The L-Fuzzy Nakano Hypergroup

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Abstract

In this paper we start with a lattice $(X, \vee, \wedge)$ and define, in terms of $\vee$, a family of crisp hyperoperations $\sqcup_p$ (one hyperoperation for each $p \in X$). We show that, for every p, the hyperalgebra $(X, \sqcup_p)$ is a join space and the hyperalgebra $(X, \sqcup_p, \wedge)$ is very similar to a hyperlattice. Then we use the hyperoperations $\sqcup_p$ as p-cuts to introduce an L-fuzzy hyperoperation $\sqcup$ and show that $(X, \sqcup)$ is an L-fuzzy join space and the hyperalgebra $(X, \sqcup_p, \wedge)$ is very similar to an L-fuzzy hyperlattice.

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1 Introduction

In this paper we study a family of *crisp hyperoperations*, each of which gives rise to a *join space*; then we use this family to synthesize a *fuzzy hyperoperation* which gives rise to a *fuzzy join space*. Let us explain each of these terms and also present some brief bibliographic remarks.

Given a reference set X, a crisp[1] hyperoperation maps each pair of elements of X to a nonempty subset of X. A *hyperalgebra* is a set X endowed with a hyperoperation. The most prominent example of a hyperalgebra is a *hypergroup*; an extensive account of hypergroup theory appears in [3]. A join space is a special type of hypergroup. Join spaces were introduced (with a geometric point of view) by Prenowitz; a comprehensive account appears in [8, 20, 21]. Two particular hyperoperations which have been the subject of extensive study are the so-called "Nakano hyperoperations". These were introduced in [2, 15, 18] and studied further in [1, 11, 14, 17]. The Nakano hyperoperations have many remarkable properties, not least of which is that each of them gives rise to a join space.

In recent years there has been considerable interest in the connections between hyperalgebras and fuzzy sets. One can distinguish two main approaches. The first approach defines *crisp* hyperoperations through fuzzy sets. Examples of this approach are [4, 5, 6, 9, 24]. However, this approach will not concern us here.

The second approach involves the definition and study of *fuzzy hyperoperations*. In analogy to a crisp hyperoperation, a fuzzy hyperoperation maps a pair of elements of a set X to a *fuzzy* subset of X. Corsini and Zahedi introduced some fuzzy algebraic hyperstructures in [7, 27, 28]. In particular, [7] gives a general definition of fuzzy hypergroups. Kehagias has studied a particular example of fuzzy join space in [10].

The rest of this paper is organized as follows. In Section 2 we summarize some basic concepts which will be used throughout the paper. In Section 3 we start with a lattice $(X, \vee, \wedge)$, introduce the family of crisp hyperoperations $\sqcup_p$ (one hyperoperation for each $p \in X$) and show that for every p the hyperalgebra $(X, \sqcup_p)$ is a fuzzy join space and the hyperalgebra $(X, \sqcup_p, \wedge)$ is very similar to a hyperlattice[13, 22, 23]. In Section 4 we use the hyperoperations $\sqcup_p$ as p-cuts to introduce a fuzzy

[1] "Cirsp" is used here in contradistinction to "fuzzy"; see below.

hyperoperation $\sqcup$ and show $(X, \sqcup)$ is a fuzzy join space and the hyperalgebra $(X, \sqcup, \wedge)$ is very similar to a fuzzy hyperlattice. Finally, in Section 5 we present some future research directions.

2 Preliminaries

In this section we present some basic definitions, notations and propositions (without proofs) which will be used throughout this paper. In what follows $(X, \leq, \vee, \wedge, ')$ will always denote a *generalized deMorgan lattice*, defined as follows.

Definition 2.1 *A* generalized deMorgan lattice *is a structure* $(X, \leq, \vee, \wedge, ')$*, where* $(X, \leq, \vee, \wedge)$ *is a complete distributive lattice with minimum element 0 and maximum element 1; the symbol* $'$ *denotes a unary operation ("complement"); and the following properties are satisfied.*

1. *For all* $x \in X$*,* $Y \subseteq X$ *we have* $x \wedge (\vee_{y \in Y} y) = \vee_{y \in Y}(x \wedge y)$*,* $x \vee (\wedge_{y \in Y} y) = \wedge_{y \in Y}(x \vee y)$*.* (Complete distributivity).
2. *For all* $x \in X$ *we have:* $(x')' = x$*.* (Negation is involutory).
3. *For all* $x, y \in X$ *we have:* $x \leq y \Rightarrow y' \leq x'$*.* (Negation is order reversing).
4. *For all* $Y \subseteq X$ *we have* $(\vee_{y \in Y} y)' = \wedge_{y \in Y} y'$*,* $\quad (\wedge_{y \in Y} y)' = \vee_{y \in Y} y'$ (Complete deMorgan laws).

A "classical" set will often be called a *crisp set*, in contradistinction to a *fuzzy* or *L-fuzzy set* (these will be defined presently). Obviously there is a 1-to-1 correspondence between sets and their characteristic functions. We will use the following convention.

Notation 2.2 *Given a crisp set* $A \subseteq X$*, its characteristic function will be denoted by* $\widetilde{A}(x)$*.* $\widetilde{A}(x)$ *takes values in* $\{0, 1\}$*.*

Definition 2.3 *A* fuzzy set *is a function* $\widetilde{A} : X \to [0, 1]$*, where* $[0, 1]$ *is an interval of* real numbers*.*

Generally speaking an *L-fuzzy set* is a function $\widetilde{A} : X \to L$ where $(L, \preccurlyeq)$ is some lattice. In the context of this paper $(L, \preccurlyeq)$ will always be the original $(X, \leq)$ deMorgan lattice. The fact that the domain and range of $\widetilde{A}$ are the same lattice does not create any problems.

Definition 2.4 *An L-fuzzy set is a function* $\widetilde{A} : X \to X$*.*

Since in this paper we do not deal with fuzzy sets in the sense of Definition 2.3, we will sometimes use the shorter term "fuzzy set" in place of "L-fuzzy set". Note that a crisp set A or, more precisely, the corresponding characteristic function $\widetilde{A}(x)$ can be seen as a special case of L-fuzzy set, since $\widetilde{A} : X \to \{0, 1\} \subseteq X$.

Notation 2.5 *The collection of all crisp subsets of* X *is denoted by* $\mathbf{P}(X)$ *(the* power set *of* X*); the collection of all L-fuzzy sets (i.e. functions* $\widetilde{A} : X \to X$*) is denoted by* $\mathbf{F}(X)$*.*

Definition 2.6 *Given a L-fuzzy set* $\widetilde{A} : X \to X$*, the* p-cut *of* $\widetilde{A}$ *is denoted by* A_p *and defined by* $A_p \doteq \{x : \widetilde{A}(x) \geq p\}$*.*

For basic properties of p-cuts see [10, 19]. Two important facts are (for details see [19]):

1. a fuzzy set $\widetilde{A}$ is uniquely specified by the collection of its p-cuts $\{A_p\}_{p \in X}$;

2. given a collection of sets that have the "p-cut properties", a fuzzy set can be synthesized from this family.

These facts are easily generalized to L-fuzzy sets and can be described by the following propositions.

Proposition 2.7 *Suppose that the L-fuzzy sets $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ satisfy $A_p = B_p$ for every $p \in X$. Then $\widetilde{A} = \widetilde{B}$.*

Proposition 2.8 *Suppose that a family of crisp sets $\{M_p\}_{p \in X} \subseteq \mathbf{P}(X)$ is given which satisfies the following (p-cut properties).*

1. *$M_0 = X$.*
2. *Given $p, q \in X$ we have: $p \leq q \Rightarrow M_q \subseteq M_p$.*
3. *Given $P \subseteq X$ we have: $\cap_{p \in P M_p} = M_{\vee P}$.*

Now define the L-fuzzy set $\widetilde{A}$ by its value for every $x \in X$: $\widetilde{A}(x) = \vee \{p : x \in M_p\}$. Then for every $p \in X$ we have $A_p = M_p$.

