

L-fuzzy \vee and \wedge Hyperoperations and the Associated L-fuzzy Hyperalgebras

Ath. Kehagias

February 16, 2003

Abstract

In this paper we study two fuzzy hyperoperations, denoted by \vee (which can be seen as a generalization of \vee) and \wedge (which can be seen as a generalization of \wedge). \vee is obtained from a family of crisp \vee_p hyperoperations and \wedge is obtained from a family of crisp \wedge_p hyperoperations. The hyperstructure (X, \vee, \wedge) resembles a *hyperlattice* and the hyperstructure (X, \vee, \wedge) resembles a *dual hyperlattice*.

AMS Classification: 06B99, 06D30, 08A72, 03E72, 20N20.

1 Introduction

In this paper we equip a lattice (X, \vee, \wedge) with two *L-fuzzy hyperoperations* (denoted by \vee and \wedge). Then we study the properties of \vee and \wedge . \vee is a *fuzzified* version of the \vee operation; the structure (X, \vee, \wedge) can be seen as a fuzzy *hyperlattice*. Similarly, \wedge is a fuzzified version of the \wedge operation; the structure (X, \wedge, \vee) can be seen as a fuzzy *dual hyperlattice*. The work presented here lies in the intersection of two fields: (a) algebraic hyperstructures and (b) fuzzy algebras. It will be useful to review these fields briefly and explain some basic terms which will be used throughout the paper.

An *algebraic hyperstructure* (henceforth simply called a *hyperalgebra*) is a set endowed with one or more *hyperoperations*, i.e. *multi-valued* operations mapping a pair of elements to a set of elements. Hyperalgebras are generalizations of classical algebras, e.g. hypergroup is a generalization of group [3]; hyperlattice and superlattice are generalizations of lattice [11, 15, 16] and so on.

The distinction between *fuzzy* and *L-fuzzy* quantities will be explained in Section 2; for the time being we use the two terms as synonyms. Sometimes non-fuzzy quantities will be called *crisp*. The concept of a *fuzzy algebra* is best explained in terms of two examples. First, consider two crisp lattices (X, \vee, \wedge, \leq) and $(L, \sqcup, \sqcap, \sqsubseteq, ')$, where $(L, \sqcup, \sqcap, \sqsubseteq, ')$ is a *complete de Morgan lattice*. A *fuzzy lattice* [1, 18, 19] is a fuzzy set $M : X \rightarrow L$ which has the following property: for every $p \in L$ the structure (M_p, \vee, \wedge) is a sublattice of (X, \vee, \wedge) (where $M_p = \{x : p \sqsubseteq M(x)\}$ is the *p-cut* of M). Similarly, given a crisp group $(X, *)$ and a complete de Morgan lattice $(L, \sqcup, \sqcap, \sqsubseteq, ')$, a *fuzzy group* [14] is a fuzzy set $M : X \rightarrow L$ which has the following property: for every $p \in L$ the structure $(M_p, *)$ is a subgroup of $(X, *)$. The same approach has been taken in connection to other algebraic structures; an extensive study of fuzzy algebras can be found in [12].

Fuzzy hyperalgebras have been introduced rather recently. Zahedi and Hasankhani have studied *fuzzy polygroups* in [6, 21, 22] and *fuzzy hyperrings* in [5]; Corsini and Tofan have studied *fuzzy hypergroups* in [4]; Kehagias has studied *L-fuzzy join spaces* in [8]. All of these works are based on the use of *fuzzy hyperoperations*, i.e. a *fuzzy hyperalgebra* is a set equipped with one or more fuzzy hyperoperations. The concept of fuzzy hyperoperation is a natural extension of crisp hyperoperation:

as already mentioned, a crisp hyperoperation maps a pair of elements to a crisp subset of elements; a fuzzy hyperoperation maps a pair of elements to a *fuzzy* subset.

Hence we discern the following difference between fuzzy algebras and fuzzy hyperalgebras:

1. in a fuzzy algebra, fuzziness concerns the membership of the elements of the carrier set; however these elements are combined by *crisp* operations;
2. in a fuzzy hyperalgebra, the elements are combined by fuzzy (hyper)operations.

As mentioned, in this paper we study two particular fuzzy hyperoperations, Υ and Λ , which can be seen as fuzzifications of the classical \vee and \wedge operations. Hence this paper falls within the general field of fuzzy hyperalgebras. On the other hand, most of the analysis is based on the use of p -cuts, a technique borrowed from the area of fuzzy algebras. In particular, Υ will be constructed from a family of crisp \vee_p hyperoperations which can be considered as the p -cuts of Υ ; similarly Λ will be constructed from a family of crisp \wedge_p hyperoperations.

2 Preliminaries

Throughout this paper we use a *generalized de Morgan lattice* defined as follows.

Definition 2.1 A generalized deMorgan lattice is a structure $(X, \leq, \vee, \wedge, ')$, where (X, \leq, \vee, \wedge) is a complete distributive lattice with minimum element 0 and maximum element 1; the symbol $'$ denotes a unary operation (“negation”); and the following properties are satisfied.

1. For all $x \in X$, $Y \subseteq X$ we have $x \wedge (\vee_{y \in Y} y) = \vee_{y \in Y} (x \wedge y)$, $x \vee (\wedge_{y \in Y} y) = \wedge_{y \in Y} (x \vee y)$. (Complete distributivity).
2. For all $x \in X$ we have: $(x')' = x$. (Negation is involutory).
3. For all $x, y \in X$ we have: $x \leq y \Rightarrow y' \leq x'$. (Negation is order reversing).
4. For all $Y \subseteq X$ we have $(\vee_{y \in Y} y)' = \wedge_{y \in Y} y'$, $(\wedge_{y \in Y} y)' = \vee_{y \in Y} y'$ (Complete deMorgan laws).

In Section 5 we will further assume that $(X, \leq, \vee, \wedge, ')$ is a *generalized Boolean lattice*.

Definition 2.2 A generalized Boolean lattice is a generalized deMorgan lattice $(X, \leq, \vee, \wedge, ')$ in which every $x \in X$ satisfies: $x \vee x' = 1$, $x \wedge x' = 0$.

The following definitions and notation will be used in the sequel.

1. A *fuzzy set* is a function $M : X \rightarrow [0, 1]$, where $[0, 1]$ is an interval of *real* numbers; an *L-fuzzy set* is a function $M : X \rightarrow X$.
2. The collection of all crisp subsets of X is denoted by $\mathbf{P}(X)$ (the *power set* of X); the collection of all *L-fuzzy sets* (i.e. functions $M : X \rightarrow X$) is denoted by $\mathbf{F}(X)$.
3. The collection of all closed *lattice intervals* of X is denoted by $\mathbf{I}(X)$. I.e. $\mathbf{I}(X)$ contains all sets of the form $[a, b] = \{x : a \leq x \leq b\}$; these include $X = [0, 1]$ and the *empty interval* \emptyset , which can be written as $[a, b]$ for any a, b such that $a \not\leq b$.
4. A (*crisp*) *hyperoperation* is a mapping $\circ : X \times X \rightarrow \mathbf{P}(X)$; a *L-fuzzy hyperoperation* is a mapping $\circ : X \times X \rightarrow \mathbf{F}(X)$.

5. Given a L -fuzzy set $M : X \rightarrow X$, the p -cut of M is denoted by M_p and defined by $M_p \doteq \{x : M(x) \geq p\}$. For some basic properties of p -cuts see [13, 8]. Two particularly important facts are (for details see [8]):

- (a) a fuzzy set is uniquely determined by its p -cuts;
- (b) a family of sets $\{\widetilde{M_p}\}_{p \in X}$ which has certain properties (“ p -cut properties”) can be used to define a fuzzy set M in a manner such that for every $p \in X$ we have $\widetilde{M_p} = M_p$.

6. For any set $P \subseteq X$ we will use the notation $\vee P \doteq \vee_{x \in P} x$, $\wedge P \doteq \wedge_{x \in P} x$.

7. \cup, \cap will denote the usual set-theoretic union and intersection. In addition, we will use $\dot{\cup}$ to denote the following set operation: $A \dot{\cup} B \doteq \cap_{C: A \subseteq C, B \subseteq C} C$. Let $[a_1, a_2], [b_1, b_2]$ be two closed intervals of (X, \leq) ; then we have:

$$[a_1, a_2] \cap [b_1, b_2] = [a_1 \vee b_1, a_2 \wedge b_2], \quad [a_1, a_2] \dot{\cup} [b_1, b_2] = [a_1 \wedge b_1, a_2 \vee b_2].$$

Further, $(I(X), \dot{\cup}, \cap, \subseteq, ')$ is a generalized deMorgan lattice (here $'$ denotes set complementation).

8. Since (X, \leq, \vee, \wedge) is a distributive lattice, the following properties hold (for all $a, b, x, y \in X$ such that $x \leq y, a \leq b$):

$$\begin{aligned} a \vee [x, y] &= [a \vee x, a \vee y]; & a \wedge [x, y] &= [a \wedge x, a \wedge y]; \\ [a, b] \vee [x, y] &= [a \vee x, b \vee y]; & [a, b] \wedge [x, y] &= [a \wedge x, b \wedge y]. \end{aligned} \tag{1}$$

3 The \vee and \wedge L-fuzzy Hyperoperations

In order to construct \vee and \wedge we first introduce *crisp* hyperoperations (which will later be used as the p -cuts of \vee and \wedge).

3.1 The Family of \vee_p Hyperoperations

First we introduce the *family* of crisp hyperoperations \vee_p which will later be used as p -cuts of \vee .

Definition 3.1 For all $p \in X$ we define the hyperoperation $\vee_p : X \times X \rightarrow \mathbf{P}(X)$ as follows:

$$\forall a, b \in X : a \vee_p b \doteq [(a \vee b) \wedge p, (a \vee b) \vee p']$$

Using standard hyperoperation notation, we define $a \vee_p [b, c] \doteq \cup_{x \in [b, c]} a \vee_p x$. In the sequel we will use repeatedly the following auxiliary proposition.

Proposition 3.2 For all $a, b, c, p \in X$ we have: $a \vee_p [b, c] = [(a \vee b) \wedge p, (a \vee c) \vee p']$.

