ and so $y \in a \sqcup_p b$. ■

Corollary 3.31 *For every $x, a, b, p \in X$ we have: $a \sqcup_p b = \cup_{x \in a \sqcup_p b} \rho_p(x)$.*

Proof. Straightforward. ■

Proposition 3.32 *For every $x, p \in X$ we have $\rho_p(x) = [a, x \vee p']$.*

Proof. Choose any $x, p \in X$. Define $a = \wedge \rho_p(x)$ and $b = \vee \rho_p(x)$. I.e.

$$a = \wedge\{y : x \vee p' = y \vee p'\}, \qquad b = \vee\{y : x \vee p' = y \vee p'\}.$$

Then

$$\begin{aligned} &\forall y \in \rho_p(x) : x \vee p' = y \vee p' \Rightarrow \\ &x \vee p' = \wedge_{y \in \rho_p(x)} \left(y \vee p'\right) \Rightarrow \\ &x \vee p' = \left(\wedge_{y \in \rho_p(x)} y\right) \vee p' = a \vee p' \Rightarrow \\ &a \in \rho_p(x) \end{aligned}$$

Similarly we show that $b \in \rho_p(x)$. Since $\rho_p(x)$ is also a convex sublattice, we have $\rho_p(x) = [a, b]$. Now $x \in \rho_p(x) = \rho_p(x \vee p') \Rightarrow x \vee p' \in [a, b] \Rightarrow x \vee p' \leq b$. On the other hand, $b \leq b \vee p' = x \vee p'$. Hence $b = x \vee p'$. ■

Since *every* class of ρ_p is an interval, the following corollary of Proposition 3.21 is immediate.

Corollary 3.33 *For all $p, x, y \in X$ we have $\rho_p(x \vee y) = \rho_p(x) \vee \rho_p(y)$.*

Given some $x \in X$, we say that an element $y \in X$ is an *opposite* of x iff $0 \in x \sqcup_p y$. In general an element will have more than one opposites. It is easy to see that every $x \in X$ is an opposite of itself (*auto-opposite*, see [8]). The following proposition shows that all opposites of x are contained in the class of one such opposite, in particular in the class of x.

Proposition 3.34 *For every $x, p \in X$ we have: $0 \in x \sqcup_p y \Rightarrow y \in \rho_p(x)$.*

Proof. $0 \in x \sqcup_p y \Rightarrow x \vee y \vee p' = x \vee 0 \vee p' = y \vee 0 \vee p' \Rightarrow x \vee p' = y \vee p' \Rightarrow y \in \rho_p(x)$. ■

4 Families of Quotient Hyperalgebras

Since R_p is a congruence with respect to $\vee, \wedge, \sqcup_p$ it is straightforward to define corresponding operations/ hyperoperations on classes, which will be denoted by $\underline{\vee}_p, \underline{\wedge}_p, \underline{\sqcup}_p$. As will turn out, $\underline{\sqcup}_p$ can be associated with some interesting quotient hyperalgebras.

Definition 4.1 *For every $x, y, p \in X$ we define*

$$R_p(x) \underline{\vee}_p R_p(y) = R_p(x \vee y), \quad R_p(x) \underline{\wedge}_p R_p(y) = R_p(x \wedge y), \quad R_p(x) \underline{\sqcup}_p R_p(y) = R_p(x \sqcup_p y).$$

The following are immediate consequences of the corresponding properties of $\sqcup_p$.

Proposition 4.2 *For every $p, x, y, z \in X$ we have:*

1. $R_p(z) \in R_p(x)\underline{\sqcup}_p R_p(y) \Leftrightarrow R_p\,(z)\,\underline{\vee}_p R_p(p') \in R_p(x)\underline{\sqcup}_p R_p(y)$.

2. $R_p(x)\underline{\sqcup}_p R_p(y) = R_p(x \vee p')\underline{\sqcup}_p R_p\,(y \vee p')\,.$

Proof. For 1, note that $R_p(z) = R_p(z \vee p')= R_p(z)\underline{\vee}_p R_p(p')$. Similarly, to prove 2 we use $R_p(x) = R_p(x \vee p')$ and $R_p(y) = R_p(y \vee p')$. ■

Also, in certain circumstances, $\underline{\vee}_p$ and $\underline{\sqcup}_p$ can be obtained from "pointwise" operations.

Proposition 4.3 *If for some $p \in X$ every $A \in X/R_p$ is an interval, then for every x, y we have:*

1. $R_p(x)\underline{\vee}_p R_p(y) = R_p(x) \vee R_p(y)$.