Definition 2.9 *For all $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ define the L-fuzzy set $\widetilde{A} \cup \widetilde{B}$ by specifying its value for every $x \in X$:*

$$(\widetilde{A} \cup \widetilde{B})(x) = \widetilde{A}(x) \vee \widetilde{B}(x);$$

define the L-fuzzy set $\widetilde{A} \cap \widetilde{B}$ by:

$$(\widetilde{A} \cap \widetilde{B})(x) = \widetilde{A}(x) \wedge \widetilde{B}(x);$$

define the L-fuzzy set $\widetilde{A}'$ by:

$$\widetilde{A}'(x) = \left(\widetilde{A}(x)\right)'.$$

Notation 2.10 *For all $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ we write $\widetilde{A} \subseteq \widetilde{B}$ iff for all $x \in X$ we have $\widetilde{A}(x) \leq \widetilde{B}(x)$.*

Proposition 2.11 *For all $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$: $\widetilde{A} \subseteq \widetilde{B}$ iff for all $p \in X$ we have $A_p \subseteq B_p$.*

Proposition 2.12 *$\subseteq$ is an order on $\mathbf{F}(X)$ and $(\mathbf{F}(X), \subseteq, \cup, \cap, ')$ is a generalized deMorgan lattice.*

Notation 2.13 *For all $A, B \in \mathbf{P}(X)$ we write $A \sim B$ iff $A \cap B \neq \emptyset$. For all $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ and $p \in X$ we write $\widetilde{A} \sim_p \widetilde{B}$ iff $\exists x \in X : \widetilde{A}(x) \wedge \widetilde{B}(x) \geq p$.*

It is clear that the symbols $\cup, \cap, '$ when applied to fuzzy sets play the same role (union, intersection, complement) as when applied (in the standard manner) to crisp sets; indeed if $A, B \in \mathbf{P}(X)$ then the characteristic functions $\widetilde{A}, \widetilde{B}$ are a special case of L-fuzzy set and $\widetilde{A} \cup \widetilde{B}$, $\widetilde{A} \cap \widetilde{B}$, $\widetilde{A}'$ correspond to the same sets as $A \cup B$, $A \cap B$, A'. In the fuzzy set literature it is usual to denote fuzzy-set union by $\widetilde{A} \vee \widetilde{B}$ and fuzzy-set intersection by $\widetilde{A} \wedge \widetilde{B}$. However, as will be explained now, in the current paper the interpretation of the symbols $\vee, \wedge$ is quite different . Recall that

1. a *(crisp) operation* $\circ$ maps a pair of elements to an element: $\circ : X \times X \rightarrow X$;
2. a *(crisp) hyperoperation* $\circ$ maps a pair of elements to a crisp set: $\circ : X \times X \rightarrow \mathbf{P}(X)$;
3. an *L-fuzzy hyperoperation* $\circ$ maps a pairs of elements to an L-fuzzy set: $\circ : X \times X \rightarrow \mathbf{F}(X)$.

Note that if $\circ$ is an L-fuzzy hyperoperartion then $a \circ b$ is a function; we will often use the notation $(a \circ b)(x)$ (i.e. the value of $a \circ b$ at the point x).

The most general of the above concepts is that of *L-fuzzy hyperoperation*: a crisp hyperoperation can be seen as a special case where the L-fuzzy set becomes a crisp set; a crisp operation can be seen as an even more special case where the crisp set becomes a singleton. Now, given some $\circ$ (crisp/fuzzy operation/hyperoperation) we will often need to apply $\circ$ to *sets* rather than elements. The basic definition is the following.

Definition 2.14 *Let* $\circ : X \times X \to \mathbf{F}(X)$ *be a* L-fuzzy *hyperoperation (*hence $a \circ b$ is a fuzzy set, i.e. a function*).*

1. *For all* $x \in X$*,* $\widetilde{A} \in \mathbf{F}(X)$ *we define the* L-fuzzy *set* $x \circ \widetilde{A}$ *by*

$$(x \circ \widetilde{A})(z) \doteq \vee_{y \in X} \left(\widetilde{A}(y) \wedge [(x \circ y)(z)] \right)$$

2. *For all* $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ *we define the* L-fuzzy *set* $\widetilde{A} \circ \widetilde{B}$ *by*

$$(\widetilde{A} \circ \widetilde{B})(z) \doteq \vee_{x \in X, y \in X} \left\{ \widetilde{A}(x) \wedge \widetilde{B}(y) \wedge [(x \circ y)(z)] \right\}.$$

As will be argued, the following definitions are special cases of Definition Definition 2.14.

Definition 2.15 *Let* $\circ : X \times X \to \mathbf{P}(X)$ *be a* crisp *operation.*

1. *For all* $x \in X$*,* $A \in \mathbf{P}(X)$ *we define the* crisp *set* $x \circ A = \cup_{y \in A} \{x \circ y\}$.
2. *For all* $A, B \in \mathbf{P}(X)$ *we define the* crisp *set* $A \circ B = \cup_{x \in A, y \in B} \{x \circ y\}$

Definition 2.16 *Let* $\circ : X \times X \to \mathbf{P}(X)$ *be a* crisp *hyperoperation.*

1. *For all* $x \in X$*,* $A \in \mathbf{P}(X)$ *we define the* crisp *set* $x \circ A = \cup_{y \in A} x \circ y$*.*
2. *For all* $A, B \in \mathbf{P}(X)$ *we define the* crisp *set* $A \circ B = \cup_{x \in A, y \in B} x \circ y$*..*

Definition 2.17 *Let* $\circ : X \times X \to \mathbf{P}(X)$ *be a* crisp *hyperoperation.*

1. *For all* $x \in X$*,* $\widetilde{A} \in \mathbf{F}(X)$ *we define the* L-fuzzy *set* $x \circ \widetilde{A}$ *by*

$$(x \circ \widetilde{A})(z) \doteq \vee_{y \in X} \left(\widetilde{A}(y) \wedge [(x \circ y)(z)] \right).$$

2. *For all* $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ *we define the* L-fuzzy *set* $\widetilde{A} \circ \widetilde{B}$ *by*

$$(\widetilde{A} \circ \widetilde{B})(z) \doteq \vee_{x \in X, y \in X} \left\{ \widetilde{A}(x) \wedge \widetilde{B}(y) \wedge [(x \circ y)(z)] \right\}.$$

(Where $(x \circ y)(z)$ *is the value of the* characteristic function of $x \circ y$ *at* z*.)*

Definition 2.14 subsumes the remaining ones as special cases. For example, the case of a crisp hyperoperation $\circ$ can be seen as a special case in Definition 2.14: i.e. for all $x, y, z \in X$ we have that $(x \circ y)(z)$ (the characteristic function of the crisp set $x \circ y = \{z : (x \circ y)(z) > 0\}$) is either 1 or 0. Similarly, if $\widetilde{A}$ is the characteristic function of the *crisp* set A, then

$$\left(x \circ \widetilde{A} \right)(z) = \vee_{y \in X} \left(\widetilde{A}(y) \wedge [(x \circ y)(z)] \right) = \vee_{y : z \in x \circ y} \widetilde{A}(y)$$

is the characteristic function of the crisp set $x \circ A = \cup_{y \in A} x \circ y$.

3 A Family of Crisp Hyperoperations

3.1 Definition and Properties

We start our investigation by recalling the definition of the $\sqcup_1$ hyperoperation.

Definition 3.1 *For every $p \in X$ we define the hyperoperation $\sqcup_1 : X \times X \rightarrow \mathbf{P}(X)$ as follows:*

$$\forall x, y \in X : x \sqcup_1 y \doteq \{z : x \vee y = x \vee z = y \vee z\}$$

The above hyperoperation was first introduced by Comer in [2]. However, the dual hyperoperation defined by $x \sqcap_1 y \doteq \{x : x \wedge y = x \wedge z = y \wedge z\}$ was introduced earlier by Nakano [18]. Hence we refer to both $\sqcup_1$ and $\sqcap_1$as the "Nakano hyperoperations". The reason for the use of the subscript 1 will become obvious in the sequel. The following properties of $\sqcup_1$ have been established by several authors.

Proposition 3.2 *For all $x, y, z \in X$ the following hold.*

1. $x \in x \sqcup_1 x$.
2. $x \sqcup_1 y = y \sqcup_1 x$.
3. $X = x \sqcup_1 X$.
4. $(x \sqcup_1 y) \sqcup_1 z = x \sqcup_1 (y \sqcup_1 z)$.
5. $x \in (x \sqcup_1 y) \wedge x$, $x \in (x \wedge y) \sqcup_1 x$.
6. $x \leq y \Leftrightarrow x \in x \sqcup_1 y$.