Proof. By definition, $a \vee_p [b, c] = \cup_{b \leq z \leq c} a \vee_p z = \cup_{b \leq z \leq c} [(a \vee z) \wedge p, (a \vee z) \vee p']$. Take any $u \in a \vee_p [b, c]$. Then there exists some z such that: $b \leq z \leq c$ and $(a \vee z) \wedge p \leq u \leq (a \vee z) \vee p'$. Hence

$$(a \vee b) \wedge p \leq (a \vee z) \wedge p \leq u \leq (a \vee z) \vee p' \leq (a \vee c) \vee p',$$

i.e. $u \in [(a \vee b) \wedge p, (a \vee c) \vee p']$. So $a \vee_p [b, c] \subseteq [(a \vee b) \wedge p, (a \vee c) \vee p']$. On the other hand, take any $u \in [(a \vee b) \wedge p, (a \vee c) \vee p']$ and define $z = (u \vee b) \wedge c = (u \wedge c) \vee b$ (by distributivity). Clearly $b \leq z \leq c$. Also

$$z \vee a \vee p' = (u \wedge c) \vee b \vee a \vee p' = (u \vee b \vee a \vee p') \wedge (c \vee b \vee a \vee p').$$

Since $u \leq u \vee b \vee a \vee p'$ and $u \leq c \vee a \vee p' = c \vee b \vee a \vee p'$, it follows that $u \leq z \vee a \vee p'$. Also

$$(z \vee a) \wedge p = ((u \wedge c) \vee b \vee a) \wedge p = (u \wedge c \wedge p) \vee ((b \vee a) \wedge p).$$

Since $u \wedge c \wedge p \leq u$ and $(b \vee a) \wedge p \leq u$, it follows that $(z \vee a) \wedge p \leq u$. Hence we have shown

$$(z \vee a) \wedge p \leq u \leq z \vee a \vee p' \Rightarrow u \in a \vee_p z \subseteq a \vee_p [b, c].$$

I.e. $[(a \vee b) \wedge p, (a \vee c) \vee p'] \subseteq a \vee_p [b, c]$. In short we have shown, as required, that $[(a \vee b) \wedge p, (a \vee c) \vee p'] = a \vee_p [b, c]$. ■

Proposition 3.3 *For all $a, b, c, p \in X$ the following hold.*

$$\mathbf{A1} \quad a \in a \vee_p a.$$

$$\mathbf{A2} \quad a \vee_p b = b \vee_p a.$$

$$\mathbf{A3} \quad (a \vee_p b) \vee_p c = a \vee_p (b \vee_p c).$$

$$\mathbf{A4} \quad a \in (a \vee_p b) \wedge a.$$

$$\mathbf{A5.1} \quad b \leq a \Rightarrow a \in a \vee_p b.$$

$$\mathbf{A5.2} \quad b \wedge p \leq a \wedge p \Leftrightarrow a \in a \vee_p b.$$

Proof. **A1** and **A2** are obvious. For **A3** we have:

$$\begin{aligned} (a \vee_p b) \vee_p c &= \cup_{x \in [(a \vee b) \wedge p, (a \vee b) \vee p']} [(x \vee c) \wedge p, (x \vee c) \vee p'] \\ &= [(((a \vee b) \wedge p) \vee c) \wedge p, (((a \vee b) \vee p') \vee c) \vee p'] \\ &= [(a \vee b \vee c) \wedge p, (a \vee b \vee c) \vee p'] \\ &= a \vee_p (b \vee_p c), \end{aligned}$$

where we have used Proposition 3.2. For **A4** we have

$$\begin{aligned} (a \vee_p b) \wedge a &= [(a \vee b) \wedge p, (a \vee b) \vee p'] \wedge a \\ &= [(a \vee b) \wedge a \wedge p, ((a \vee b) \vee p') \wedge a] \\ &= [a \wedge p, a] \ni a, \end{aligned}$$

where we have used (1). For **A5.1** we have $b \leq a \Rightarrow [(a \vee b) \wedge p, (a \vee b) \vee p'] = [a \wedge p, a \vee p'] \ni a$. Note that $a \in a \vee_p b = [(a \vee b) \wedge p, (a \vee b) \vee p'] \Rightarrow (a \vee b) \wedge p \leq a \Rightarrow (a \wedge p) \vee (b \wedge p) \leq a$, which *does not* necessarily imply $b \leq a$. Regarding **A5.2** assume $b \wedge p \leq a \wedge p$. Then we have $b \wedge p \leq a$. Also $a \wedge p \leq a$. Hence

$$(b \wedge p) \vee (a \wedge p) \leq a \Rightarrow (b \vee a) \wedge p \leq a \Rightarrow a \in [(b \vee a) \wedge p, (b \vee a) \vee p'] = a \vee_p b.$$

On the other hand, assume $a \in a \vee_p b$. Then

$$(b \vee a) \wedge p \leq a \leq (b \vee a) \vee p' \Rightarrow (b \vee a) \wedge p \leq a \wedge p \Rightarrow (b \wedge p) \vee (a \wedge p) \leq a \wedge p \Rightarrow b \wedge p \leq a \wedge p.$$

■

Remark. The properties which define a hyperlattice can be found in [11] (and they are briefly listed in this paper, in Table 1, Section 3.5). Comparing these to the properties listed in the above proposition, we see that the hyperstructure (X, \wedge, \vee_p) is very similar to a hyperlattice. The only difference is that in a hyperlattice one would have the property $b \leq a \Leftrightarrow a \in a \vee_p b$, rather than **A5.1**, **A5.2**. However, the similarity can be emphasized by using the following notation.

Definition 3.4 For all $a, b, p \in X$ we write $a \leq_p b$ (and $b \geq_p a$) iff $a \wedge p \leq b \wedge p$.

Remark. Using Definition 3.4 we can rewrite **A5.2** as

$$\mathbf{A5} \quad b \leq_p a \Leftrightarrow a \in a \vee_p b.$$

Now the similarity of (X, \wedge, \vee_p) to a hyperlattice becomes obvious from properties (**A1–A4**, **A5**).

Proposition 3.5 The following properties hold for all $a, b \in X$.

$$\mathbf{B1} \quad a \vee_1 b = \{a \vee b\}, a \vee_0 b = [0, 1].$$

$$\mathbf{B2} \quad \text{For all } p, q \in X: p \leq q \Rightarrow a \vee_q b \subseteq a \vee_p b.$$

$$\mathbf{B3.1} \quad \text{For all } p, q \in X: a \vee_{p \vee q} b = (a \vee_p b) \cap (a \vee_q b).$$

$$\mathbf{B3.2} \quad \text{For all } P \subseteq X: a \vee_{\vee P} b = \bigcap_{p \in P} (a \vee_p b).$$

$$\mathbf{B4} \quad \text{For all } p, q \in X: a \vee_{p \wedge q} b = (a \vee_p b) \dot{\cup} (a \vee_q b) \text{ (where } (a \vee_p b) \dot{\cup} (a \vee_q b) \doteq \bigcap_{t: t \leq p, t \leq q} (a \vee_t b) \text{)}.$$

Proof. **B1** is obvious. For **B2** assume $p \leq q$. Then also $q' \leq p'$ and

$$\left. \begin{array}{l} (a \vee b) \wedge p \leq (a \vee b) \wedge q \\ (a \vee b) \vee q' \leq (a \vee b) \wedge p' \end{array} \right\} \Rightarrow [(a \vee b) \wedge q, (a \vee b) \vee q'] \subseteq [(a \vee b) \wedge p, (a \vee b) \wedge p']$$

and we are done. Next, we will prove **B3.2** (**B3.1** is a special case of **B3.2**). We have

$$\begin{aligned} \bigcap_{p \in P} (a \vee_p b) &= \bigcap_{p \in P} [(a \vee b) \wedge p, (a \vee b) \wedge p'] \\ &= [\bigvee_{p \in P} ((a \vee b) \wedge p), \bigwedge_{p \in P} ((a \vee b) \wedge p')] \\ &= [(a \vee b) \wedge (\bigvee_{p \in P} p), (a \vee b) \wedge (\bigwedge_{p \in P} p')] \\ &= [(a \vee b) \wedge (\bigvee P), (a \vee b) \wedge (\bigvee P)'] = a \vee_{\vee P} b. \end{aligned}$$

Finally, with respect to **B4** we have

$$\begin{aligned} a \vee_{p \wedge q} b &= [(a \vee b) \wedge (p \wedge q), (a \vee b) \vee (p \wedge q)'] \\ &= [((a \vee b) \wedge p) \wedge ((a \vee b) \wedge q), ((a \vee b) \vee p') \vee ((a \vee b) \vee q')] \\ &= [(a \vee b) \wedge p, (a \vee b) \vee p'] \dot{\cup} [(a \vee b) \wedge q, (a \vee b) \vee q'] = (a \vee_p b) \dot{\cup} (a \vee_q b). \end{aligned}$$

■

Remark. The above proposition shows that for every $a, b \in X$ the family $\{a \vee_p b\}_{p \in X}$ has the “ p -cut properties”. In particular note **B2** (smaller values of p result in “less definite” behavior of \vee_p). Furthermore, properties **B1**, **B2**, **B3.1**, **B4** show that for fixed $a, b \in X$, the structure $(\{a \vee_p b\}_{p \in X}, \dot{\cup}, \cap, \subseteq)$ is a lattice with respect to the set inclusion order.

Proposition 3.6 For all $a, b \in X: a \in a \vee_a b$.

$$\mathbf{Proof.} \quad a \vee_a b = [(a \vee b) \wedge a, (a \vee b) \vee a'] = [a, (a \vee b) \vee a'] \ni a. \quad \blacksquare$$

Proposition 3.7 For all $a, b, c, p \in X$ the following properties hold.

$$1. \quad a \vee_p (b \wedge c) = (a \vee_p b) \wedge (a \vee_p c).$$

2. $a \wedge (b \vee_p c) = (a \wedge b) \vee_p (a \wedge c)$.
3. $a \vee (b \vee_p c) \subseteq a \vee_p b \vee_p c$.

Proof. The proof of this proposition is postponed, because it will follow as a special case of Proposition 3.19, proved in Section 3.3. ■

Recall that $\mathbf{I}(X)$, the set of lattice intervals, is a lattice with respect to the set inclusion order \subseteq . We can introduce an alternative order on $\mathbf{I}(X)$, as seen by the following definition.

Definition 3.8 For every $[a_1, a_2], [b_1, b_2] \in \mathbf{I}(X)$, we write $[a_1, a_2] \preceq [b_1, b_2]$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$.

Proposition 3.9 \preceq is an order on $\mathbf{I}(X)$ and $(\mathbf{I}(X), \preceq)$ is a lattice.