2. $R_p(x)\underline{\sqcup}_p R_p(y) = \{R_p(z) : R_p(x) \vee R_p(y) = R_p(x) \vee R_p(z) = R_p(y) \vee R_p(z)\ \}\,.$

Proof. 1 is simply a restatement of Proposition 3.21. Regarding 2, let us tentatively define a hyperoperation $\underline{\underline{\sqcup}}_p$by

$$R_p(x)\underline{\underline{\sqcup}}_p R_p(y) = \{R_p(z) : R_p(x) \vee R_p(y) = R_p(x) \vee R_p(z) = R_p(y) \vee R_p(z)\ \}\,.$$

We note the following.

(i) $A \in R_p(x)\underline{\sqcup}_p R_p(y) = R_p(x \sqcup_p y)$ implies that there exists some z_0 such that $z_0 \in A$ and $z_0 \in x \sqcup_p y$. Hence

$$\begin{gathered}
x \vee y \vee p' = z_0 \vee x \vee p' = z_0 \vee y \vee p' \Rightarrow \\
R_1(x \vee y \vee p') = R_1\left(z_0 \vee x \vee p'\right) = R_1\left(z_0 \vee y \vee p'\right) \Rightarrow \\
R_p(x \vee y) = R_p(z_0 \vee x) = R_p(z_0 \vee y) \Rightarrow \\
R_p(x) \vee R_p(y) = R_p(z_0) \vee R_p(x) = R_p(z_0) \vee R_p(y).
\end{gathered}$$

Hence $A = R_p(z_0) \in R_p(x)\underline{\underline{\sqcup}}_p R_p(y)$, i.e. $R_p(x)\underline{\sqcup}_p R_p(y) \subseteq R_p(x)\underline{\underline{\sqcup}}_p R_p(y)$.

(ii) On the other hand, take some $A \in R_p(x)\underline{\underline{\sqcup}}_p R_p(y)$. Then there exists some z_0 such that $z_0 \in A$ and

$$R_p(x) \vee R_p(y) = R_p(z_0) \vee R_p(x) = R_p(z_0) \vee R_p(y)$$

Now $z_0 \vee x \in R_p(z_0 \vee x)$ and so there exist $z_1 \in R_p(z_0)$, $y_1 \in R_p(y)$ such that $z_0 \vee x = z_1 \vee y_1$. Similarly, ther exist $x_2 \in R_p(x)$, $y_2 \in R_p(y)$ such that $z_0 \vee x = x_2 \vee y_2$. Now:

$$z_0 \vee x = z_1 \vee y_1 \Rightarrow z_0 \vee x \vee z_1 = z_1 \vee y_1 \vee z_1 = z_1 \vee y_1 = x_2 \vee y_2 \tag{9}$$

and

$$z_0 \vee x = z_1 \vee y_1 \Rightarrow z_0 \vee x \vee z_0 = z_1 \vee y_1 \vee z_0 = z_0 \vee x_2 \vee y_2 = x_2 \vee y_2 \tag{10}$$

(in the last step we have used that $z_0 \leq z_0 \vee x = x_2 \vee y_2$.) Now, (9) implies

$$z_0 \vee x \vee z_1 = x_2 \vee y_2 \Rightarrow z_0 \vee x \vee z_1 \vee x_2 = x_2 \vee y_2 \tag{11}$$

and (10) implies

$$z_0 \vee y_1 \vee z_1 = x_2 \vee y_2 \Rightarrow z_0 \vee x \vee z_1 \vee y_2 = x_2 \vee y_2. \tag{12}$$

But $x \leq z_0 \vee x = x_2 \vee y_2$ and $y_1 \leq z_1 \vee y_1 = x_2 \vee y_2$ imply

$$x \vee y_1 \leq x_2 \vee y_2 \Rightarrow x_2 \vee y_2 = x \vee y_1 \vee x_2 \vee y_2. \tag{13}$$

Hence (11) and (13) imply

$$z_0 \vee x \vee z_1 \vee x_2 = x \vee y_1 \vee x_2 \vee y_2 \Rightarrow$$

$$(z_0 \vee z_1) \vee (x \vee x_2) \vee p' = (x \vee x_2) \vee (y_1 \vee y_2) \vee p'; \tag{14}$$

similarly (12) and (13) imply

$$z_0 \vee y_1 \vee z_1 \vee y_2 = x \vee y_1 \vee x_2 \vee y_2 \Rightarrow$$

$$(z_0 \vee z_1) \vee (y_1 \vee y_2) \vee p' = (x \vee x_2) \vee (y_1 \vee y_2) \vee p'. \tag{15}$$