Proof. 1 and 2 are obvious; 3 follows from 1; 4 has been proved by Comer [2]; 5 and 6 have been proved by Mittas and Konstantinidou [16] . ■

Now we generalize $\sqcup_1$ by defining a *family* of hyperoperations $\sqcup_p$ (one hyperoperation for each $p \in X$). Note that in the following definition using $p = 1$ we recover the $\sqcup_1$ hyperoperation of Definition 3.1.

Definition 3.3 *For every $p \in X$ we define the hyperoperation $\sqcup_p : X \times X \rightarrow \mathbf{P}(X)$ as follows:*

$$\forall x, y \in X : x \sqcup_p y \doteq \{z : x \vee y \vee p' = x \vee z \vee p' = y \vee z \vee p'\}$$

The following proposition gives two properties of $\sqcup_p$ which will be very useful in the sequel. The proof is immediate, hence omitted.

Proposition 3.4 *For every $p, x, y, z \in X$ we have:*

1. $z \in x \sqcup_p y \Leftrightarrow z \vee p' \in x \sqcup_p y$.
2. $x \sqcup_p y = (x \vee p') \sqcup_p (y \vee p') = (x \vee p') \sqcup_1 (y \vee p')$.

The next proposition shows that $x \sqcup_p y$ is a closed interval.

Proposition 3.5 *For all $x, y, p \in X$ there exists some $a \in X$ such that $x \sqcup_p y = [a, x \vee y \vee p']$.*

Proof. Choose any $x, y, p \in X$. First let us show that $x \sqcup_p y$ is a convex sublattice. Indeed, taking any $z, u \in x \sqcup_p y$ we have

$$\left.\begin{array}{l} x \vee y \vee p' = x \vee z \vee p' = y \vee z \vee p' \\ x \vee y \vee p' = x \vee u \vee p' = y \vee u \vee p' \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x \vee y \vee p' = x \vee z \vee u \vee p' = y \vee z \vee u \vee p' \\ x \vee y \vee p' = x \vee (z \wedge u) \vee p' = y \vee (z \wedge u) \vee p' \end{array}\right.$$

which implies that $z \vee u \in x \sqcup_p y$ and $z \wedge u \in x \sqcup_p y$. Hence $x \sqcup_p y$ is a sublattice. To show convexity, take any z, u, w such that $z \leq u \leq w$ and $z, w \in x \sqcup_p y$. Then

$$\begin{array}{c} x \vee y \vee p' = x \vee z \vee p' = y \vee z \vee p' \Rightarrow \\ x \vee y \vee u \vee p' = x \vee z \vee u \vee p' = y \vee z \vee u \vee p' \Rightarrow \\ x \vee y \vee u \vee p' = x \vee u \vee p' = y \vee u \vee p'; \end{array} \tag{1}$$

and

$$\begin{array}{c} x \vee y \vee p' = x \vee w \vee p' = y \vee w \vee p' \Rightarrow \\ x \vee y \vee u \vee p' = x \vee y \vee w \vee u \vee p' = y \vee w \vee u \vee p' \Rightarrow \\ x \vee y \vee u \vee p' = x \vee w \vee p' = y \vee w \vee p' = x \vee y \vee p'. \end{array} \tag{2}$$

Combining (1) and (2) we obtain $x \vee y \vee p' = x \vee u \vee p' = y \vee u \vee p'$ which shows that $u \in x \sqcup_p y$. Next we show that $x \sqcup_p y$ contains its infimum and supremum. We have

$$\begin{array}{l} \forall z \in x \sqcup_p y : x \vee y \vee p' = z \vee x \vee p' = z \vee y \vee p' \Rightarrow \\ x \vee y \vee p' = \wedge_{z \in x \sqcup_p y} \left(z \vee x \vee p'\right) = \wedge_{z \in x \sqcup_p y} \left(z \vee y \vee p'\right) \Rightarrow \\ x \vee y \vee p' = \left(\wedge_{z \in x \sqcup_p y} z\right) \vee x \vee p' = \left(\wedge_{z \in x \sqcup_p y} z\right) \vee y \vee p' \Rightarrow \\ \wedge_{z \in x \sqcup_p y} z \in x \sqcup_p y \end{array}$$

Similarly we can show $\vee_{z \in x \sqcup_p y} z \in x \sqcup_p y$. Hence, letting $a = \wedge_{z \in x \sqcup_p y} z$, $b = \vee_{z \in x \sqcup_p y} z$, we have $x \sqcup_p y = [a, b]$. Now, since $x \vee y \vee p' \in x \sqcup_p y$, we have $x \vee y \vee p' \leq b$. On the other hand $b \in x \sqcup_p y$ and so $x \vee y \vee p' = x \vee b \vee p' = y \vee b \vee p' \geq b$. Hence $b = x \vee y \vee p'$. ■

Corollary 3.6 *For all $x, y, z \in X$ we have: $x \leq y \Rightarrow (\forall w \in x \sqcup_p z : \exists u \in y \sqcup_p z$ such that $w \leq u)$.*

In Definition 3.7 we introduce a family of relationships between elements of X; as stated in Proposition 3.8, each of these relations is a preorder (the proof is immediate and hence omitted).

Definition 3.7 *For all $x, y, p \in X$ we write:*

1. *$x \leq_p y$ (and $y \geq_p x$) iff $x \vee p' \leq y \vee p'$.*
2. *$x =_p y$ iff $x \vee p' = y \vee p'$.*

Proposition 3.8 *For all $p \in X$: $\leq_p$ is a preorder on X and $=_p$ is the associated equivalence relation (i.e. $x =_p y \Leftrightarrow (x \leq_p y$ and $y \leq_p x)$).*

Proposition 3.9 summarizes the basic properties of $\sqcup_p$ and hence is analogous to Proposition 3.2.

Proposition 3.9 *For all $x, y, z, p \in X$ we have:*

1. *$x \in x \sqcup_p x$;*

2. $x \sqcup_p y = y \sqcup_p x$;

3. $X = x \sqcup_p X$;

4. $(x \sqcup_p y) \sqcup_p z = x \sqcup_p (y \sqcup_p z)$;

5. $x \in (x \sqcup_p y) \wedge x,\ x \in (x \wedge y) \sqcup_p x$;

6. $y \leq_p x \Leftrightarrow x \in x \sqcup_p y$.

Proof. 1 and 2 are obvious. For 3, clearly we have $x \sqcup_p X \subseteq X$. On the other hand, take any $z \in X$. Then $x \vee (x \vee z) \vee p' = x \vee z \vee p' = (x \vee z) \vee z \vee p'$, hence $z \in x \sqcup_p (x \vee z) \subseteq x \sqcup_p X$.

For 4 take any $u \in (x \sqcup_p y) \sqcup_p z$; then there exists $w \in x \sqcup_p y$ such that $u \in w \sqcup_p z$. But $w \in x \sqcup_p y$ implies $w \vee p' \in (x \vee p') \sqcup_1 (y \vee p')$ and $u \in w \sqcup_p z$ implies $u \in (w \vee p') \sqcup_1 (z \vee p')$ .Now

$$\begin{aligned} u &\in \left(w \vee p'\right) \sqcup_1 \left(z \vee p'\right) \\ &\subseteq \left(\left(x \vee p'\right) \sqcup_1 \left(y \vee p'\right)\right) \sqcup_1 \left(z \vee p'\right) \\ &= \left(x \vee p'\right) \sqcup_1 \left(\left(y \vee p'\right) \sqcup_1 \left(z \vee p'\right)\right) \\ &= \cup_{v \in y \sqcup_p z} \left(x \vee p'\right) \sqcup_1 v \\ &= \cup_{v \in y \sqcup_p z} \left(x \vee p'\right) \sqcup_1 \left(v \vee p'\right) \\ &= \cup_{v \in y \sqcup_p z} x \sqcup_p v = x \sqcup_p (y \sqcup_p z) . \end{aligned}$$

So we have shown $(x \sqcup_p y) \sqcup_p z \subseteq x \sqcup_p (y \sqcup_p z)$. Similarly we show $x \sqcup_p (y \sqcup_p z) \subseteq (x \sqcup_p y) \sqcup_p z$ and we have proved 4.

For 5 we have $x = ((x \vee p') \vee (y \vee p')) \wedge x \in ((x \vee p') \sqcup_1 (y \vee p')) \wedge x = (x \sqcup_p y) \wedge x$. Also

$$(x \wedge y) \vee x \vee p' = \ (x \wedge y) \vee x \vee p' = \ x \vee x \vee p'$$

and hence $x \in (x \wedge y) \sqcup_p x$. For 6, we have $x \in x \sqcup_p y \Leftrightarrow x \vee y \vee p' = x \vee x \vee p' = y \vee x \vee p' \Leftrightarrow y \vee p' \leq x \vee p'$ $\Leftrightarrow y \leq_p x$.[2] ∎

Next we show that $x \sqcup_p y$ has (for every $x, y, p \in X$) the p-cut properties.

