The L-Fuzzy Corsini Join Hyperoperation

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Abstract

Corsini has defined a hyperoperation $\cdot$ through a fuzzy set and has shown $\cdot$ to be a *join* hyperoperation. This hyperoperation can be generalized so that it can be defined in terms of an *L-fuzzy* set. We explore the generalized hyperoperation and give sufficient conditions for the resulting hyperstructure to be a hypergroup and / or a join space.

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In [3] Corsini used a fuzzy set to define a hyperoperation $\cdot$ and showed that it is a *join* hyperoperation. This hyperoperation was further studied by Corsini in [4] and by Ameri and Zahedi in [1]. It is easy to generalize Corsini's hyperoperation so that it can be defined in terms of an *L-fuzzy* set. In this paper we explore the generalized hyperoperation and give sufficient conditions for the resulting hyperstructure to be a hypergroup and / or a join space.

For the purposes of this paper, an L-fuzzy set is a function from any set to a complete lattice (for more details on L-fuzzy sets see [7]). The notions of hypergroup and join space is described in [2]. The following will remain fixed for the remainder of the paper: X is a set; $\mathbf{P}(X)$ denotes the power set of X; $(L, \leq)$ is a complete lattice; $M : X \to L$ is an L-fuzzy set; $M(X)$ is the image of X under M.

Definition 1 *We define the relationship $J_M \subseteq X \times X$ by: $(x, y) \in J_M$ iff $M(x) = M(y)$.*

Proposition 2 *J_M is an equivalence relation.*

Proof. (i) $M(x) = M(x) \Rightarrow (x, x) \in J_M$.
(ii) $(x, y) \in J_M \Rightarrow M(x) = M(y) \Rightarrow (y, x) \in J_M$.
(iii)$((x, y) \in J_M, (y, z) \in J_M) \Rightarrow (M(x) = M(y), M(y) = M(z)) \Rightarrow (M(x) = M(z)) \Rightarrow (x, z) \in J_M$.
■

Definition 3 *The* classes *of J_M are denoted by $\overline{x}$ and defined by $\overline{x} \doteq \{y : M(x) = M(y)\}$.*

Definition 4 *The* quotient *of X with respect to J_M is denoted by X/M and defined by $X/M \doteq \{\overline{x}\}_{x \in X}$.*

We now use the L-fuzzy set M to introduce a hyperoperation on X, and an associated hyperoperation on the quotient X/M.

Definition 5 *We define the hyperoperation $\cdot : X \times X \to \mathbf{P}(X)$ by $x \cdot y = \{z : M(x) \wedge M(y) \leq M(z) \leq M(x) \vee M(y)\}$.*

Definition 6 *We define the hyperoperation $\circ : X/M \times X/M \to \mathbf{P}(X/M)$ by $\overline{x} \circ \overline{y} \doteq \{\overline{z} : M(x) \wedge M(y) \leq M(z) \leq M(x) \vee M(y)\} = \{\overline{z} : z \in x \cdot y\}$.*

The two hyperoperations are "equivalent" as can be seen by the following.

Proposition 7 *For all $x, y, z \in X$ we have $\overline{z} \in \overline{x} \circ \overline{y} \Leftrightarrow z \in x \cdot y$.*

Proof. The $\Leftarrow$ implication is immediate. For the $\Rightarrow$ implication:

$$\overline{z} \in \overline{x} \circ \overline{y} \Rightarrow \left(\exists u : \begin{array}{c} \overline{z} = \overline{u} \\ u \in x \cdot y \end{array} \right) \Rightarrow \left(\exists u : \begin{array}{c} M(z) = M(u) \\ M(x) \wedge M(y) \leq M(u) \leq M(x) \vee M(y) \end{array} \right) \Rightarrow$$

$$M(x) \wedge M(y) \leq M(z) \leq M(x) \vee M(y) \Rightarrow z \in x \cdot y.$$

■

Proposition 8 *For every $x, y \in X$ we have: (i) $\overline{x} = \overline{y} \Rightarrow x \cdot y = \overline{x}$,, (ii) $x \cdot x = \overline{x}$.*

Proof. For (i) note that $\overline{x} = \overline{y} \Rightarrow M(x) = M(y)$. Then $z \in x \cdot y \Leftrightarrow M(x) \wedge M(y) \leq M(z) \leq M(x) \vee M(y) \Leftrightarrow M(x) \leq M(z) \leq M(x) \Leftrightarrow M(x) = M(z) \Leftrightarrow z \in \overline{x}$. Hence $x \cdot y = \overline{x}$. (ii) follows from (i) taking $y = x$ ■

The next proposition shows that J_M is a congruence with respect to $\circ$.