Proof. In fact \preceq is the order $\leq \times \leq$ of the product lattice $(X, \leq) \times (X, \leq)$. ■

The \vee_p hyperoperation is isotone in the sense of the following proposition.

Proposition 3.10 For all $a, b, c, p \in X$ we have: $a \leq b \Rightarrow a \vee_p c \preceq b \vee_p c$.

Proof. Indeed

$$a \leq b \Rightarrow \left\{ \begin{array}{l} (a \vee c) \wedge p \leq (b \vee c) \wedge p \\ (a \vee c) \vee p' \leq (b \vee c) \vee p' \end{array} \right\} \Rightarrow [(a \vee c) \wedge p, (a \vee c) \vee p'] \preceq [(b \vee c) \wedge p, (b \vee c) \vee p'].$$

■

3.2 The Family of \wedge_p Hyperoperations

Next we introduce the family of crisp hyperoperations \wedge_p . The following propositions are analogous to the ones of Section 3.1 and hence proofs are omitted.

Definition 3.11 For all $p \in X$ we define the hyperoperation $\vee_p : X \times X \rightarrow \mathbf{P}(X)$ as follows:

$$\forall a, b \in X : a \wedge_p b = [(a \wedge b) \wedge p, (a \wedge b) \vee p']$$

We define $a \wedge_p [b, c] = \cup_{x \in [b, c]} a \wedge_p x$. Similarly to Proposition 3.2 we have:

Proposition 3.12 For all $a, b, c, p \in X$ we have: $a \wedge_p [b, c] = [(a \wedge b) \wedge p, (a \wedge c) \vee p']$.

Proposition 3.13 For all $a, b, c, p \in X$ the following hold.

C1 $a \in a \wedge_p a$.

C2 $a \wedge_p b = b \wedge_p a$.

C3 $(a \wedge_p b) \wedge_p c = a \wedge_p (b \wedge_p c)$.

C4 $a \in (a \wedge_p b) \vee a$.

C5.1 $b \leq a \Rightarrow b \in a \wedge_p b$.

C5.2 $b \vee p' \leq a \vee p' \Leftrightarrow b \in a \wedge_p b$.

Remark. We see from the above proposition that the hyperstructure (X, \vee, \wedge_p) is very similar to a dual hyperlattice [15]. The similarity can be emphasized by defining the following relation.

Definition 3.14 For all $a, b, p \in X$ we write $a \leq^p b$ (and $b \geq^p a$) iff $a \vee p' \leq b \vee p'$.

Remark. If we rewrite **C5.2** as

$$\mathbf{C5} \quad b \leq^p a \Leftrightarrow a \in a \wedge_p b,$$

the similarity of (X, \wedge_p, \vee) to a dual hyperlattice becomes obvious from properties (**C1–C4, C5**).

Proposition 3.15 The following properties hold for all $a, b \in X$.

$$\mathbf{D1} \quad a \wedge_1 b = \{a \wedge b\}, a \wedge_0 b = [0, 1].$$

$$\mathbf{D2} \quad \text{For all } p, q \in X: p \leq q \Rightarrow a \wedge_q b \subseteq a \wedge_p b.$$

$$\mathbf{D3.1} \quad \text{For all } p, q \in X: a \wedge_{p \vee q} b = (a \wedge_p b) \cap (a \wedge_q b).$$

$$\mathbf{D3.2} \quad \text{For all } P \subseteq X: a \wedge_{\bigvee P} b = \bigcap_{p \in P} (a \wedge_p b).$$

$$\mathbf{D4} \quad \text{For all } p, q \in X: a \wedge_{p \wedge q} b = (a \wedge_p b) \dot{\cup} (a \wedge_q b)$$

Remark. The remarks following Proposition 3.5 also apply to Proposition 3.15.

Proposition 3.16 For all $a, b \in X: a \in a \wedge_{a'} b$.

Proof. $a \wedge_{a'} b = [(a \wedge b) \wedge a', (a \wedge b) \vee a] = [(a \wedge b) \wedge a', a] \ni a$. ■

Proposition 3.17 For all $a, b, c, p \in X$ the following properties hold.

$$1. \quad a \vee (b \wedge_p c) = (a \vee b) \wedge_p (a \vee c).$$

$$2. \quad a \wedge_p (b \vee c) = (a \wedge_p b) \vee (a \wedge_p c).$$

$$3. \quad a \wedge (b \wedge_p c) \subseteq a \wedge_p b \wedge_p c.$$

The \wedge_p hyperoperation is isotone in the sense of the following proposition.

Proposition 3.18 For all $a, b, c, p \in X$ we have: $a \leq b \Rightarrow a \wedge_p c \preceq b \wedge_p c$.

3.3 Further Properties of \vee_p and \wedge_p

Let us list some additional properties of the \vee_p and \wedge_p hyperoperations.

Proposition 3.19 For all $a, b, c, p, q \in X$ the following properties hold.

$$1. \quad a \vee_p (b \wedge_q c) \subseteq (a \vee_{p \wedge q} b) \wedge_{p \vee q} (a \vee_{p \wedge q} c) \text{ (when } p \leq q, \text{ the } \subseteq \text{ becomes } =).$$

$$2. \quad a \wedge_p (b \vee_q c) \subseteq (a \wedge_{p \wedge q} b) \vee_{p \vee q} (a \wedge_{p \wedge q} c) \text{ (when } p \leq q, \text{ the } \subseteq \text{ becomes } =).$$

$$3. \quad (a \vee_{p \vee q} b) \wedge_{p \wedge q} (a \vee_{p \vee q} c) = (a \wedge_{p \wedge q} b) \vee_{p \vee q} (a \wedge_{p \wedge q} c).$$

$$4. \quad (a \wedge_{p \wedge q} b) \vee_{p \vee q} (a \wedge_{p \wedge q} c) = (a \wedge_{p \vee q} b) \vee_{p \wedge q} (a \wedge_{p \vee q} c).$$

$$5. \quad a \vee_{p \vee q} b \vee_{p \vee q} c \subseteq \left\{ \begin{array}{l} (a \vee_q b) \vee_p c \\ a \vee_q (b \vee_p c) \end{array} \right\} \subseteq a \vee_{p \wedge q} b \vee_{p \wedge q} c.$$

$$6. \ a \wedge_{p \vee q} b \wedge_{p \vee q} c \subseteq \left\{ \begin{array}{l} (a \wedge_q b) \wedge_p c \\ a \wedge_q (b \wedge_p c) \end{array} \right\} \subseteq a \wedge_{p \wedge q} b \wedge_{p \wedge q} c.$$

Proof.

$$1. \ a \vee_p (b \wedge_q c) = a \vee_p [(b \wedge c) \wedge q, (b \wedge c) \vee q',] = [x, y] \text{ with}$$

$$\begin{aligned} x &= (a \vee ((b \wedge c) \wedge q)) \wedge p \\ y &= a \vee (b \wedge c) \vee q' \vee p' \end{aligned}$$

(note the use of Proposition 3.2). Also, $(a \vee_{p \wedge q} b) \wedge_{p \vee q} (a \vee_{p \wedge q} c) =$

$$[(a \vee b) \wedge p \wedge q, (a \vee b) \vee p' \vee q'] \wedge_{p \vee q} [(a \vee c) \wedge p \wedge q, (a \vee c) \vee p' \vee q'] = [u, w]$$

with

$$u = (a \vee b) \wedge (p \wedge q) \wedge (a \vee c) \wedge (p \vee q) \quad (2)$$

$$\begin{aligned} w &= ((a \vee b \vee p' \vee q') \wedge (a \vee c \vee p' \vee q')) \vee (p \vee q)' \\ &= (a \vee b \vee p' \vee q') \wedge (a \vee c \vee p' \vee q'). \end{aligned} \quad (3)$$

(Since $p \wedge q \leq p \vee q$ it follows that $p' \vee q' = (p \wedge q)' \geq (p \vee q)'$). Now, $x = (a \vee (b \wedge c)) \wedge (a \vee q) \wedge p$ and $u = (a \vee (b \wedge c)) \wedge p \wedge q$. Since $q \leq a \vee q$, it follows that $u \leq x$. If however $p \leq q$, then $p \wedge q = p$ and $(a \vee q) \wedge p = p$, hence $u = x$. Regarding y and w we have

$$\begin{aligned} y &= a \vee (b \wedge c) \vee q' \vee p' \\ &= a \vee (b \wedge c) \vee (p' \vee q') \\ &= ((a \vee b) \wedge (a \vee c)) \vee (p' \vee q') \\ &= ((a \vee b \vee p' \vee q') \wedge (a \vee c \vee p' \vee q')) = w. \end{aligned}$$

In short we have shown: $u \leq x \leq y = w$ and, when $p \leq q$, $u = x \leq y = w$ which yield the required result.

2. Is proved dually to 1.

3. We have

$$\begin{aligned} (a \vee_{p \vee q} b) &= [(a \vee b) \wedge (p \vee q), (a \vee b) \vee (p \vee q)'] \\ (a \vee_{p \vee q} c) &= [(a \vee c) \wedge (p \vee q), (a \vee c) \vee (p \vee q)'] \end{aligned}$$

Hence $(a \vee_{p \vee q} b) \wedge_{p \wedge q} (a \vee_{p \vee q} c) = [f, g]$ with

$$\begin{aligned} f &= (a \vee b) \wedge (a \vee c) \wedge (p \vee q) \wedge (p \wedge q) \\ g &= (((a \vee b) \vee (p \vee q)') \wedge ((a \vee c) \vee (p \vee q)')) \vee (p \wedge q)' \\ &= ((a \vee b) \vee (p \vee q)' \vee (p \wedge q)') \wedge ((a \vee b) \vee (p \vee q)' \vee (p \wedge q)') \\ &= (a \vee b \vee p' \vee q') \wedge (a \vee c \vee p' \vee q'). \end{aligned}$$

Also $(a \vee_{p \wedge q} b) \wedge_{p \vee q} (a \vee_{p \wedge q} c) = [u, w]$ where u, w are given by (2), (3); it can be seen that $u = f$ and $w = g$.