From (14) and (15) we see that $z_0 \vee z_1 \in (x \vee x_2) \sqcup_p (y_1 \vee y_2)$. Then it follows that

$$R_p(z_0 \vee z_1) \in R_p((x \vee x_2) \sqcup_p (y_1 \vee y_2)) = R_p(x \vee x_2) \underline{\sqcup}_p R_p(y_1 \vee y_2). \tag{16}$$

Finally

$$z_0, z_1 \in R_p(z_0) \Rightarrow z_0 \vee z_1 \in R_p(z_0) \Rightarrow R_p(z_0 \vee z_1) = R_p(z_0) \tag{17}$$

$$x, x_2 \in R_p(x) \Rightarrow x \vee x_2 \in R_p(x) \Rightarrow R_p(x \vee x_2) = R_p(x) \tag{18}$$

$$y_1, y_2 \in R_p(y) \Rightarrow y_1 \vee y_2 \in R_p(y) \Rightarrow R_p(y_1 \vee y_2) = R_p(y) \tag{19}$$

and (17), (18), (19) in conjunction with (16) imply that $R_p(z_0) \in R_p(x) \underline{\sqcup}_p R_p(y)$ and hence $R_p(x) \underline{\underline{\sqcup}}_p R_p(y)_p \subseteq R_p(x) \sqcup_p R_p(y)$. This, in conjunction with the conclusion of (i.1) means that

$$R_p(x) \underline{\underline{\sqcup}}_p R_p(y)_p = R_p(x) \underline{\sqcup}_p R_p(y)$$

and so

$$R_p(x) \underline{\sqcup}_p R_p(y) = \{ R_p(z) : R_p(x) \vee R_p(y) = R_p(x) \vee R_p(z) = R_p(y) \vee R_p(z) \}.$$

■

Now we turn to the hyperalgebras associated with $\underline{\sqcup}_p$. First, for every value of p, the resulting quotient hyperalgebra is a hypergroup.

Proposition 4.4 *For all $p \in X$, $(X/R_p, \underline{\sqcup}_p)$ is a* commutative hypergroup, *with neutral element $R_p(0)$, i.e. for all $x, y, z \in X$ the following hold.*

1. $R_p(x) \underline{\sqcup}_p X/R_p = X/R_p$.
2. $R_p(x) \underline{\sqcup}_p R_p(y) = R_p(y) \underline{\sqcup}_p R_p(x)$.
3. $\left(R_p(x) \underline{\sqcup}_p R_p(y) \right) \underline{\sqcup}_p R_p(z) = R_p(x) \underline{\sqcup}_p \left(R_p(y) \underline{\sqcup}_p R_p(z) \right)$.
4. $R_p(x) \in R_p(x) \underline{\sqcup}_p R_p(0)$.
5. $R_p(0) \in R_p(x) \underline{\sqcup}_p R_p(x)$.

Proof. Regarding 1:

$$R_p(x)\underline{\sqcup}_p X/R_p = \cup_{z\in X} R_p(x)\underline{\sqcup}_p R_p(z) = \cup_{z\in X} R_p(x\sqcup_p z) = R_p(x\sqcup_p X) = R_p(X) = X/R_p,$$

where we have used Proposition 2.4. As for 2, it is immediate. Regarding 3, first note that for every $x, y, z \in X$ we have: $R_p\left(\cup_{u\in y\sqcup_p z} x\sqcup_p u\right) = \cup_{u\in y\sqcup_p z} R_p\left(x\sqcup_p u\right)$ where we have used again Proposition 2.4. Now

$$\begin{aligned} R_p(x)\underline{\sqcup}_p\left(R_p(y)\underline{\sqcup}_p R_p(z)\right) &= R_p(x)\underline{\sqcup}_p R_p(y\sqcup_p z) = \cup_{u\in y\sqcup_p z} R_p\left(x\sqcup_p u\right) = R_p\left(\cup_{u\in y\sqcup_p z} x\sqcup_p u\right) \\ &= R_p\left(x\sqcup_p y\sqcup_p z\right). \end{aligned}$$

Similarly we can show $\left(R_p(x)\underline{\sqcup}_p R_p(y)\right)\underline{\sqcup}_p R_p(z)= R_p\left(x\sqcup_p y\sqcup_p z\right)$ and this completes the proof of 3. Finally, regarding 4, $x\in x\sqcup_p 0 \Rightarrow R_p\left(x\right)\in R_p\left(x\sqcup_p 0\right) = R_p\left(x\right)\underline{\sqcup}_p R_p\left(0\right)$; 5 is proved similarly. ■

In fact, $(X/R_p, \underline{\sqcup}_p)$ is not simply a hypergroup, but a join space. To show this we will prove a sequence of propositions.