Proposition 3.10 *The following properties hold for all $x, y, p, q \in X$, $P \subseteq X$.*

1. $x \sqcup_0 y = [0, 1]$.

2. $p \leq q \Rightarrow x \sqcup_q y \subseteq x \sqcup_p y$.

3. $x \sqcup_{p \vee q} y = (x \sqcup_p y) \cap (x \sqcup_q y)$; *more generally* $x \sqcup_{\vee P} y = \cap_{p \in P} (x \sqcup_p y)$.

Proof. 1 is obvious. For 2: $p \leq q \Rightarrow q' \leq p'$. Now

$$\begin{gathered} z \in x \sqcup_q y \Rightarrow \\ x \vee y \vee q' = x \vee z \vee q' = y \vee z \vee q' \Rightarrow \\ x \vee y \vee q' \vee p' = x \vee z \vee q' \vee p' = y \vee z \vee q' \vee p' \Rightarrow \\ x \vee y \vee p' = x \vee z \vee p' = y \vee z \vee p' \Rightarrow \\ z \in x \sqcup_p y. \end{gathered}$$

[2] We have also: $y \leq x \Rightarrow x \in x \sqcup_p y$; but the reverse implication generally does not hold.

Regarding 3 we will prove the (more general) part: $x \sqcup_{\vee P} y = \cap_{p\in P}(x \sqcup_p y)$. Take any $P \subseteq X$. Since for every $p \in P$ we have $p \leq \vee P$, it follows from 2 that: $\forall p \in P : x \sqcup_{\vee P} y \subseteq x \sqcup_p y$. Hence $x \sqcup_{\vee P} y \subseteq \cap_{p\in P}(x \sqcup_p y)$. On the other hand

$$\begin{gathered}
z \in \cap_{p\in P}(x \sqcup_p y) \Rightarrow \\
\forall p \in P : z \in x \sqcup_p y \Rightarrow \\
\forall p \in P : x \vee y \vee p' = x \vee z \vee p' = y \vee z \vee p' \Rightarrow \\
\wedge_{p\in P}\left(x \vee y \vee p'\right) = \wedge_{p\in P}\left(x \vee z \vee p'\right) = \wedge_{p\in P}\left(y \vee z \vee p'\right) \Rightarrow \\
x \vee y \vee \left(\wedge_{p\in P} p'\right) = x \vee z \vee \left(\wedge_{p\in P} p'\right) = y \vee z \vee \left(\wedge_{p\in P} p'\right) \Rightarrow \\
x \vee y \vee (\vee_{p\in P} p)' = x \vee z \vee (\vee_{p\in P} p)' = y \vee z \vee (\vee_{p\in P} p)' \Rightarrow \\
z \in x \sqcup_{\vee P} y
\end{gathered}$$

(where we have used complete distributivity and the fact that $\wedge_{p\in P} p' = (\vee_{p\in P} p)' = (\vee P)'$). Hence $\cap_{p\in P}(x \sqcup_p y) \subseteq x \sqcup_{\vee P} y$ and the proof is complete. ■

Corollary 3.11 *For all* $x, y \in X$, $\{x \sqcup_p y\}_{p\in X}$ *is a* closure system *and* $\left(\{x \sqcup_p y\}_{p\in X}, \dot{\cup}, \cap, \subseteq\right)$ *is a lattice (where* $(x \sqcup_p y)\,\dot{\cup}\,(x \sqcup_q y) = \cap_{r: x\sqcup_p y \subseteq x \sqcup_r y, x \sqcup_p y \subseteq x \sqcup_r y}(x \sqcup_r y)$*).*

Distributivity between $\sqcup_p$, $\wedge$ and $\vee$ holds in a weak sense. We first cite Proposition 3.12 which regards $\sqcup_1$ and has been proved in [14]. Then, in Proposition 3.13 we generalize the weak distributivity properties to $\sqcup_p$.

Proposition 3.12 *For all* $x, y, z \in X$ *the following properties hold.*

1. $x \wedge (y \sqcup_1 z) \subseteq (x \wedge y) \sqcup_1 (x \wedge z)$.
2. $x \vee (y \sqcup_1 z) \subseteq (x \vee y) \sqcup_1 (x \vee z)$.
3. $(x \sqcup_1 y) \vee (x \sqcup_1 z) \subseteq x \sqcup_1 (y \vee z)$.

Proposition 3.13 *For all* $x, y, z, p \in X$ *the following properties hold.*

1. $x \wedge (y \sqcup_p z) \subseteq (x \wedge y) \sqcup_p (x \wedge z)$.
2. $x \vee (y \sqcup_p z) \subseteq (x \vee y) \sqcup_p (x \vee z)$.
3. $(x \sqcup_p y) \vee (x \sqcup_p z) \subseteq x \sqcup_p (y \vee z)$.

Proof. For 1:

$$\begin{aligned}
x \wedge (y \sqcup_p z) &= x \wedge \left(\left(y \vee p'\right) \sqcup_1 \left(z \vee p'\right)\right) \\
&\subseteq \left(x \wedge \left(y \vee p'\right)\right) \sqcup_1 \left(x \wedge \left(z \vee p'\right)\right) \\
&= \left((x \wedge y) \vee \left(x \wedge p'\right)\right) \sqcup_1 \left((x \wedge z) \vee \left(x \wedge p'\right)\right) \\
&= \left((x \wedge y) \vee \left(x' \vee p\right)'\right) \sqcup_1 \left((x \wedge z) \vee \left(x' \vee p\right)'\right) \\
&= (x \wedge y) \sqcup_{x' \vee p} (x \wedge z) \\
&\subseteq (x \wedge y) \sqcup_p (x \wedge z).
\end{aligned}$$

Similarly, for 2:

$$\begin{aligned} x \vee (y \sqcup_p z) &= x \vee \left(\left(y \vee p'\right) \sqcup_1 \left(z \vee p'\right)\right) \\ &\subseteq \left(x \vee y \vee p'\right) \sqcup_1 \left(x \vee z \vee p'\right) \\ &= (x \vee y) \sqcup_p (x \vee z) \,. \end{aligned}$$

Similarly, for 3:

$$\begin{aligned} &(x \sqcup_p y) \vee (x \sqcup_p z) \\ &= \left(\left(x \vee p'\right) \sqcup_1 \left(y \vee p'\right)\right) \vee \left(\left(x \vee p'\right) \sqcup_1 \left(z \vee p'\right)\right) \\ &\subseteq \left(x \vee p'\right) \sqcup_1 \left(\left(y \vee p'\right) \vee \left(z \vee p'\right)\right) \\ &= \left(x \vee p'\right) \sqcup_1 \left(y \vee z \vee p'\right) \\ &= x \sqcup_p (y \vee z) \,. \end{aligned}$$

■

Corollary 3.14 *For all $x, y, z, p \in X$ we have: $y \leq z \Rightarrow (x \sqcup_p y) \vee (x \sqcup_p z) \subseteq x \sqcup_p z$.*

3.2 A Family of Crisp Join Spaces

In this section we put together the results of Section 3.1 to show that for every $p \in X$ the hyperalgebra $(X, \sqcup_p)$ is a join space in the sense of Prenowitz and Jantosciak [20, 21].

Proposition 3.15 *For all $p \in X$, $(X, \sqcup_p)$ is a* commutative hypergroup *with 0 being its neutral element. In other words, the following hold for all $x, y, z \in X$.*

1. $x \sqcup_p X = X$.

2. $x \sqcup_p y = y \sqcup_p x$.

3. $(x \sqcup_p y) \sqcup_p z = x \sqcup_p (y \sqcup_p z)\,.$

4. $x \in x \sqcup_p 0$, $0 \in x \sqcup_p x$.