4. Is proved dually to 3.

5. We have $a \vee_{p \vee q} b \vee_{p \vee q} c = [(a \vee b \vee c) \wedge (p \vee q), (a \vee b \vee c) \vee (p \vee q)']$. Also

$$\begin{aligned} a \vee_q (b \vee_p c) &= \cup_{x \in [(b \vee c) \wedge p, b \vee c \vee p']} [(a \vee x) \wedge q, a \vee x \vee q'] \\ &= [(a \vee b \vee c) \wedge (a \vee p) \wedge q, a \vee b \vee c \vee p' \vee q']. \end{aligned}$$

Since $(a \vee p) \wedge q \leq q \leq p \vee q$ and $(p \vee q)' \leq (p \wedge q)' = p' \vee q'$, we have shown $a \vee_{p \vee q} b \vee_{p \vee q} c \subseteq a \vee_q (b \vee_p c)$. Also, $a \vee_{p \wedge q} b \vee_{p \wedge q} c = [(a \vee b \vee c) \wedge (p \wedge q), (a \vee b \vee c) \vee (p \wedge q)']$. Since $p \wedge q \leq (a \vee p) \wedge q$ and $p' \vee q' = (p \wedge q)'$, we have shown $a \vee_q (b \vee_p c) \subseteq a \vee_{p \wedge q} b \vee_{p \wedge q} c$. Similarly we can show $a \vee_{p \vee q} b \vee_{p \vee q} c \subseteq (a \vee_q b) \vee_p c \subseteq a \vee_{p \wedge q} b \vee_{p \wedge q} c$.

6. Is proved dually to 5.

■

It is well known that in de Morgan and Boolean algebras the three operations $\vee, \wedge, '$ are not independent: \wedge can be defined in terms of \vee and $'$; \vee can be defined in terms of \wedge and $'$. A similar relation holds between \vee_p, \wedge_p and $'$.

Definition 3.20 For every $A \in \mathbf{P}(X)$, we define $A' \doteq \{x'\}_{x \in A}$.

Proposition 3.21 For every $p, a, b \in X$ we have: (1) $(a \vee_p b)' = a' \wedge_p b'$, (2) $(a \wedge_p b)' = a' \vee_p b'$.

Proof. For 1 we have

$$\begin{aligned} (a \vee_p b)' &= \{x' : (a \vee b) \wedge p \leq x \leq a \vee b \vee p'\} \\ &= \{x' : ((a \vee b) \wedge p)' \geq x' \geq (a \vee b \vee p')'\} \\ &= \{z : a' \wedge b' \wedge p \leq z \leq (a' \wedge b') \vee p'\} = a' \wedge_p b'. \end{aligned}$$

Similarly we can prove 2. ■

Remark. It is possible to introduce a *fuzzified negation*, but we will not pursue this matter further here.

3.4 The Construction of Υ and \wedge

We now construct the L-fuzzy hyperoperations Υ and \wedge . Following a standard approach, we will construct Υ and \wedge through their p -cuts, which will be the \vee_p and \wedge_p families studied previously.

Definition 3.22 For all $a, b \in X$ we define the L-fuzzy set $a \Upsilon b$ by defining for every $x \in X$:

$$(a \Upsilon b)(x) \doteq \vee \{q : x \in a \vee_q b\}; \quad (4)$$

we define the L-fuzzy set $a \wedge b$ by defining for every $x \in X$:

$$(a \wedge b)(x) \doteq \vee \{q : x \in a \wedge_q b\}; \quad (5)$$

Proposition 3.23 For all $a, b \in X$ and $p \in X$ we have

$$(a \Upsilon b)_p = a \vee_p b, \quad (a \wedge b)_p = a \wedge_p b.$$

Proof. Follows from the construction of $a \Upsilon b, a \wedge b$ as given in Definition 3.22 (for details see [13]).

■

Definition 3.24 We say $M : X \rightarrow X$ is a L -fuzzy interval of (X, \leq) iff

$$\forall p \in X : M_p \text{ is a closed interval of } (X, \leq).$$

Definition 3.25 We denote the collection of L -fuzzy intervals of X by $\tilde{\mathbf{I}}(X)$.

Proposition 3.26 For all $a, b \in X$, the L -fuzzy sets $a \vee b$ and $a \wedge b$ are L -fuzzy intervals.

Proof. As already mentioned (Proposition 3.23), for any $p \in X$ the p -cut of $a \vee b$ is $(a \vee b)_p = a \vee_p b$ and by construction $a \vee_p b$ is an interval. The same is true for $a \wedge b$. ■

Remark. Some properties of L -fuzzy intervals are listed in [9, 8].

Before proceeding, we will need some auxiliary definitions and propositions.

Definition 3.27 Let $\circ : X \times X \rightarrow \mathbf{F}(X)$ be an L -fuzzy hyperoperation.

1. For all $a \in X$, $B \in \mathbf{F}(X)$ we define the L -fuzzy set $a \circ B$ by

$$(a \circ B)(x) \doteq \bigvee_{b: B(b) > 0} (a \circ b)(x)$$

2. For all $A, B \in \mathbf{F}(X)$ we define the L -fuzzy set $A \circ B$ by

$$(A \circ B)(x) \doteq \bigvee_{a: A(a) > 0, b: B(b) > 0} (a \circ b)(x).$$

Remark. The above definition has been used by Corsini and Tofan in [4] and we use it here to preserve compatibility with the above authors. However, it must be remarked that $a \circ B$ according to this definition is somewhat “too big” in the sense that every b such that $B(b) > 0$ contributes equally to the membership of x in $a \circ B$, irrespective of how small $B(b)$ is. A somewhat more restrictive (and perhaps more appropriate) definition would be

$$(a \circ B)(x) \doteq \bigvee_{b \in X} (B(b) \wedge (a \circ b)(x))$$

Definition 3.28 Let $\cdot : X \times X \rightarrow \mathbf{P}(X)$ be a crisp hyperoperation.

1. For all $a \in X$, $B \in \mathbf{F}(X)$ we define the crisp set $a \cdot B$ as follows

$$x \in a \cdot B \text{ iff } \exists b : B(b) > 0 \text{ and } x \in a \cdot b.$$

2. For all $A, B \in \mathbf{F}(X)$ we define the crisp set $A \cdot B$ by

$$x \in A \cdot B \text{ iff } \exists a, b : A(a) > 0 \text{ and } B(b) > 0 \text{ and } x \in a \cdot b.$$

Remark. The following facts can be checked easily.

1. If the set B is crisp, then Definition 3.28 reduces to the classical hyperoperation definition $a \cdot B = \bigcup_{b \in B} a \cdot b$; similarly for $A \cdot B$ when A, B crisp sets.
2. Definitions 3.27 and 3.28 are compatible. More specifically, in case that \circ in Definition 3.27 is a “degenerate” L -fuzzy hyperoperation (i.e. for all $a, b, x \in X$ we have that $(a \circ b)(x)$ is either 1 or 0) then $a \circ B$ is the characteristic function of the set $a \cdot B$ (where for every a, b we have $a \cdot b = \{x : (a \circ b)(x) > 0\}$).

3. Definition 3.28 can be used also in the case where \cdot in is an operation (i.e. for every $a, b \in X$ the set $a \cdot b$ is a singleton). In this case $x \in a \cdot B$ iff $\exists b : B(b) > 0$ and $x = a \cdot b$ (and similarly for $A \cdot B$).

Proposition 3.29 *For all $a, p \in X$, for all $A, B \in \mathbf{F}(X)$ we have*

1. $a \vee_p B_p \subseteq (a \vee B)_p$.
2. $A_p \vee_p B_p \subseteq (A \vee B)_p$.
3. $a \wedge_p B_p \subseteq (a \wedge B)_p$.
4. $A_p \wedge_p B_p \subseteq (A \wedge B)_p$.

Proof. We only prove 1, since the remaining items are proved similarly. Choose any $x \in a \vee_p B_p$. If $p = 0$, then $a \vee_0 B_0 = X = (a \vee B)_0$. If $p > 0$ then exists $b \in B_p$ such that $x \in a \vee_p b$. Now $x \in a \vee_p b = (a \vee b)_p$ implies that $(a \vee b)(x) \geq p$. Also, since $B(b) \geq p > 0$ it follows that $b \in \{u : B(u) > 0\}$. Hence

$$(a \vee B)(x) = \vee\{(a \vee u)(x), u : B(u) > 0\} \geq (a \vee b)(x) \geq p$$

which implies that $x \in (a \vee B)_p$. We have thus shown that $a \vee_p B_p \subseteq (a \vee B)_p$. ■

Remark. By examining the above proof we see that to obtain equality (i.e. $(a \vee B)_p = a \vee_p B_p$), we would need some element u_0 such that $\vee\{(a \vee u)(x), u : B(u) > 0\} = (a \vee u_0)(x)$. Such an element would depend on a, b, x and its existence cannot be guaranteed; hence in general $a \vee_p B_p$ will be a proper subset of $(a \vee B)_p$. However note that in certain cases $a \vee_p B_p = (a \vee B)_p$ does hold, for example when X is finite.

Let us first prove some simple properties of \vee, \wedge .

Proposition 3.30 *For all $a \in X$ the following hold.*

1. $(1 \vee a)(1) = 1, (1 \wedge a)(a) = 1$.
2. $(0 \vee a)(a) = 1, (0 \wedge a)(0) = 1$.
3. $(a \wedge b)(a \wedge b) = 1, (a \vee b)(a \vee b) = 1$.
4. $a \in ((a \wedge b) \vee a), a \in ((a \vee b) \wedge a)$.

Proof. For 1 we have: $(1 \vee a)(1) \doteq \vee\{q : 1 \in 1 \vee_q a\}$. Since $1 \in 1 \vee_1 a = [(1 \vee a) \wedge 1, (1 \vee a) \vee 1']$, it follows that $1 \in \{q : 1 \in 1 \vee_q a\}$ and so $(1 \vee a)(1) = 1$. The remaining part of 1, as well as 2 are proved similarly.

Regarding 3, we note that $(a \wedge b)(a \wedge b) = \vee\{q : a \wedge b \in a \wedge_q b\} = 1$ (since $a \wedge b \in a \wedge_1 b$). $(a \vee b)(a \vee b) = 1$ is proved similarly. For 4, since $a \wedge b \in a \wedge_1 b$, we have $(a \wedge b)(a \wedge b) = 1 > 0$. Also $a = a \vee (a \wedge b)$. Hence, by Definition 3.28 we have $a \in ((a \wedge b) \vee a)$. Similarly we can prove $((a \vee b) \wedge a)(a) = 1$. ■

We are now ready to establish some basic properties of \vee and \wedge .