Proposition 9 *For all $x, y, z \in X$ we have $\overline{x} = \overline{y} \Rightarrow \overline{x} \circ \overline{z} = \overline{y} \circ \overline{z}$.*

Proof. For all $x, y, z \in X$ we have $\overline{x} = \overline{y} \Rightarrow M(x) = M(y)$. Take any $\overline{u} \in \overline{x} \circ \overline{z}$. Then

$$\begin{array}{l} M(x) \wedge M(z) \leq M(u) \leq M(x) \vee M(z) \Rightarrow \\ M(y) \wedge M(z) \leq M(u) \leq M(y) \vee M(z) \Rightarrow \overline{u} \in \overline{y} \circ \overline{z}. \end{array}$$

So we have shown $\overline{u} \in \overline{x} \circ \overline{z} \Rightarrow \overline{u} \in \overline{y} \circ \overline{z}$. Similarly we can show the converse and we are done. ■

Definition 10 *For all $A \in \mathbf{P}(X)$, $\overline{A} \doteq \{\overline{x}\}_{x \in A}$.*

Remark. It follows from the above definition that $\overline{x \cdot y} = \{\overline{z} : z \in x \cdot y\} = \overline{x} \circ \overline{y}$.

Definition 11 *$\overline{M} : X/M \to L$ is defined by $\overline{M}(\overline{x}) \doteq M(x)$.*

Proposition 12 *$\overline{M}$ is well defined and 1-1, onto $M(X)$.*

Proof. It is well defined and 1-1 because $\overline{x} = \overline{y} \Leftrightarrow M(x) = M(y) \Leftrightarrow \overline{M}(\overline{x}) = \overline{M}(\overline{y})$. It is onto $M(X)$ because: $a \in M(X) \Rightarrow \exists x \in X : a = M(x) \Rightarrow \exists \overline{x} \in X/M : a = \overline{M}(\overline{x})$. ■

We now introduce one more hyperoperation, which is the *restriction* of a join hyperoperation introduced in [6].

Definition 13 *We define the hyperoperation $* : M(X) \times M(X) \to \mathbf{P}(M(X))$ by $a * b = [a \wedge b, a \vee b] \cap M(X)$ (where $a, b \in M(X)$, i.e. $\exists x, y \in X$ such that $a = M(x)$, $b = M(y)$).*

Let us now establish the connection between $\cdot, \circ$ and $*$.

Proposition 14 *The following are equivalent for all $x, y, z \in X$.*

(i) $z \in x \cdot y$;

(ii) $\overline{z} \in \overline{x} \circ \overline{y}$;

(iii) $M(z) \in M(x) * M(y)$;

(iv) $\overline{M}(\overline{z}) \in \overline{M}(\overline{x}) * \overline{M}(\overline{y})$.

Proof. (i) is equivalent to (ii) by Proposition 7; (iii) is equivalent to (iv) by Definition 11. Let us show that (i) is equivalent to (iii). We have

$$z \in x \cdot y \Leftrightarrow \begin{pmatrix} M(z) \in [M(x) \wedge M(y), M(x) \vee M(y)] \\ M(z) \in M(X) \end{pmatrix}$$

$$\begin{aligned} &\Leftrightarrow M(z) \in [M(x) \wedge M(y), M(x) \vee M(y)] \cap M(X) \\ &\Leftrightarrow M(z) \in M(x) * M(y). \end{aligned}$$

■

We now introduce an order on X/M and then use it to establish an isomorphism between the domain and range of $\overline{M}$.

Definition 15 *We define $\sqsubseteq$ on X/M by: $\overline{x} \sqsubseteq \overline{y} \Leftrightarrow M(x) \leq M(y)$.*

Proposition 16 *$\sqsubseteq$ is an order on X/M.*

Proof. Clearly $\overline{x} \sqsubseteq \overline{y} \Leftrightarrow \overline{M}(\overline{x}) \leq \overline{M}(\overline{y})$. Also, $\overline{M}$ is 1-1 from X/M onto $M(X)$. Finally, since $\leq$ is an order on L it is also an order on $M(X) \subseteq L$. ■

Remark. One could define $\preceq$ on X by $x \preceq y$ iff $M(x) \leq M(y)$. In this case, $\preceq$ is a *preorder* on X and the classes generated by this preorder are exactly the elements of X/M.

Proposition 17 $(X/M, \sqsubseteq, \cdot) \stackrel{\overline{M}}{\to} (M(X), \leq, *)$ *is an order isomorphism, i.e.:*

(i) $\overline{M}$ *is 1-1, onto;*

(ii) $\overline{x} \sqsubseteq \overline{y} \Leftrightarrow M(x) \leq M(y)$*;*

(iii) $\overline{M}(\overline{x} \circ \overline{y}) = M(x) * M(y)$.