Proposition 4.5 *For all $x, y, z, p \in X$ we have*

$$R_p\left(z\right)\in R_p\left(x\right)\underline{\sqcup}_p R_p\left(y\right) \Leftrightarrow R_p\left(x\right)\in R_p\left(y\right)\underline{\sqcup}_p R_p\left(z\right) \Leftrightarrow R_p\left(y\right)\in R_p\left(z\right)\underline{\sqcup}_p R_p\left(x\right).$$

Proof. We only show the first equivalence (the second is proved in identical manner). We have $R_p\left(z\right)\in R_p\left(x\right)\underline{\sqcup}_p R_p\left(y\right) \Leftrightarrow \left(\exists u: \begin{array}{c} u\in x\sqcup_p y \\ R_p(u)=R_p(z)\end{array}\right) \Leftrightarrow \left(\exists u: \begin{array}{c} x\in y\sqcup_p u \\ R_p(u)=R_p(z)\end{array}\right) \Leftrightarrow R_p\left(x\right)\in R_p\left(y\right)\underline{\sqcup}_p R_p\left(z\right)$, where we have used the property $z\in x\sqcup_p y \Leftrightarrow x\in y\sqcup_p z$, established in [8]. ■

In the standard manner of join spaces we can define for every $p\in X$ the extension hyperoperation $/_p$ by: $x/_p y = \{z : x\in z\sqcup_p y\}$ (this definition actually appears in [8]). Then we can also define the corresponding extension hyperoperation on the quotient X/R_p.

Definition 4.6 *For every $x, y, p\in X$ we define $R_p\left(x\right)//_p R_p\left(y\right) = R_p(x/_p y)$.*

The extension hyperoperation $//_p$ is identical to $\underline{\sqcup}_p$.

Proposition 4.7 *For every $x, y, p\in X$ we have $R_p\left(x\right)//_p R_p\left(y\right) = R_p\left(x\right)\underline{\sqcup}_p R_p\left(y\right)$.*

Proof. As already shown in [8], for every $x, y, p\in X$ we have $x/_p y = x\sqcup_p y$, from which the required result follows immediately. ■

Proposition 4.8 *For all $x, y, z, u, p\in X$, the following holds.*

$$(R_p(x)//_p R_p(y))\cap(R_p(u)//_p R_p(z))\neq\emptyset \Rightarrow \left(R_p(x)\underline{\sqcup}_p R_p\left(z\right)\right)\cap\left(R_p\left(y\right)\underline{\sqcup}_p R_p\left(u\right)\right)\neq\emptyset.$$

Proof. We already know that $(R_p(x)//_p R_p(y))\cap(R_p(u)//_p R_p(z)) = \left(R_p(x)\underline{\sqcup}_p R_p(y)\right)\cap\left(R_p(u)\underline{\sqcup}_p R_p(z)\right) = \left(R_p(x)\underline{\sqcup}_p R_p(y)\right)\cap\left(R_p(z)\underline{\sqcup}_p R_p(u)\right)$. Now

$$\left(R_p(x)\underline{\sqcup}_p R_p(y)\right)\cap\left(R_p(z)\underline{\sqcup}_p R_p(u)\right)\neq\emptyset \Rightarrow \left(\exists v, w: \begin{array}{c} v\in x\sqcup_p y \\ w\in z\sqcup_p u \\ R_p(v)=R_p(w)\end{array}\right).$$

Now

$$\begin{gathered} v\in x\sqcup_p y \Rightarrow y\in x\sqcup_p v \Rightarrow \\ R_p(y)\in R_p\left(x\sqcup_p v\right) = R_p\left(x\sqcup_p w\right). \end{gathered} \tag{20}$$

Also, since $w \in z \sqcup_p u$ it follows that

$$R_p\left(x \sqcup_p v\right) \subseteq R_p\left(x \sqcup_p \left(z \sqcup_p u\right)\right) = R_p\left(x \sqcup_p z \sqcup_p u\right) = R_p\left(\left(x \sqcup_p z\right) \sqcup_p u\right) \tag{21}$$