Proposition 3.31 *For all $a, b, c, p \in X$ the following hold.*

E1 $(a \vee a)(a) = 1, (a \wedge a)(a) = 1$.

$$\mathbf{E2} \quad a \vee b = b \vee a, \quad a \wedge b = b \wedge a.$$

$$\mathbf{E3} \quad a \vee_p b \vee_p c \subseteq (a \vee (b \vee c))_p \cap ((a \vee b) \vee c)_p; \quad a \wedge_p b \wedge_p c \subseteq ((a \wedge b) \wedge c) \cap (a \wedge (b \wedge c)).$$

$$\mathbf{E4} \quad ((a \wedge b) \vee a)(a) = 1, \quad ((a \vee b) \wedge a)(a) = 1.$$

$$\mathbf{E5} \quad b \leq_p a \Leftrightarrow (a \vee b)(a) \geq p; \quad b \leq^p a \Leftrightarrow (a \wedge b)(b) \geq p.$$

Proof.

1. To show $(a \vee a)(a) = 1$ it suffices to note that $a \in [a, a] = a \vee_1 a = (a \vee a)_1$ and so $(a \vee a)(a) \geq 1$. Similarly we can show $(a \wedge a)(a) = 1$ and we have proved **E1**.
2. **E2** is obvious.
3. To prove **E3**, we apply Proposition 3.29.1 using $B = a \vee b$; in this manner we show that $a \vee_p b \vee_p c = a \vee_p (b \vee_p c) = a \vee_p (b \vee c)_p \subseteq (a \vee (b \vee c))_p$. Similarly $a \vee_p b \vee_p c \subseteq ((a \vee b) \vee c)_p$ and we are done.
4. We prove the first part of **E4** (the second is proved similarly) by considering two cases.
 - (a) If $a < 1$, then $a' > 0$. Since $a \in a \wedge_{a'} b$ (Proposition 3.16) it follows that $(a \wedge b)(a) \geq a' > 0$. Hence $a \in \{z : (a \wedge b)(z) > 0\}$. Now $((a \wedge b) \vee a)(a) = \vee_{z: (a \wedge b)(z) > 0} (z \vee a)(a) \geq (a \vee a)(a) = 1$ which implies $((a \wedge b) \vee a)(a) = 1$.
 - (b) For the case $a = 1$, we must show $((1 \wedge b) \vee 1)(1) = 1$. First recall that $(1 \wedge b)(b) = 1$; from this follows that $b \in \{z : (1 \wedge b)(z) > 0\}$. Hence $((1 \wedge b) \vee 1)(1) = \vee_{z: (1 \wedge b)(z) > 0} (z \vee 1)(1) \geq (b \vee 1)(1)$. But, as mentioned, $(b \vee 1)(1) = (1 \vee b)(1) = 1$. Hence $((1 \wedge b) \vee 1)(1) = 1$.
5. Finally, we prove the first part of **E5** (the second is proved similarly) as follows.
 - (i) $b \leq_p a \Rightarrow b \wedge p \leq a \wedge p \Rightarrow a \in a \vee_p b \Rightarrow p \in \{q : a \in a \vee_q b\}$. Hence $(a \vee b)(a) = \vee \{q : a \in a \vee_q b\} \geq p$.
 - (ii) Also $(a \vee b)(a) \geq p \Rightarrow a \in (a \vee b)_p = a \vee_p b$. Hence $(a \vee b) \wedge p \leq a \Rightarrow (a \vee b) \wedge p \leq a \wedge p \Rightarrow (a \wedge p) \vee (b \wedge p) \leq a \wedge p \Rightarrow b \wedge p \leq a \wedge p \Rightarrow$ i.e. $b \leq_p a$.

■

Proposition 3.32 *For all $a, b, c, p \in X$ we have*

1. $a \vee_p (b \wedge_p c) \subseteq (a \vee (b \wedge c))_p \cap ((a \vee b) \wedge (a \vee c))_p$.
2. $a \wedge_p (b \vee_p c) \subseteq (a \wedge (b \vee c))_p \cap ((a \wedge b) \vee (a \wedge c))_p$.

Proof. From Proposition 3.29.1 we have

$$a \vee_p (b \wedge_p c) \subseteq (a \vee (b \wedge c))_p. \quad (6)$$

From Proposition 3.19.1, with $p = q$, we have

$$a \vee_p (b \wedge_p c) = (a \vee_p b) \wedge_p (a \vee_p c) = (a \vee b)_p \wedge_p (a \vee c)_p \subseteq ((a \vee b) \wedge (a \vee c))_p. \quad (7)$$

From (6) and (7) follows the first part of the proposition; the second part can be proved similarly. ■

Remark. Property **E3** shows that the associativity of \vee, \wedge holds in a limited sense. Proposition 3.32 shows a limited form of distributivity. These limitations can be seen as consequences of Proposition 3.29 (they are lifted in the case of finite X). On the other hand, the next proposition shows that an important property of distributive lattices also holds for \vee and \wedge .

Proposition 3.33 For all $a, b, c \in X$ we have

$$\left. \begin{array}{l} a \vee c = b \vee c \\ a \wedge c = b \wedge c \end{array} \right\} \Rightarrow a = b.$$

Proof. $a \vee c = b \vee c \Rightarrow (\forall p \in X : (a \vee c)_p = (b \vee c)_p) \Rightarrow (\forall p \in X : a \vee_p c = b \vee_p c) \Rightarrow a \vee_1 c = b \vee_1 c \Rightarrow a \vee c = b \vee c$; also $a \wedge c = b \wedge c \Rightarrow a \wedge c = b \wedge c$; and $(a \vee c = b \vee c, a \wedge c = b \wedge c) \Rightarrow a = b$. ■

We have introduced an order on crisp intervals with Definition 3.9. We now extend this order to $\tilde{\mathbf{I}}(X)$, the collection of all L-fuzzy intervals of X .

Definition 3.34 For every $A, B \in \tilde{\mathbf{I}}(X)$, we write $A \preceq B$ iff for all $p \in X$ we have $A_p \preceq B_p$.

Proposition 3.35 \preceq is an order on $\tilde{\mathbf{I}}(X)$ and $(\tilde{\mathbf{I}}(X), \preceq)$ is a lattice.

Proof. This follows from the fact that a fuzzy set is uniquely specified by its p -cuts. ■

The \vee, \wedge hyperoperations are isotone in the sense of the following proposition.

Proposition 3.36 For all $a, b \in X$ we have: $a \leq b \Rightarrow \left\{ \begin{array}{l} a \vee c \preceq b \vee c, \\ a \wedge c \preceq b \wedge c. \end{array} \right.$

Proof. Take any $p \in X$. Then

$$a \leq b \Rightarrow a \vee c \leq b \vee c \Rightarrow \left\{ \begin{array}{l} (a \vee c) \wedge p \leq (b \vee c) \wedge p \\ (a \vee c) \vee p' \leq (b \vee c) \vee p' \end{array} \right\} \Rightarrow a \vee_p c \preceq b \vee_p c \Rightarrow (a \vee c)_p \preceq (b \vee c)_p.$$

Since the above is true for every $p \in X$, it follows that $a \vee c \preceq b \vee c$. Similarly we show that $a \wedge c \preceq b \wedge c$. ■

\vee, \wedge and $'$ are interrelated as seen by the following proposition.

Definition 3.37 For every $A \in \mathbf{F}(X)$ define A' by its p -cuts, i.e. A' is the (unique) fuzzy set which for every $p \in X$ satisfies

$$(A')_p = (A_p)' = \{x'\}_{x \in A_p}.$$

Proposition 3.38 For every $a, b \in X$ we have: (i) $(a \vee b)' = a' \wedge b'$, (ii) $(a \wedge b)' = a' \vee b'$.

Proof. Choose any $p \in X$. Then $((a \vee b)')_p = ((a \vee b)_p)' = (a \vee_p b)' = a' \wedge_p b' = (a' \wedge b')_p$. Since for all $p \in X$ the fuzzy sets $(a \vee b)'$ and $a' \wedge b'$ have the same cuts, we have $(a \vee b)' = a' \wedge b'$. ■

3.5 The L-fuzzy Hyperstructures (X, \vee, \wedge) and (X, \wedge, \vee)

Let us discuss briefly the similarities and differences of (X, \vee, \wedge) to a crisp hyperlattice. (X, \leq, \vee, \wedge) denotes the lattice used throughout this paper (but the negation operator will not be used in this section). Take ∇ to be some crisp hyperoperation defined on X . (X, ∇, \wedge) is a crisp hyperlattice if it satisfies the properties (axioms) listed in the first column of Table 1. The second column of Table 1 lists the corresponding properties of (X, \vee, \wedge) as described in Propositions 3.30, 3.31.

(X, ∇, \wedge)	(X, \vee, \wedge)
$a \in a \nabla a, a = a \wedge a$	$(a \vee a)(a) = 1, a = a \wedge a$
$a \nabla b = b \nabla a, a \wedge b = b \wedge a$	$a \vee b = b \vee a, a \wedge b = b \wedge a$
$(a \nabla b) \nabla c = a \nabla (b \nabla c)$	$(a \vee (b \vee c))_p \cap ((a \vee b) \vee c)_p \neq \emptyset$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$a \in (a \nabla b) \wedge a$	$a \in ((a \vee b) \wedge a)$
$a \in (a \wedge b) \nabla a$	$((a \wedge b) \vee a)(a) = 1$
$a \in a \nabla b \Leftrightarrow b = a \wedge b$	$(a \vee b)(a) \geq p \Leftrightarrow b \wedge p = a \wedge b \wedge p$

Table 1

The correspondence between the properties of (X, ∇, \wedge) and (X, γ, \wedge) is rather obvious. Hence (X, γ, \wedge) can justifiably be considered as an L-fuzzy relative of (X, ∇, \wedge) . Note however that: (a) γ has a weak form of associativity (similar to H_v associativity [17, 20]) and (b) the ordering property induced by γ concerns the preorder \leq_p rather than the order \leq .

Similar remarks can be made regarding the similarities and differences of (X, λ, \vee) to a crisp dual hyperlattice (X, Δ, \vee) . Table 2 shows the correspondence of the properties of the two structures.