Proof. 1, 2 and 3 have been proved in Proposition 3.9. To verify 4, simply note that $x \vee 0 \vee p' = x \vee x \vee p' = 0 \vee x \vee p'$ and that $x \vee x \vee p' = x \vee 0 \vee p' = x \vee 0 \vee p'$. ■

Remark. If we call the element y an *opposite* of x when $0 \in x \sqcup_p y$, then $0 \in x \sqcup_p x$ indicates that x is "auto-opposite". However, note that x has many opposite elements, i.e. in general there exist y's such that $0 \in x \sqcup_p y$ and $x \neq y$. In particular, we have $0 \in x \sqcup_p y \Leftrightarrow y \vee p' = x \vee p'$.

Proposition 3.16 *For all $x, y, z, p \in X$ we have:*

$$z \in x \sqcup_p y \Leftrightarrow x \in y \sqcup_p z \Leftrightarrow y \in z \sqcup_p x.$$

Proof. We only show that $z \in x \sqcup_p y \Leftrightarrow x \in y \sqcup_p z$. Indeed: $z \in x \sqcup_p y \Leftrightarrow x \vee y \vee p' = x \vee z \vee p' = y \vee z \vee p' \Leftrightarrow x \in y \sqcup_p z$. ■

If we consider $\sqcup_p$ to be a *join* hyperoperation, then we can define the corresponding *extension* hyperoperation $/_p$ in the standard manner of the theory of *join spaces*.

Definition 3.17 *For every $p \in X$ we define the hyperoperation $/_p : X \times X \to \mathbf{P}(X)$ as follows:*

$$x/_p y \doteq \{z : x \in y \sqcup_p z\}\,.$$

Proposition 3.18 *For all $x, y, p \in X$ we have $x/_p y = x \sqcup_p y$.*

Proof. Obvious by Proposition 3.16. ■

The next proposition establishes that the pair $\sqcup_p$, $/_p$ is a *join-extension* pair in the sense of the theory of *join spaces*. (3) is the so-called *transposition* property.

Proposition 3.19 *For all $x, y, z, u, p \in X$, we have*

$$(x/_p y) \sim (z/_p u) \Rightarrow (x \sqcup_p u) \sim (z \sqcup_p y)\,. \tag{3}$$

Proof. In light of Proposition 3.18, (3) is equivalent to

$$(x \sqcup_p y) \sim (z \sqcup_p u) \Rightarrow (x \sqcup_p u) \sim (z \sqcup_p y)\,. \tag{4}$$

Choose any $x, y, z, u, p \in X$. If $(x \sqcup_p y) \sim (z \sqcup_p u)$ then there exists some w such that $w \in x \sqcup_p y$ and $w \in z \sqcup_p u$. Then

$$\begin{aligned} w \in x \sqcup_p y &\Rightarrow y \in x \sqcup_p w \subseteq x \sqcup_p (z \sqcup_p u) = z \sqcup_p (x \sqcup_p u) \\ &\Rightarrow (\exists v \in x \sqcup_p u : y \in z \sqcup_p v) \\ &\Rightarrow (\exists v \in x \sqcup_p u : v \in z \sqcup_p y) \\ &\Rightarrow v \in (x \sqcup_p u) \cap (z \sqcup_p y) \\ &\Rightarrow (x \sqcup_p u) \cap (z \sqcup_p y) \neq \emptyset. \end{aligned}$$

Hence $(x \sqcup_p u) \sim (z \sqcup_p y)\,.$ ■

In light of Propositions 3.15 and 3.19, the next corollary follows immediately.

Corollary 3.20 *For every $p \in X$, $(X, \sqcup_p)$ is a* join space*.*

Here are some additional propositions regarding the family of subhypergroups of $(X, \sqcup_p)$.

Proposition 3.21 *For every $x, y, p \in X$ we have:*

1. $z \in x \sqcup_p x \Rightarrow z \in (x \vee y) \sqcup_p (x \vee y)$.
2. $(x \sqcup_p x) \cap (y \sqcup_p y) \subseteq (x \vee y) \sqcup_p (x \vee y)$.

Proof. For 1 note that $z \in x \sqcup_p x \Rightarrow z \leq x \leq x \vee y \vee p' \Rightarrow z \in (x \vee y) \sqcup_p (x \vee y)$. For 2, take any $z \in (x \sqcup_p x) \cap (y \sqcup_p y)$. Then

$$\left.\begin{array}{l} z \leq x \vee p' \\ z \leq y \vee p' \end{array}\right\} \Rightarrow z \leq x \vee y \vee p' \Rightarrow z \in (x \vee y) \sqcup_p (x \vee y)\,.$$

■

Proposition 3.22 *For every $x, y, p \in X$:*

1. *$(x \sqcup_p x) \cap (y \sqcup_p y)$ is a join subspace of $(X, \sqcup_p)$;*

2. $x \sqcup_p x$ *is a join subspace of* $(X, \sqcup_p)$.

Proof. We prove 1 (obviously 2 is a special case). Clearly $(x \sqcup_p y) \sim (x \sqcup_p y)$ hence (from Proposition 3.19) $(x \sqcup_p x) \sim (y \sqcup_p y)$ as well. Take any z_1, $z_2 \in (x \sqcup_p x) \cap (y \sqcup_p y)$. Then

$$z_1 \leq x \vee p', z_1 \leq y \vee p', z_2 \leq x \vee p', z_2 \leq y \vee p'.$$

Now take any $w \in z_1 \sqcup_p z_2$. We will have

$$\begin{aligned} z_1 \vee z_2 \vee p' &= w \vee z_1 \vee p' = w \vee z_2 \vee p' \\ &\Rightarrow \left\{ \begin{array}{c} w \leq z_1 \vee z_2 \vee p' \leq x \vee p' \Rightarrow w \in x \sqcup_p x \\ w \leq z_1 \vee z_2 \vee p' \leq y \vee p' \Rightarrow w \in y \sqcup_p y \end{array} \right\} \\ &\Rightarrow w \in (x \sqcup_p x) \cap (y \sqcup_p y) . \end{aligned}$$

Hence $(x \sqcup_p x) \cap (y \sqcup_p y)$ is closed with respect to $\sqcup_p$.

Furthermore, for any $z \in (x \sqcup_p x) \cap (y \sqcup_p y)$ we will show

$$((x \sqcup_p x) \cap (y \sqcup_p y)) \sqcup_p z = (x \sqcup_p x) \cap (y \sqcup_p y) . \tag{5}$$

Indeed, take any $w \in (x \sqcup_p x) \cap (y \sqcup_p y)$, then from closedness with respect to $\sqcup_p$ we obtain

$$\begin{gathered} w \sqcup_p z \subseteq (x \sqcup_p x) \cap (y \sqcup_p y) \Rightarrow \\ ((x \sqcup_p x) \cap (y \sqcup_p y)) \sqcup_p z = \cup_{w \in (x \sqcup_p x) \cap (y \sqcup_p y)} w \sqcup_p z \subseteq (x \sqcup_p x) \cap (y \sqcup_p y) . \end{gathered} \tag{6}$$

On the other hand, take any $u \in (x \sqcup_p x) \cap (y \sqcup_p y)$; we have

$$\left. \begin{array}{c} z \in (x \sqcup_p x) \cap (y \sqcup_p y) \\ u \in (x \sqcup_p x) \cap (y \sqcup_p y) \end{array} \right\} \Rightarrow z \vee u \in (x \sqcup_p x) \cap (y \sqcup_p y)$$

and $z \in (z \vee u) \sqcup_p z \subseteq ((x \sqcup_p x) \cap (y \sqcup_p y)) \sqcup_p z$. Hence

$$(x \sqcup_p x) \cap (y \sqcup_p y) \subseteq ((x \sqcup_p x) \cap (y \sqcup_p y)) \sqcup_p z. \tag{7}$$

From (6) and (7) we obtain (5) and we conclude that $(x \sqcup_p x) \cap (y \sqcup_p y)$ is a subhypergroup of $(X, \sqcup_p)$. It is also a join subspace, since $/_p = \sqcup_p$ possesses the transposition property and $(x \sqcup_p x) \cap (y \sqcup_p y)$ is closed with respect to $/_p$. ■

Proposition 3.23 *For every* $x, y, p \in X$ *we have:* $(x \sqcup_p x) \vee (y \sqcup_p y) \subseteq (x \vee y) \sqcup_p (x \vee y)$.

Proof. Take any $z \in (x \sqcup_p x) \vee (y \sqcup_p y)$, then there exists some $w \in x \sqcup_p x$ and some $u \in y \sqcup_p y$ such that $z = w \vee u$; since $w \leq x \vee p'$ and $u \leq y \vee p'$ it follows that $z = w \vee u \leq x \vee y \vee p'$ and so $z \in (x \vee y) \sqcup_p (x \vee y)$. ■

3.3 A Family of Crisp "Almost-HyperLattices"

Let us also make some brief remarks about the hyperalgebra $(X, \sqcup_p, \wedge, \leq_p)$. The following proposition summarizes its properties.