From (20) and (21) follows that $R_p(y) \in R_p\left(\left(x \sqcup_p z\right) \sqcup_p u\right)$ and hence there exist a and b such that

$$a \in x \sqcup_p z \tag{22}$$

$$b \in a \sqcup_p u \tag{23}$$

$$R_p(b) = R_p(y) \tag{24}$$

From (22) follows that $R_p(a) \in R_p\left(x \sqcup_p z\right)$; from (23) and (24) follows that $R_p(y) \in R_p\left(a \sqcup_p u\right)$ and hence (from Proposition 4.5) that $R_p(a) \in R_p\left(y \sqcup_p u\right)$. In short $\left(R_p\left(x\right) \underline{\sqcup}_p R_p\left(z\right)\right) \cap \left(R_p(y) \underline{\sqcup}_p R_p\left(u\right)\right) \neq \emptyset$. ■

Corollary 4.9 *For every $p \in X$, $(X/R_p, \underline{\sqcup}_p)$ is a join space.*

Finally, we will show that the quotient hyperoperation $\underline{\sqcup}_p$, in conjunction with the quotient operation $\underline{\triangle}_p$, generates a hyperlattice. To establish this fact, let us present some order-related properies of $\underline{\sqcup}_p$.

Proposition 4.10 *For all $x, y, z, p \in X$ we have the following:*

1. $R_p(x) \in R_p(x) \underline{\sqcup}_p R_p(x)$
2. $R_p(x) \underline{\sqcup}_p R_p(y) = R_p(y) \underline{\sqcup}_p R_p(x)$.
3. $\left(R_p(x) \underline{\sqcup}_p R_p(y)\right) \underline{\sqcup}_p R_p(z) = R_p\left(x\right) \underline{\sqcup}_p \left(R_p(y) \underline{\sqcup}_p R_p(z)\right)$.
4. $R_p(x) \in \left(R_p(x) \underline{\sqcup}_p R_p(y)\right) \underline{\triangle}_p R_p(x)$, $R_p(x) \in \left(R_p(x) \underline{\triangle}_p R_p(y)\right) \underline{\sqcup}_p R_p(x)$.
5. $R_p(x) \in R_p(x) \underline{\sqcup}_p R_p(y) \Leftrightarrow R_p(y) \preceq_p R_p(x)$.

Proof. Regarding 1 we have $x \in x \sqcup_p x$ and so $R_p(x) \in R_p(x \sqcup_p x) = R_p(x) \underline{\sqcup}_p R_p(x)$. Parts 2 and 3 have already been proved in Proposition 4.4. Regarding part 4, since $x \vee y \in x \sqcup_p y$, it follows that $x = (x \vee y) \wedge x \in \left(R_p(x) \underline{\sqcup}_p R_p(y)\right) \underline{\triangle}_p R_p(x)$. Also $\left(R_p(x) \underline{\triangle}_p R_p(y)\right) \underline{\sqcup}_p R_p(x) = \left(R_p(x \wedge y)\right) \underline{\sqcup}_p R_p(x)$ which contains $(x \wedge y) \vee x = x$. Finally, regarding 5, if $R_p(x) \in R_p(x) \underline{\sqcup}_p R_p(y)$ then there exists some u such that $R_p(x) = R_p(u)$ and $x \vee y \vee p' = x \vee u \vee p' = y \vee u \vee p'$. From this follows that

$$R_p\left(x \vee y\right) = R_p\left(x \vee u\right) \in R_p\left(x \sqcup_p u\right) = R_p\left(x\right) \underline{\sqcup}_p R_p\left(u\right) = R_p\left(x\right) \underline{\sqcup}_p R_p\left(x\right) = R_p\left(x\right)$$

hence $R_p(y) \preceq_p R_p(x)$. Conversely, $R_p(y) \preceq_p R_p(x) \Rightarrow R_p(x) = R_p(x \vee y) \in R_p(x \sqcup_p y)$. ■

Corollary 4.11 *For every $p \in X$, $(X/R_p, \underline{\sqcup}_p, \underline{\triangle}_p)$ is a hyperlattice.*

From the above corollary we see that, in particular, $(X/\rho_p, \underline{\sqcup}_p, \underline{\triangle}_p, \preceq_p)$ is a hyperlattice. Recall that in [8] we had mentioned that the hyperalgebra $(X, \sqcup_p, \wedge, \leq_p)$ closely resembles a hyperlattice except for the fact that $\leq_p$ is not an order but a *preorder*. Now we see that if we use the relationship ρ_p (which is the natural equivalence generated from $\leq_p$) we obtain in a "natural" manner the hyperlattice $(X/\rho_p, \underline{\sqcup}_p, \underline{\triangle}_p, \preceq_p)$ which can be seen as the "quotient hyperlattice" which corresponds to $(X, \sqcup_p, \wedge, \leq_p)$ under ρ_p.

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