(X, Δ, \vee)	(X, λ, \vee)
$a \in a \Delta a, a = a \vee a$	$(a \lambda a)(a) = 1, a = a \vee a$
$a \Delta b = b \Delta a, a \vee b = b \vee a$	$a \lambda b = b \lambda a, a \vee b = b \vee a$
$(a \Delta b) \Delta c = a \Delta (b \Delta c)$	$((a \lambda b) \lambda c)_p \cap (a \lambda (b \lambda c))_p \neq \emptyset$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$a \in (a \Delta b) \vee a$	$a \in ((a \lambda b) \vee a)$
$a \in (a \vee b) \Delta a$	$((a \vee b) \lambda a)(a) = 1$
$a \in a \Delta b \Leftrightarrow b = a \vee b$	$(a \lambda b)(a) \geq p \Leftrightarrow a \leq^p b$

Table 2

4 L-fuzzy Similarity Relations Obtained from \leq_p

In Section 3.1 we have introduced the family of preorders \leq_p . As is well known, every preorder generates an equivalence relationship; in this section we will study the equivalences generated by the \leq_p preorders and will use these to construct a *fuzzy equivalence*.

4.1 The Family of Crisp Equivalence Relations $\{\sigma_p\}_{p \in X}$

Definition 4.1 For every $p \in X$ we define the relation $\sigma_p \subseteq X \times X$ as follows

$$(a, b) \in \sigma_p \text{ iff } a \wedge p = b \wedge p.$$

Definition 4.2 We will also use the notation $a =_p b$, i.e.

$$a =_p b \Leftrightarrow a \wedge p = b \wedge p \Leftrightarrow (a, b) \in \sigma_p.$$

Proposition 4.3 For all $p \in X$, σ_p is an equivalence, i.e. we have the following.

1. For all $a \in X$: $(a, a) \in \sigma_p$.
2. For all $a, b \in X$: $(a, b) \in \sigma_p \Leftrightarrow (b, a) \in \sigma_p$.
3. For all $a, b, c \in X$: $((a, b) \in \sigma_p, (b, c) \in \sigma_p) \Rightarrow (a, c) \in \sigma_p$.

Furthermore, the equivalence classes of σ_p are the same as the ones generated by the preorder \leq_p .

Proof. Easy. ■

Proposition 4.4 The following properties hold

1. $\sigma_1 = \{(a, a)\}_{a \in X}$, $\sigma_0 = X \times X$

2. For all $p, q \in X$: $p \leq q \Rightarrow \sigma_q \subseteq \sigma_p$.

3.1 For all $p, q \in X$: $\sigma_{p \vee q} = \sigma_p \cap \sigma_q$.

3.2 For all $P \subseteq X$: $\sigma_{\vee P} = \bigcap_{p \in P} \sigma_p$.

4. For all $p, q \in X$: $\sigma_{p \wedge q} = \sigma_p \dot{\cup} \sigma_q$ (where $\sigma_p \dot{\cup} \sigma_q \doteq \bigcap_{t: t \leq p, t \leq q} \sigma_t$).

Proof. 1 is immediate. For 2, take some $p, q \in X$ such that $p \leq q$ and choose any $(a, b) \in \sigma_q$. Then $a \wedge q = b \wedge q \Rightarrow a \wedge q \wedge p = b \wedge q \wedge p \Rightarrow a \wedge p = b \wedge p \Rightarrow (a, b) \in \sigma_p$, which concludes the proof of 2.

Next we prove 3.2 (3.1 is a special case of 3.1). Choose any $P \subseteq X$. We have:

$$(\forall p \in P : p \leq \vee P) \Rightarrow (\forall p \in P : \sigma_p \supseteq \sigma_{\vee P}) \Rightarrow \bigcap_{p \in P} \sigma_p \supseteq \sigma_{\vee P}.$$

On the other hand, take any $(x, y) \in \bigcap_{p \in P} \sigma_p$. Then

$$\begin{aligned} (x, y) \in \bigcap_{p \in P} \sigma_p &\Rightarrow (\forall p \in P : x \wedge p = y \wedge p) \\ &\Rightarrow \bigvee_{p \in P} (x \wedge p) = \bigvee_{p \in P} (y \wedge p) \Rightarrow x \wedge (\bigvee_{p \in P} p) = y \wedge (\bigvee_{p \in P} p) \\ &\Rightarrow x \wedge (\vee P) = y \wedge (\vee P) \Rightarrow (x, y) \in \sigma_{\vee P} \Rightarrow \bigcap_{p \in P} \sigma_p \subseteq \sigma_{\vee P}. \end{aligned}$$

Hence $\bigcap_{p \in P} \sigma_p = \sigma_{\vee P}$.

Finally we prove 4. First choose any $p, q \in X$. Define $R_{pq} = \{r : r \in X, r \leq p, r \leq q\}$. Then

$$(\forall r \in R_{pq} : r \leq p \wedge q) \Rightarrow (\forall r \in R_{pq} : \sigma_{p \wedge q} \subseteq \sigma_r) \Rightarrow \sigma_{p \wedge q} \subseteq \bigcap_{r \in R_{pq}} \sigma_r.$$

On the other hand, since $p \wedge q \in R_{pq}$, we have $\bigcap_{r \in R_{pq}} \sigma_r \subseteq \sigma_{p \wedge q}$. Hence $\sigma_{p \wedge q} = \bigcap_{r \in R_{pq}} \sigma_r = \sigma_p \dot{\cup} \sigma_q$. ■

Proposition 4.5 Define $f : X \rightarrow \{\sigma_p\}_{p \in X}$ by $f(p) = \sigma_p$. Then f is an anti-homomorphism: $(X, \vee, \wedge, \leq) \xrightarrow{f} (\{\sigma_p\}_{p \in X}, \cap, \dot{\cup}, \subseteq)$. More specifically we have

1. $p \leq q \Rightarrow f(q) \subseteq f(p)$.
2. $f(p \wedge q) = f(p) \dot{\cup} f(q)$.
3. $f(p \vee q) = f(p) \cap f(q)$.

Proof. This follows immediately from Proposition 4.4. ■

4.2 The Equivalence Classes of σ_p

Definition 4.6 For all $p, a \in X$ we define the set \bar{a}^p by: $\bar{a}^p \doteq \{x : (a, x) \in \sigma_p\}$.

Proposition 4.7 For every $a, p \in X$, \bar{a}^p is an interval. More specifically

$$\bar{a}^p = [a \wedge p, \vee \bar{a}^p].$$

Proof. We have

$$(\forall x \in \bar{a}^p : x \wedge p = a \wedge p) \Rightarrow \bigwedge_{x \in \bar{a}^p} (x \wedge p) = a \wedge p \Rightarrow (\bigwedge_{x \in \bar{a}^p} x) \wedge p = a \wedge p \Rightarrow (\bigwedge \bar{a}^p) \wedge p = a \wedge p.$$

Hence $\bigwedge \bar{a}^p \in \bar{a}^p$; similarly we show $\bigvee \bar{a}^p \in \bar{a}^p$. Hence $\bar{a}^p = [\bigwedge \bar{a}^p, \bigvee \bar{a}^p]$. It remains to show that $\bigwedge \bar{a}^p = a \wedge p$. Since $a \wedge p \in \bar{a}^p$ it follows that $a \wedge p \geq \bigwedge \bar{a}^p$. On the other hand, for every $x \in \bar{a}^p$ we have $x \geq x \wedge p = a \wedge p$ and so $\bigwedge_{x \in \bar{a}^p} x \geq a \wedge p$ which implies $\bigwedge \bar{a}^p \geq a \wedge p$. In short, $\bigwedge \bar{a}^p = a \wedge p$. ■

Proposition 4.8 For all $p, a, b, c \in X$ we have:

1. $\bar{a}^p = \bar{b}^p \Rightarrow \overline{a \wedge c}^p = \overline{b \wedge c}^p$.
2. $\bar{a}^p = \bar{b}^p \Rightarrow \overline{a \wedge_p c}^p = \overline{b \wedge_p c}^p$.
3. $\bar{a}^p = \bar{b}^p \Rightarrow \overline{a \vee c}^p = \overline{b \vee c}^p$.
4. $\bar{a}^p = \bar{b}^p \Rightarrow \overline{a \vee_p c}^p = \overline{b \vee_p c}^p$.

Proof. Choose any $p, a, b, c \in X$ and keep them fixed for the rest of the proof. We have $a \in \bar{a}^p = \bar{b}^p \Rightarrow a \wedge p = b \wedge p \Rightarrow a \wedge c \wedge p = b \wedge c \wedge p$.

Now take any $y \in \overline{a \wedge c}^p$. Then $y \wedge p = a \wedge c \wedge p = b \wedge c \wedge p \Rightarrow y \in \overline{b \wedge c}^p$, hence $\overline{a \wedge c}^p \subseteq \overline{b \wedge c}^p$. We can show $\overline{b \wedge c}^p \subseteq \overline{a \wedge c}^p$ in exactly the same manner. Hence we have proved 1.

By definition, $\overline{a \wedge_p c}^p = \{\bar{x}^p : x \in a \wedge_p c\}$. Choose any $\bar{x}^p \in \overline{a \wedge_p c}^p$; then there exists some y such that $y \in a \wedge_p c$ and $x \in \bar{y}^p$. I.e.

$$\begin{aligned} (a \wedge c) \wedge p \leq y &\leq (a \wedge c) \vee p' \Rightarrow \\ (a \wedge c) \wedge p \leq y \wedge p &\leq ((a \wedge c) \vee p') \wedge p \Rightarrow \\ a \wedge c \wedge p \leq y \wedge p &\leq (a \wedge c \wedge p) \vee (p \wedge p') \Rightarrow \\ b \wedge c \wedge p \leq y \wedge p &\leq (b \wedge c \wedge p) \vee (p \wedge p') \leq (b \wedge c) \vee p'. \end{aligned}$$

Set $z = y \wedge p$; we have just shown that $z \in b \wedge_p c$. Also, $z = y \wedge p \in \bar{y}^p = \bar{x}^p$, and so $\bar{x}^p = \bar{z}^p$. Hence there exists a z such that: $\bar{x}^p = \bar{z}^p$ and $z \in b \wedge_p c$, i.e. $\bar{x}^p \in \overline{b \wedge_p c}^p$. In short, we have shown $\overline{a \wedge_p c}^p \subseteq \overline{b \wedge_p c}^p$. In exactly the same way we can show $\overline{b \wedge_p c}^p \subseteq \overline{a \wedge_p c}^p$ and so $\overline{a \wedge_p c}^p = \overline{b \wedge_p c}^p$; the proof of 2 is complete. 3 is proved similarly to 1 and 4 is proved similarly to 2. ■

Proposition 4.9 The following hold for every $a \in X$.