Proposition 3.24 *For all* $x, y, z, p \in X$ *the following hold.*

1. $x = x \wedge x$, $x \in x \sqcup_p x$.

2. $x \wedge y = y \wedge x$, $x \sqcup_p y = y \sqcup_p x$.

3. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \sqcup_p y) \sqcup_p z = x \sqcup_p (y \sqcup_p z)$.

4. $x \in (x \sqcup_p y) \wedge x$, $x \in (x \wedge y) \sqcup_p x$,

5. $y \leq x \Leftrightarrow x = x \wedge y$, $y \leq_p x \Leftrightarrow x \in x \sqcup_p y$.

Proof. The first parts of 1-3 and 5 are classical; 4 as well as the second parts of 1-3 and 5 have been proved in Proposition 3.9. ∎

Mittas and Konstantinidou defined *hyperlattices* in [13] as hyperalgebras which satisfy five axioms[3]. Four of these axioms are identical to properties 1-4 of Proposition 3.24. The fifth axiom in [13] is similar to property 5 of Proposition 3.24, but involves an *order* relation. Hence $(X, \sqcup_p, \wedge, \leq_p)$ (which involves the *preorder* $\leq_p$) is "almost" a hyperlattice. Similarly to hyperlattices, $(X, \sqcup_p, \wedge, \leq_p)$ possesses many properties which are analogous to classical lattice properties. A complete study of $(X, \sqcup_p, \wedge, \leq_p)$ will be presented elsewhere; here we only give one property which is analogous to a classical property of distributive lattices.

Proposition 3.25 *For all $x, y, z, p \in X$ we have*

$$\left.\begin{array}{c} x \sqcup_p z = y \sqcup_p z \\ x \wedge z = y \wedge z \end{array}\right\} \Rightarrow x \vee p' = y \vee p'.$$

Proof. We have seen that there exist a, b such that $x \sqcup_p z = [a, x \vee z \vee p']$ and $y \sqcup_p z = [b, y \vee z \vee p']$. Hence $x \vee z \vee p' = y \vee z \vee p'$. Also, since $x \wedge z = y \wedge z$ we have $(x \wedge z) \vee p' = (y \wedge z) \vee p'$. Then we get

$$\begin{array}{c} \left(x \vee p'\right) \vee \left(z \vee p'\right) = \left(y \vee p'\right) \vee \left(z \vee p'\right) \\ \left(x \vee p'\right) \wedge \left(z \vee p'\right) = \left(y \vee p'\right) \wedge \left(z \vee p'\right) \end{array}$$

and so $x \vee p' = y \vee p'$ by distributivity. ∎

4 The L-Fuzzy Nakano Hyperoperation

4.1 Definition and Properties

We now proceed to synthesize the L-fuzzy hyperoperation $\sqcup$ using the crisp hyperoperations $\sqcup_p$ as its p-cuts. We will use a form of the classical construction presented in Section 2.

Definition 4.1 *For all $x, y \in X$ we define the L-fuzzy set $x \sqcup y$ by defining for every $z \in X$:*

$$(x \sqcup y)(z) \doteq \vee\{q : z \in x \sqcup_q y\}. \tag{8}$$

Proposition 4.2 *For all $x, y, p \in X$ we have: $(x \sqcup y)_p = x \sqcup_p y$.*

Proof. This follows from Definition 4.1 and Proposition 2.8. ∎

Proposition 4.3 *For all $x, p \in X$, for all $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ we have:*

1. $x \sqcup_p A_p \subseteq \left(x \sqcup \widetilde{A}\right)_p$;

[3]This axiomatic definition was intended to generalize the classical lattice concept. Further work on hyperlattices appears in [22, 23] and several other places.

2. $A_p \sqcup_p B_p \subseteq \left(\widetilde{A} \sqcup \widetilde{B}\right)_p$.

Proof. For 1, suppose $y \in x \sqcup_p A_p$, then there exists some $z \in A_p$ such that $y \in x \sqcup_p z = (x \sqcup z)_p$. Hence $\widetilde{A}(z) \geq p$ and $(x \sqcup z)(y) \geq p$. Now

$$\left(x \sqcup \widetilde{A}\right)(y) = \vee_{a \in X} \left(\widetilde{A}(a) \wedge (a \sqcup x)(y)\right) \geq \widetilde{A}(z) \wedge (z \sqcup x)(y) \geq p \Rightarrow y \in \left(x \sqcup \widetilde{A}\right)_p$$

and we have proved 1.

For 2, suppose $y \in A_p \sqcup_p B_p$, then there exist $x \in A_p$ and $z \in B_p$ such that $y \in x \sqcup_p z = (x \sqcup z)_p$. Hence $\widetilde{A}(x) \geq p$, $\widetilde{B}(z) \geq p$ and $(x \sqcup z)(y) \geq p$. Now

$$\left(\widetilde{A} \sqcup \widetilde{B}\right)(y) = \vee_{a,b \in X} \left(\widetilde{A}(a) \wedge \widetilde{B}(b) \wedge (a \sqcup b)(y)\right) \geq \widetilde{A}(x) \wedge \widetilde{B}(z) \wedge (x \sqcup z)(y) \geq p \Rightarrow y \in \left(\widetilde{A} \sqcup \widetilde{B}\right)_p$$

and we have proved 2. ■

Proposition 4.4 *For all $x, y, z, p \in X$ we have:*

$$(x \sqcup y)(z) \geq p \Leftrightarrow (x \sqcup y)(z \vee p') \geq p \Leftrightarrow ((x \vee p') \sqcup (y \vee p'))(z) \geq p.$$

Proof. $(x \sqcup y)(z) \geq p \Leftrightarrow z \in x \sqcup_p y \Leftrightarrow z \vee p' \in x \sqcup_p y \Leftrightarrow (x \sqcup y)(z \vee p') \geq p$. Also, $(x \sqcup y)(z) \geq p \Leftrightarrow z \in x \sqcup_p y \Leftrightarrow z \in (x \vee p') \sqcup_p (y \vee p') \Leftrightarrow z \in ((x \vee p') \sqcup (y \vee p'))_p \Leftrightarrow ((x \vee p') \sqcup (y \vee p'))(z) \geq p$. ■

Proposition 4.5 *For all $x, y, z, u \in X$ we have:*

1. $u \in x \sqcup_1 y \Rightarrow (x \sqcup y)(u) = 1$.
2. $u \in (x \sqcup_1 y) \wedge z \Rightarrow ((x \sqcup y) \wedge z)(u) = 1$

Proof. 1 is immediate. For 2, take any $u \in (x \sqcup_1 y) \wedge z$. Then there exists some w such that $w \in x \sqcup_1 y$ and $u = w \wedge z$. Then

$$((x \sqcup y) \wedge z)(u) = \vee_{v \in X} ((x \sqcup y)(v) \wedge (v \wedge z)(u)) = \vee_{v: u = v \wedge z} (x \sqcup y)(v) \geq (x \sqcup y)(w) = 1.$$

■

Proposition 4.6 *For all $x, y, p \in X$ the following hold.*

1. $(1 \sqcup x)(1) = 1$; $(0 \sqcup x)(x) = 1$; $(x \sqcup x)(x) = 1$.
2. $(x \sqcup y)(x \vee y) = 1$; $((x \wedge y) \sqcup x)(x) = 1$; $((x \sqcup y) \wedge x)(x) = 1$.

Proof. Regarding 1, by the first part of Proposition 4.5 we have: $(1 \sqcup x)(1) = 1$ (because $1 \in 1 \sqcup_1 x$); $(0 \sqcup x)(x) = 1$ (because $x \in 0 \sqcup_1 x$); $(x \sqcup x)(x) = 1$ (because $x \in x \sqcup_1 x$).

Regarding 2, by the first part of Proposition 4.5 we have: $(x \sqcup y)(x \vee y) = 1$ (because $x \vee y \in x \sqcup_1 y$); $((x \wedge y) \sqcup x)(x) = 1$ (because $x \in (x \wedge y) \sqcup_1 x$). Also, by the second part of Proposition 4.5 $((x \sqcup y) \wedge x)(x) = 1$, because $x \in (x \sqcup_1 y) \wedge x$. ■

Proposition 4.7 *For all $x, y, z, p \in X$ the following hold.*

1. $(x \sqcup x)(x) = 1$.