1. $\bar{a}^1 = \{a\}, \bar{a}^0 = [0, 1]$.
2. For all $p, q \in X$: $p \leq q \Rightarrow \bar{a}_q \subseteq \bar{a}_p$
- 3.1 For all $p, q \in X$: $\bar{a}_{p \vee q} = \bar{a}_p \cap \bar{a}_q$.
- 3.2 For all $P \subseteq X$: $\bar{a}_{\vee P} = \cap_{p \in P} \bar{a}_p$.
4. For all $p, q \in X$: $\bar{a}_{p \wedge q} = \bar{a}_p \dot{\cup} \bar{a}_q$ (where $\bar{a}_p \dot{\cup} \bar{a}_q = \cap_{t: t \leq p, t \leq q} \bar{a}_t$).

Proof. 1 follows from the definition of σ_1, σ_0 . Regarding 2, take any $p, q \in X$ such that $p \leq q$. Then $x \in \bar{a}_q \Rightarrow x \wedge q = a \wedge q \Rightarrow x \wedge q \wedge p = a \wedge q \wedge p \Rightarrow x \wedge p = a \wedge p \Rightarrow x \in \bar{a}_p$. Hence $\bar{a}_q \subseteq \bar{a}_p$. Next we prove 3.2 (3.1 is a special case of 3.2). We have

$$(\forall p \in P : p \leq \vee P) \Rightarrow (\forall p \in P : \bar{a}^p \supseteq \bar{a}^{\vee P}) \Rightarrow \cap_{p \in P} \bar{a}^p \supseteq \bar{a}^{\vee P}.$$

On the other hand, take any $x \in \cap_{p \in P} \bar{a}^p$. Then

$$(\forall p \in P : x \in \bar{a}^p) \Rightarrow (\forall p \in P : x \wedge p = a \wedge p) \Rightarrow \vee_{p \in P} (x \wedge p) = \vee_{p \in P} (a \wedge p) \Rightarrow x \wedge (\vee P) = a \wedge (\vee P).$$

Hence $x \in \bar{a}^{\vee P}$ and we conclude that $\cap_{p \in P} \bar{a}^p \subseteq \bar{a}_{\vee P}$. Hence $(\cap_{p \in P} \bar{a}^p) = \bar{a}_{\vee P}$.

Finally let us prove 4. Choose any $p, q \in X$ and define $R_{pq} = \{r : r \leq p, r \leq q\}$. We have

$$(\forall r \in R_{pq} : r \leq p \wedge q) \Rightarrow (\forall r \in R_{pq} : \bar{a}^r \supseteq \bar{a}^{p \wedge q}) \Rightarrow \cap_{r \in R_{pq}} \bar{a}^r \supseteq \bar{a}^{p \wedge q}.$$

But also $p \wedge q \in R_{pq}$ and so $(\cap_{r \in R_{pq}} \bar{a}^r) \subseteq \bar{a}^{p \wedge q}$. Hence $\cap_{r \in R_{pq}} \bar{a}^r = \bar{a}^{p \wedge q}$. ■

Proposition 4.10 ,Choose any $a \in X$ and define $g : X \rightarrow \{\bar{a}^p\}_{p \in X}$ by $g(p) = \bar{a}^p$. Then g is an anti-homomorphism: $(X, \vee, \wedge, \leq) \xrightarrow{g} (\{\bar{a}^p\}_{p \in X}, \cap, \dot{\cup}, \subseteq)$. More specifically we have

1. $p \leq q \Rightarrow g(q) \subseteq g(p)$.
2. $g(p \wedge q) = g(p) \dot{\cup} g(q)$.
3. $g(p \vee q) = g(p) \cap g(q)$.

Proof. This follows from Proposition 4.9. ■

4.3 The L-fuzzy Relation σ

An L-fuzzy relation σ is a mapping of $X \times X$ to X . We define an L-fuzzy relation σ in such a manner that for every $p \in X$, the p -cut of σ is σ_p . As will be seen presently, σ is an L-fuzzy equivalence.

Definition 4.11 We define the L-fuzzy relation $\sigma : X \times X \rightarrow X$ as follows

$$\sigma(a, b) = \vee \{p : (a, b) \in \sigma_p\}.$$

Proposition 4.12 For all $p \in X$: $\{(a, b) : \sigma(a, b) \geq p\} = \sigma_p$.

Proof. Follows from Definition 4.11 and the standard properties of p -cuts. ■

Proposition 4.13 σ is an L-fuzzy equivalence, i.e.

1. For all $a \in X$: $\sigma(a, a) = 1$.
2. For all $a, b \in X$: $\sigma(a, b) = \sigma(b, a)$.
3. For all $a, b, c \in X$: $\sigma(a, b) \geq \sigma(a, c) \wedge \sigma(c, b)$.

Proof. We will prove the proposition using p -cut properties. Since $a \wedge 1 = a \wedge 1$ we have $(a, a) \in \sigma_1$ and so $\sigma(a, a) \geq 1$; this proves 1. Also, since for any $a, b, p \in X$ we have $\sigma_p(a, b) = \sigma_p(b, a)$, it follows that $\sigma(a, b) = \vee \{p : (a, b) \in \sigma_p\} = \vee \{p : (b, a) \in \sigma_p\} = \sigma(b, a)$ which proves 2. Finally, take any $a, b, c \in X$. Set $p_1 = \sigma(a, c)$, $p_2 = \sigma(c, b)$. Then $(a, c) \in \sigma_{p_1}$, $(c, b) \in \sigma_{p_2}$, and

$$\begin{aligned} p_1 \geq p_1 \wedge p_2 &\Rightarrow \sigma_{p_1} \subseteq \sigma_{p_1 \wedge p_2} \Rightarrow (a, c) \in \sigma_{p_1 \wedge p_2} \\ p_2 \geq p_1 \wedge p_2 &\Rightarrow \sigma_{p_2} \subseteq \sigma_{p_1 \wedge p_2} \Rightarrow (c, b) \in \sigma_{p_1 \wedge p_2} \end{aligned}$$

But, since $\sigma_{p_1 \wedge p_2}$ is a (crisp) equivalence, we have: $((a, c) \in \sigma_{p_1 \wedge p_2}, (c, b) \in \sigma_{p_1 \wedge p_2}) \Rightarrow (a, b) \in \sigma_{p_1 \wedge p_2} \Rightarrow \sigma(a, b) \geq p_1 \wedge p_2 = \sigma(a, c) \wedge \sigma(c, b)$. ■

Definition 4.14 The p -classes of σ are denoted by \tilde{a}^p and defined as follows:

$$\tilde{a}^p = \{b : \sigma(a, b) \geq p\}.$$

Proposition 4.15 For all $p \in X$, $\tilde{a}^p = \bar{a}^p$.

Proof. Indeed $x \in \tilde{a}^p \Leftrightarrow \sigma(a, x) \geq p \Leftrightarrow (a, x) \in \sigma_p \Leftrightarrow x \wedge p = a \wedge p \Leftrightarrow x \in \bar{a}^p$. ■

Definition 4.16 For every $a \in X$ define the fuzzy set $\bar{a} : X \rightarrow X$ by

$$\bar{a}(x) = \vee \{p : x \in \bar{a}^p\}.$$

Proposition 4.17 For all $a \in X$, \bar{a} is an L-fuzzy interval.

Proof. Follows immediately from the fact that for every $p \in X$ the set \bar{a}^p is an interval. ■

4.4 An Additional Family of Crisp Equivalence Relations

Let us briefly note that we can define a family of crisp equivalence relations $\{\tau_p\}_{p \in X}$ as follows.

Definition 4.18 For every $p \in X$ we define the relation $\tau_p \subseteq X \times X$ as follows

$$(a, b) \in \tau_p \text{ iff } a \vee p' = b \vee p'.$$

Using the above definition we can prove various properties of τ_p and construct a fuzzy equivalence relation τ . The analysis is similar to the one regarding σ_p and σ ; the details are omitted for brevity.

5 The Boolean Case

In this Section we assume that $(X, \vee, \wedge, \leq, ')$ is a generalized *Boolean* lattice according to the Definition 2.2. In other words, the following additional assumption is made.

$$\forall p \in X \text{ we have } p' \vee p = 1, p' \wedge p = 0.$$

The above assumption has the following important consequence.

Proposition 5.1 For all $a, b, p \in X$ we have:

1. $a \leq_p b \Leftrightarrow a \wedge p \leq b \wedge p \Leftrightarrow a \vee p' \leq b \vee p' \Leftrightarrow a \leq^p b$.
2. $a =_p b \Leftrightarrow a \wedge p = b \wedge p \Leftrightarrow a \vee p' = b \vee p'$

Proof. Indeed $a \leq_p b \Leftrightarrow a \wedge p \leq b \wedge p$. Now $a \wedge p \leq b \wedge p \Rightarrow (a \wedge p) \vee p' \leq (b \wedge p) \vee p' \Rightarrow (a \vee p') \wedge (p \vee p') \leq (b \vee p') \wedge (p \vee p') \Rightarrow a \vee p' \leq b \vee p' \Leftrightarrow a \leq^p b$. We can show analogously that $a \vee p' \leq b \vee p' \Rightarrow a \wedge p \leq b \wedge p$; this completes the proof of 1. For 2 we use the fact that $a =_p b \Leftrightarrow (a \leq_p b \text{ and } b \leq_p a)$. ■

Proposition 5.1 has considerable ramifications which are presented in the following propositions.

Proposition 5.2 For all $a, p \in X$ we have: $\bar{a}^p = [a \wedge p, a \vee p']$.

Proof. We already know from Proposition 4.7 that $\bar{a}^p = [a \wedge p, \vee \bar{a}^p]$. So it remains to show $a \vee p' = \vee \bar{a}^p$. We have $(a \vee p') \wedge p = (a \wedge p) \vee (p' \wedge p) = a \wedge p$. Hence $a \vee p' \in \bar{a}^p$ and so $a \vee p' \leq \vee \bar{a}^p$. Now take any $x \in \bar{a}^p$. Then we have

$$x \wedge p = a \wedge p \Rightarrow (x \wedge p) \vee p' = (a \wedge p) \vee p' \Rightarrow x \vee p' = a \vee p'.$$

Hence $x \leq x \vee p' = a \vee p'$. In particular, setting $x = \vee \bar{a}^p$ we get $\vee \bar{a}^p \leq a \vee p'$ and so $\vee \bar{a}^p = a \vee p'$. ■

The proofs of the remaining propositions are omitted (they follow from the fact $\bar{a}^p = [a \wedge p, a \vee p']$).

Proposition 5.3 For all $a, b, p \in X$: (1) $\overline{a \vee b}^p = a \vee_p b$, (2) $\overline{a \vee_p b}^p = a \vee_p b$.

Remark. Note that according to Proposition 5.3.2 $\overline{a \vee_p b}^p$ contains a *single* class, namely $a \vee_p b$.

Proposition 5.4 For all $a, b, c, p \in X$: $\bar{a}^p = \bar{b}^p \Rightarrow (\overline{a \vee c}^p = \overline{b \vee c}^p \text{ and } \overline{a \wedge c}^p = \overline{b \wedge c}^p)$.

In view of Proposition 5.4 we can now define operations on classes as follows.

Definition 5.5 For all $a, b, p \in X$ define: $\bar{a}^p \vee_p \bar{b}^p = \overline{a \vee b}^p$, $\bar{a}^p \wedge_p \bar{b}^p = \overline{a \wedge b}^p$, $(\bar{a}^p)' = \bar{a}^p$.

Proposition 5.6 For all $a, b, p \in X$ we have:

$$a \leq_p b \Leftrightarrow \bar{a}^p \bar{\wedge}_p \bar{b}^p = \bar{a}^p \Leftrightarrow \bar{b}^p = \bar{a}^p \vee_p \bar{b}^p \Leftrightarrow \bar{a}^p \preceq \bar{b}^p.$$

Proposition 5.7 For all $x, p \in X$, $Y \subseteq X$: (1) $\vee_p \{\bar{y}^p : y \in Y\} = \overline{\vee Y^p}$; (2) $\bar{\wedge}_p \{\bar{y}^p : y \in Y\} = \overline{\wedge Y^p}$.

Proposition 5.8 For all $a, b, c, p \in X$:

1. $\bar{a}^p \bar{\wedge}_p (\bar{b}^p \vee_p \bar{c}^p) = (\bar{a}^p \bar{\wedge}_p \bar{b}^p) \vee_p (\bar{a}^p \bar{\wedge}_p \bar{c}^p).$
2. $\bar{a}^p \vee_p (\bar{b}^p \bar{\wedge}_p \bar{c}^p) = (\bar{a}^p \vee_p \bar{b}^p) \bar{\wedge}_p (\bar{a}^p \vee_p \bar{c}^p).$

Proposition 5.9 For all $a, p \in X$:

1. $(\bar{a}^p)' = \bar{a}^p.$
2. $((\bar{a}^p)')' = \bar{a}^p.$
3. $\bar{a}^p \preceq \bar{b}^p \Leftrightarrow (\bar{b}^p)' \preceq (\bar{a}^p)'.$
4. $\bar{0}^p \preceq \bar{a}^p \preceq \bar{1}^p.$
5. $\bar{a}^p \vee_p (\bar{a}^p)' = \bar{1}^p.$
6. $\bar{a}^p \bar{\wedge}_p (\bar{a}^p)' = \bar{0}^p.$
7. $(\bar{a}^p \vee_p \bar{b}^p)' = (\bar{a}^p)' \bar{\wedge}_p (\bar{b}^p)'.$
8. $(\bar{a}^p \bar{\wedge}_p \bar{b}^p)' = (\bar{a}^p)' \vee_p (\bar{b}^p)'.$

Proposition 5.10 For all $x, p \in X$ and for all $Y \subseteq X$:

1. $\bar{x}^p \bar{\wedge}_p (\vee_p \{\bar{y}^p : y \in Y\}) = \vee_p \{\bar{x}^p \bar{\wedge}_p \bar{y}^p : y \in Y\}; \bar{x}^p \vee_p (\bar{\wedge}_p \{\bar{y}^p : y \in Y\}) = \bar{\wedge}_p \{\bar{x}^p \vee_p \bar{y}^p : y \in Y\}.$
2. $(\vee_p \{\bar{y}^p : y \in Y\})' = \bar{\wedge}_p \{(\bar{y}^p)'\} : y \in Y\}; (\bar{\wedge}_p \{\bar{y}^p : y \in Y\})' = \vee_p \{(\bar{y}^p)'\} : y \in Y\}.$

The following proposition summarizes the results of Propositions 5.6 – 5.10.

Proposition 5.11 For all $p \in X$ the structure $(\{\bar{a}^p\}_{a \in X}, \vee_p, \bar{\wedge}_p, \preceq, ')$ is a generalized Boolean lattice.

6 Discussion

We have introduced two L-fuzzy hyperoperations which can be seen as fuzzy analogues of the classical \vee, \wedge operations, we have presented two associated L-fuzzy hyperalgebras and discussed how these are related to hyperlattices and dual hyperlattices.

It is worth noting that \vee, \wedge are related to the L-fuzzy join hyperoperation presented in [8]. This hyperoperation is denoted by $*$; the L-fuzzy set $a * b$ is defined by its p -cuts as follows

$$(a * b)_p = [a \wedge b \wedge p, a \vee b \vee p'].$$

It is easily seen that (for all $p \in X$)

$$(a * b)_p = [\wedge (a \wedge_p b), \vee (a \vee_p b)]. \quad (8)$$

(8) is rather similar to the “classical” (crisp) join hyperoperation

$$a \circ b = [a \wedge b, a \vee b] \quad (9)$$

studied in [7]. Now, $a \circ b$ can be used to define a notion of *order convexity*. In particular $a \circ b$ can be understood as the lattice analog of a *straight line segment* in a metric space (for details see [2, pp.315-320]) and the set $Y \in \mathbf{P}(X)$ is called *order-convex* if for every pair $a, b \in Y$ we have $a \circ b \subseteq Y$. This suggests the possibility of defining a notion of *p-convexity* as follows: for some $p \in X$ we say that a set $Y \in \mathbf{P}(X)$ is *p-convex* iff for every pair $a, b \in Y$ we have $(a * b)_p \subseteq Y$. Similarly, we can define a notion of *fuzzy convexity* as follows: a fuzzy set $Y \in \mathbf{F}(X)$ is *fuzzy-convex* iff for every $p, a, b \in Y$ we have $(a * b)_p \subseteq Y_p$. It would be interesting to investigate the connection of this concept of fuzzy convexity to other work (for example in the area of convex fuzzy sets); it appears reasonable to expect that \vee and \wedge will be useful in such an investigation.

The \vee and \wedge studied in this paper are an *example* of fuzzified order operations. Several other examples and generalizations can be given. A fuzzification of the *Nakano superlattice* [10] seems particularly interesting, in view of its importance in logic applications. Also, the \vee_p, \wedge_p hyperoperations can be seen as a special case of the \vee^P, \wedge^Q hyperoperations used in connection with (P, Q) -superlattices [15, 16]. In general, we believe that the use of fuzzified operations will turn out to be useful in applications which involve uncertainty not with respect to particular objects, but with respect to the manner in which these objects are combined. These topics will be discussed in future publications.

References

- [1] N. Ajmal and K.V. Thomas. “Fuzzy lattices”. *Info. Sciences*, vol. 79, pp.271-291, 1994.
- [2] L.M. Blumenthal. *Theory and Applications of Distance Geometry*. Oxford University Press, 1953.
- [3] P. Corsini, *Prolegomena of Hypergroup Theory*, Udine: Aviani, 1993.
- [4] P. Corsini and I. Tofan. “On fuzzy hypergroups”. *PU.M.A.* vol.8, pp.29-37, 1997.
- [5] A. Hasankhani and M.M. Zahedi. “*F*-Hyperrings”. *Ital. Journal of Pure and Applied Math.*, vol. 4, pp.103-118, 1998.
- [6] A. Hasankhani and M.M. Zahedi. “On *F*-polygroups and fuzzy sub-*F*-polygroups”. *J. Fuzzy Math.*, vol. 6, pp. 97–110. 1998.
- [7] Ath. Kehagias and M. Konstantinidou. “Lattice-ordered join space: an applications-oriented example”. To appear in *Italian Journal of Pure and Applied Mathematics*.
- [8] Ath. Kehagias. “An example of L-fuzzy join space”. *Rend. Circ. Mat. Palermo*, vol.51, pp.503-526, 2002.
- [9] Ath. Kehagias. “The lattice of fuzzy intervals and sufficient conditions for its distributivity”. *arXiv:cs.OH/0206025*, at <http://xxx.lanl.gov/find/cs>.

- [10] Ath. Kehagias, K. Serafimidis and M. Konstantinidou. “A note on the congruences of the Nakano superlattice and Some Properties of the Associated Quotients”. *Rend. Circ. Mat. Palermo*, vol.51, pp.333-354, 2002.
- [11] M. Konstantinidou and J. Mittas. “An introduction to the theory of hyperlattices”. *Math. Balkanica*, vol.7, pp.187-193, 1977.
- [12] J.N. Mordeson and D.S. Malik. *Fuzzy commutative algebra*. World Scientific, 1998.
- [13] H.T. Nguyen and E.A. Walker. *A First Course on Fuzzy Logic*, CRC Press, Boca Raton, 1997.
- [14] A. Rosenfeld. “Fuzzy groups”. *J. Math. Anal. Appl.*, vol.35 , pp.512–517, 1971.
- [15] K. Serafimidis, Ath. Kehagias and M. Konstantinidou. “The structure of the (P, Q) -superlattice and order related properties”. To appear in *Italian Journal of Pure and Applied Mathematics*.
- [16] K. Serafimidis and Ath. Kehagias. “Some representation results for (P, Q) -superlattices”. To appear in *Italian Journal of Pure and Applied Mathematics*.
- [17] S. Spartalis, A. Dramalides and T. Vougiouklis. “On H_V -group rings”. *Algebras Groups Geom.*, vol.15, pp.47–54, 1998.
- [18] U.M. Swamy and D.V. Raju. “Fuzzy ideals and congruences of lattices”. *Fuzzy Sets and Systems*, vol. 95, pp.249-253, 1998.
- [19] A. Tepavcevic and G. Trajkovski. “ L -fuzzy lattices: an introduction”. *Fuzzy Sets and Systems*, vol. 123, pp.209–216. 2001.
- [20] T. Vougiouklis. “ H_V -groups defined on the same set”. *Discrete Math.* vol.155, pp.259–265, 1996.
- [21] M.M. Zahedi and A. Hasankhani. “ F -Polygroups”. *Int. J. Fuzzy Math.*, vol. 4, pp.533–548. 1996.
- [22] M.M. Zahedi and A. Hasankhani. “ F -Polygroups (II)”. *Inf. Sciences*, vol.89, pp.225-243, 1996.