2. $x \sqcup y = y \sqcup x$

3. $x \sqcup \widetilde{X} = \widetilde{X}$.

4. $x \sqcup_p y \sqcup_p z \subseteq (x \sqcup (y \sqcup z))_p \cap ((x \sqcup y) \sqcup z)_p$.

5. $y \leq_p x \Leftrightarrow (x \sqcup y)(x) \geq p$.

Proof. 1 has been proved as part of Proposition 4.6. 2 is immediate. For 3, take any $u \in X$, then

$$\left(x \sqcup \widetilde{X}\right)(u) = \vee_{y \in X} (x \sqcup y)(u) \geq (x \sqcup (x \vee u))(u) = 1$$

since $u \in x \sqcup_1 (x \vee u)$. Hence $\left(x \sqcup \widetilde{X}\right)(u) = 1 = \widetilde{X}(u)$. For 4 take any $u \in x \sqcup_p y \sqcup_p z$. Then there exists some w such that $w \in y \sqcup_p z$ and $u \in x \sqcup_p w$. Now

$$u \in x \sqcup_p w \subseteq x \sqcup_p (y \sqcup_p z) = x \sqcup_p (y \sqcup z)_p \subseteq (x \sqcup (y \sqcup z))_p$$

(from part 1 of Proposition 4.3). Hence $x \sqcup_p y \sqcup_p z \subseteq (x \sqcup (y \sqcup z))_p$. In similar manner we show $x \sqcup_p y \sqcup_p z \subseteq ((x \sqcup y) \sqcup z)_p$ and we have proved 4. For 5: $y \leq_p x \Leftrightarrow x \in x \sqcup_p y = (x \sqcup y)_p \Leftrightarrow (x \sqcup y)(x) \geq p$. ■

Remark. Property 4 of the above proposition is similar to the weak associativity of H_v structures [12, 25, 26]. Indeed, for every $p \in X$ we have $(x \sqcup (y \sqcup z))_p \sim ((x \sqcup y) \sqcup z)_p$ and so $x \sqcup (y \sqcup z) \sim_p (x \sqcup y) \sqcup z$.

We continue with some distributivity properties.

Proposition 4.8 *For all $x, y, z \in X$ the following properties hold.*

1. $x \wedge (y \sqcup z) \subseteq (x \wedge y) \sqcup (x \wedge z)$.

2. $x \vee (y \sqcup z) \subseteq (x \vee y) \sqcup (x \vee z)$.

3. $(x \sqcup y) \cap (x \sqcup z) \subseteq x \sqcup (y \vee z)$.

Proof. To prove 1, take any $w \in X$ and define $s = (x \wedge (y \sqcup z))(w)$, $t = ((x \wedge y) \sqcup (x \wedge z))(w)$. Then

$$s = \vee_{b \in X} ((y \sqcup z)(b) \wedge (x \wedge b)(w)) = \vee_{b \in B} (y \sqcup z)(b)$$

where $B = \{b : w = x \wedge b\}$. Now, for every $b \in B$ let $s_b = (y \sqcup z)(b)$. Then $s = \vee_{b \in B} s_b$. Hence

$$\begin{aligned}
&\forall b \in B : b \in y \sqcup_{s_b} z \Rightarrow \\
&\forall b \in B : b \wedge x = w \in (y \sqcup_{s_b} z) \wedge x \subseteq (y \wedge x) \sqcup_{s_b} (z \wedge x) \Rightarrow \\
&\forall b \in B : s_b \leq ((y \wedge x) \sqcup (z \wedge x))(w) = t \Rightarrow \vee_{b \in B} s_b \leq t \Rightarrow \\
&(x \wedge (y \sqcup z))(w) \leq ((x \wedge y) \sqcup (x \wedge z))(w)
\end{aligned}$$

which completes the proof of 1.

Similarly, for 2,take any $w \in X$ and define $s = (x \vee (y \sqcup z))(w)$, $t = ((x \vee y) \sqcup (x \vee z))(w)$. Then

$$s = \vee_{b \in X} ((y \sqcup z)(b) \wedge (x \vee b)(w)) = \vee_{b \in B} (y \sqcup z)(b)$$

where $B = \{b : w = x \vee b\}$. Now, for every $b \in B$ let $s_b = (y \sqcup z)(b)$. Then $s = \vee_{b \in B} s_b$. Hence

$$\begin{aligned}
&\forall b \in B : b \in y \sqcup_{s_b} z \Rightarrow \\
&\forall b \in B : b \vee x = w \in (y \sqcup_{s_b} z) \vee x \subseteq (y \vee x) \sqcup_{s_b} (z \vee x) \Rightarrow \\
&\forall b \in B : s_b \leq ((y \vee x) \sqcup (z \vee x))(w) = t \Rightarrow \vee_{b \in B} s_b \leq t \Rightarrow \\
&(x \vee (y \sqcup z))(w) \leq ((x \vee y) \sqcup (x \vee z))(w)
\end{aligned}$$

which completes the proof of 2.

Finally, for 3 to be true we must have for every $w \in X$

$$(x \sqcup y)(w) \wedge (x \sqcup z)(w) \leq (x \sqcup (y \vee z))(w).$$

Define $r = (x \sqcup y)(w)$, $s = (x \sqcup z)(w)$, $t = (x \sqcup (y \vee z))(w)$. Then

$$\begin{aligned}
w \in x \sqcup_r y \subseteq x \sqcup_{r \wedge s} y \Rightarrow y \in x \sqcup_{r \wedge s} w \\
w \in x \sqcup_r z \subseteq x \sqcup_{r \wedge s} z \Rightarrow z \in x \sqcup_{r \wedge s} w
\end{aligned}$$

hence $y \vee z \in x \sqcup_{r \wedge s} w$ which implies $w \in x \sqcup_{r \wedge s} (y \vee z)$. But then $t = (x \sqcup (y \vee z))(w) \geq r \wedge s$ and 3 has been proved. ■

4.2 An L-Fuzzy Join Space

Proposition 4.9 $(X, \sqcup)$ *is a* L-fuzzy commutative hypergroup, *with neutral element, i.e. for all $x, y, z \in X$ the following hold.*

1. $x \sqcup \widetilde{X} = \widetilde{X}$.
2. $x \sqcup y = y \sqcup x$.
3. $x \sqcup_p y \sqcup_p z \subseteq (x \sqcup (y \sqcup z))_p \cap ((x \sqcup y) \sqcup z)_p$.
4. $(x \sqcup 0)(x) = 1$.
5. $(x \sqcup x)(x) = 1$.

Proof. 1, 2 and 3 have been proved in Proposition 4.7. 4 and 5 have been proved in Proposition 4.6. ■

Proposition 4.10 *For all $x, y, z \in X$ we have:.*

$$(x \sqcup y)(z) \geq p \Leftrightarrow (y \sqcup z)(x) \geq p \Leftrightarrow (z \sqcup x)(y) \geq p.$$

Proof. $(x \sqcup y)(z) \geq p \Leftrightarrow z \in x \sqcup_p y \Leftrightarrow x \in y \sqcup_p z \Leftrightarrow (y \sqcup z)(x) \geq p$. The second equivalence is proved similarly. ■

We can define the L-fuzzy hyperoperation $/$ to be the one which has p-cuts the hyperoperations $/_p$. If we consider $\sqcup$ to be a *join* hyperoperation, then $/$ is the corresponding *extension* hyperoperation as usually defined in the theory of *join spaces*.

Definition 4.11 *The L-fuzzy hyperoperation $/$ is defined in terms of the $/_p$, hyperoperations, as follows: for all $x, y, z \in X$: we set*

$$(x/y)(z) = \vee \{p : z \in x/_p y\}.$$

Proposition 4.12 *For all $x, y, p \in X$ we have $(x/y)_p = x/_p y = x \sqcup_p y$.*

Proof. That $(x/y)_p = x/_p y$ follows from Definition 4.11; that $x/_p y = x \sqcup_p y$ has been proved in Proposition 3.18. ■

Corollary 4.13 *For all $x, y \in X$ we have $x/y = x \sqcup y$.*

Proposition 4.14 *For all $x, y, z, u, p \in X$, the following holds.*

$$(x/y) \sim_p (z/u) \Rightarrow (x \sqcup u) \sim_p (y \sqcup z) . \tag{9}$$

Proof. Similarly to the crisp case, (9) is equivalent to

$$(x \sqcup y) \sim_p (z \sqcup u) \Rightarrow (x \sqcup u) \sim_p (y \sqcup z) .$$

Now, if $(x \sqcup y) \sim_p (z \sqcup u)$ then $(x \sqcup y)_p \sim (z \sqcup u)_p$. Hence there exists some w such that $(x \sqcup y)(w) \geq p$ and $(z \sqcup u)(w) \geq p$, which means $w \in (x \sqcup_p y) \cap (z \sqcup_p u)$. Then

$$(x \sqcup_p y) \sim (z \sqcup_p u) \Rightarrow (x \sqcup_p u) \sim (y \sqcup_p z) \Rightarrow (x \sqcup u)_p \sim (y \sqcup z)_p .$$

and the proof is complete. ■

Let us now give the definition of an L-fuzzy p-join space. This definition is based on the previously mentioned definition of hypergroup [7] with the addition of a form of the extension property.

Definition 4.15 *Given a L-fuzzy hyperoperation $\circ : X \times X \to \mathbf{F}(X)$, the hyperstructure $(X, \circ)$ is called* L-fuzzy p-join space *if it is a commutative L-fuzzy hypergroup and also satisfies for all $x, y, z, u, p \in X$ the property:*

$$x \wr y \sim_p z \wr u \Rightarrow x \circ u \sim_p y \circ z,$$

where $x \wr y \doteq \{z : x \in z \circ y\}$.

In light of Propositions 4.9 and 4.14, the next proposition follows immediately.

Proposition 4.16 $(X, \sqcup)$ *is an* L-fuzzy p-join space.

Here are some additional propositions regarding the family of subhypergroups of $(X, \sqcup)$.

Proposition 4.17 *For every $x, y \in X$ we have:*

1. $x \sqcup x \subseteq (x \vee y) \sqcup (x \vee y)$.
2. $(x \sqcup x) \cap (y \sqcup y) \subseteq (x \vee y) \sqcup (x \vee y)$.

Proof. Take any $p \in X$ and any $z \in x \sqcup_p x$, then $z \vee x \vee p' = x \vee p' \Rightarrow \; z \vee x \vee y \vee p' = x \vee y \vee p'$. Hence $z \in (x \vee y) \sqcup_p (x \vee y)$ and so

$$(x \sqcup x)_p = x \sqcup_p x \subseteq (x \vee y) \sqcup_p (x \vee y) = ((x \vee y) \sqcup (x \vee y))_p .$$

Since the above holds for every $p \in X$, we have proved 1. In similar manner we show $(y \sqcup y)_p \subseteq ((x \vee y) \sqcup (x \vee y))_p$. Hence for every $p \in X$ we have

$$((x \sqcup x) \cap (y \sqcup y))_p = (x \sqcup x)_p \cap (y \sqcup y)_p \subseteq (x \vee y) \sqcup_p (x \vee y) = ((x \vee y) \sqcup (x \vee y))_p ;$$

since the above holds for every $p \in X$, we have proved 2. ■

Proposition 4.18 *For every $x, y, p \in X$ we have:* $(x \sqcup x) \vee (y \sqcup y) \subseteq (x \vee y) \sqcup (x \vee y)$.

Proof. Take any $z \in X$. From Proposition 4.17 we have $x \sqcup x \subseteq (x \vee y) \sqcup (x \vee y)$ and hence

$$(x \sqcup x)(z) \leq ((x \vee y) \sqcup (x \vee y))(z).$$

Similarly

$$(y \sqcup y)(z) \leq ((x \vee y) \sqcup (x \vee y))(z).$$

Hence

$$(x \sqcup x)(z) \vee (y \sqcup y)(z) \leq ((x \vee y) \sqcup (x \vee y))(z) \Rightarrow$$
$$((x \sqcup x) \cup (y \sqcup y))(z) \leq ((x \vee y) \sqcup (x \vee y))(z)$$

and we are done. ■

4.3 An L-Fuzzy "Almost-Hyperlattice"

Similarly to the crisp case, the L-fuzzy hyperalgebra $(X, \sqcup, \wedge)$ can be characterized as an *L-fuzzy almost-hyperlattice*. Its basic properties are given by the following proposition.

Proposition 4.19 *For all $x, y, z \in X$ the following hold.*

1. $(x \sqcup x)(x) = 1$, $x = x \wedge x$.
2. $x \sqcup y = y \sqcup x$, $x \wedge y = y \wedge x$.
3. $x \sqcup_p y \sqcup_p z \subseteq ((x \sqcup y) \sqcup z)_p \cap (x \sqcup (y \sqcup z))_p$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
4. $((x \sqcup y) \wedge x)(x) = 1$, $((x \wedge y) \sqcup x)(x) = 1$,
5. $y \leq_p x \Leftrightarrow (x \sqcup y)(x) \geq p$.

Proof. 1, 2, 3 and 5 have been proved in Proposition 4.7. 4 follows immediately from $x \in (x \sqcup_p y) \wedge x$ and $x \in (x \wedge y) \sqcup_p x$, which have been proved in Proposition 3.9. ■

We see that properties 1-5 of Proposition 4.19 are L-fuzzy analogs of the hyperlattice axioms of Mittas and Konstantinidou [13]. The only difference from these axioms is that property 5 involves a *preorder* rather than an order. The study of $(X, \sqcup, \wedge)$ will be presented in a future work, along with the study of $(X, \sqcup_p, \wedge)$ mentioned in Section 3.3. We only give here the L-fuzzy analog of Proposition 3.25.

Proposition 4.20 *For all $a, b, c \in X$ we have*

$$\left.\begin{array}{c} x \sqcup z = y \sqcup z \\ x \wedge z = y \wedge z \end{array}\right\} \Rightarrow x = y.$$

Proof. From $x \sqcup z = y \sqcup z$ follows that, for every $p \in X$, $x \sqcup_p z = y \sqcup_p z$. Hence also $x \sqcup_1 z = y \sqcup_1 z$, i.e. $x \vee z = y \vee z$. Since also $x \wedge z = y \wedge z$, by distributivity we get $x = y$. ■

5 Discussion

We have presented a family of hyperoperations $\sqcup_p$ which generalize the Nakano hyperoperation $\sqcup_1$ and used these as p-cuts to construct the L-fuzzy Nakano hyperoperation $\sqcup$. We have shown that $(X, \sqcup_p)$ is a join space and $(X, \sqcup_p)$ an L-fuzzy p-join space. Furthermore, we have seen that the hyperalgebra $(X, \sqcup_p, \wedge, \leq_p)$ is "almost" a hyperlattice and $(X, \sqcup, \wedge, \leq_p)$ is "almost" an L-fuzzy hyperlattice. Several topics suggest themselves for further investigation; let us conclude this work by briefly discussing some of them.

First, we have already mentioned that there is further scope for the study of the crisp hyperalgebra $(X, \sqcup_p, \wedge, \leq_p)$ and the L-fuzzy hyperalgebra $(X, \sqcup, \wedge, \leq_p)$.

Second, if we define the dual Nakano hyperoperations $\sqcap_p$ by $x \sqcap_p y = \{z : x \wedge y \wedge p = x \wedge y \wedge p = x \wedge y \wedge p\}$, we can show that $(X, \sqcap_p)$ is a join space; also, if we define $(x \sqcap y)(z) \doteq \vee\{q : z \in x \sqcap_q y\}$, then we can show that $(X, \sqcap)$ is an L-fuzzy join space. In addition to properties which are exactly analogous to the ones obtained here, one could investigate the interrelationship of the $\sqcup_p, \sqcap_p$ (for example their distributivity properties). Perhaps more interesting is the study of the hyperalgebra $(X, \sqcup_p, \sqcap_p)$ which may turn out to be a Nakano *superlattice* similar to the one studied in [11].

Another topic of interest is the study of congruences with respect to $\sqcup_p$ and/or $\sqcap_p$; this would be a continuation of work on the congruences with respect to $\sqcup_1$ in [17] and for congruences with respect to both $\sqcup_1$ and $\sqcap_1$ in [11].

Finally, the study of the hyperalgebras $(X, \sqcup_p)$, $(X, \sqcup_p, \wedge, \leq_p)$ etc. may be particularly fruitful in the case when $(X, \leq, \vee, \wedge, ')$ is a *Boolean* rather than a deMorgan algebra.

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