Proof. (i) follows from Proposition 12. (ii) follows from Definition 15. For (iii) note the following. First, from Proposition 14.(iv) for all $z \in \overline{x} \circ \overline{y}$ we have $\overline{M}(\overline{z}) \in \overline{M}(\overline{x}) * \overline{M}(\overline{y})$. Since $\overline{M}(\overline{x} \circ \overline{y}) = \{\overline{M}(\overline{z})\}_{\overline{z} \in \overline{x} \circ \overline{y}}$, it follows that

$$\overline{M}(\overline{x} \circ \overline{y}) \subseteq M(x) * M(y). \tag{1}$$

Second,

$$\begin{aligned} a \in M(x) * M(y) = [M(x) \wedge M(y), M(x) \vee M(y)] \cap M(X) \Rightarrow \\ \exists z : a = M(z) \in [M(x) \wedge M(y), M(x) \vee M(y)] \Rightarrow \\ z \in x \cdot y \Rightarrow \overline{z} \in \overline{x} \circ \overline{y} \Rightarrow a = M(\overline{z}) \in M(\overline{x} \circ \overline{y}) \Rightarrow \\ M(x) * M(y) \subseteq \overline{M}(\overline{x} \circ \overline{y}). \end{aligned} \tag{2}$$

From (1) and (2) follows that $M(x) * M(y) = \overline{M}(\overline{x} \circ \overline{y})$. ■

Proposition 18 *$(X/M, \sqsubseteq)$ is a (modular, distributive) lattice iff $(M(X), \leq)$ is a (modular, distributive) lattice.*

Proof. This is obvious, since $\overline{M}$ is an order isomorphism between $(X/M, \sqsubseteq)$ and $(M(X), \leq)$. ■
We also define *extension* hyperoperations obtained from the respective join hyperoperations.

Definition 19 *We define the hyperoperation $/ : X \times X \to \mathbf{P}(X)$ by $x/y = \{z : x \in y \cdot z\}$.*

Definition 20 *We define the hyperoperation $// : X/M \times X/M \to \mathbf{P}(X/M)$ by $\overline{x}//\overline{y} \doteq \{\overline{z} : \overline{x} \in \overline{y} \circ \overline{z}\}$.*

Definition 21 *We define the hyperoperation $\wr : M(X) \times M(X) \to \mathbf{P}(M(X))$ by $a \wr b = \{c \in M(X)$: $a \in b * c\}$.*

We are now ready to present conditions for $\cdot, \circ$ and $*$ to be join hyperoperations.

Proposition 22 *$(X/M, \circ)$ is a hypergroup (join space) iff $(M(X), *)$ is a hypergroup (join space).*

Proof. This follows from the isomorphism between $\circ$ and $*$.
(i) If $(X/M, \circ)$ is a join space, then for all $\overline{x}, \overline{y}, \overline{z}, \overline{u}, \overline{w} \in X/M$ we have

A1 $\overline{x} \circ \overline{x} = \overline{x}$.

A2 $\overline{x} \circ \overline{y} = \overline{y} \circ \overline{x}$.

A3 $(\overline{x} \circ \overline{y}) \circ \overline{z} = \overline{x} \circ (\overline{y} \circ \overline{z})$.

A4 $\overline{x} \circ (X/M) = X/M$.

A5 $\overline{x}//\overline{y} \sim \overline{u}//\overline{w} \Rightarrow \overline{x} \circ \overline{w} \sim \overline{y} \circ \overline{u}$.

(ii) Similarly, if $(M(X), *)$ is a join space, then for all $a, b, c, d \in M(X)$ we have

B1 $a * a = a$.

B2 $a * b = b * a$.

B3 $(a * b) * c = a * (b * c)$.

B4 $a * M(X) = M(X)$.

B5 $a/b \sim c/d \Rightarrow a * d \sim b * c$

(iii) **A1, A2, A4** are always true; similarly for **B1, B2, B4**. We will next show that **A3** is equivalent to **B3**. First we need to show that

$$\overline{M}(\overline{x} \circ \overline{y}) * \overline{M}(\overline{z}) = \overline{M}((\overline{x} \circ \overline{y}) \circ \overline{z}).$$

Note that $a \in M(X)$ implies (for some $w \in X$) that $a = M(w)$. Then, for any $a \in \overline{M}(\overline{x} \circ \overline{y}) * \overline{M}(\overline{z})$ we have

$$\left(\exists u \in X : \begin{array}{c} \overline{M}(\overline{u}) \in \overline{M}(\overline{x} \circ \overline{y}) = M(x) * M(y) \\ a \in \overline{M}(\overline{u}) * M(\overline{z}) \end{array}\right)$$

$$\Leftrightarrow \left(\exists u \in X : \begin{array}{c} M(x) \wedge M(y) \leq M(u) = \overline{M}(\overline{u}) \leq M(x) \vee M(y) \\ M(u) \wedge M(z) \leq a = M(w) \leq M(u) \vee M(z) \end{array}\right)$$

$$\Leftrightarrow \left(\exists u \in X : \begin{array}{c} \overline{u} \in \overline{x} \circ \overline{y} \\ \overline{w} \in \overline{u} \circ \overline{z} \end{array}\right)$$

$$\Leftrightarrow w \in (\overline{x} \circ \overline{y}) \circ \overline{z}$$

$$\Leftrightarrow a = \overline{M}(\overline{w}) \in \overline{M}((\overline{x} \circ \overline{y}) \circ \overline{z}).$$

Hence we have $\overline{M}(\overline{x} \circ \overline{y}) * \overline{M}(\overline{z}) = \overline{M}((\overline{x} \circ \overline{y}) \circ \overline{z})$. Now:

$$\left(\overline{M}(\overline{x}) * \overline{M}(\overline{y})\right) * \overline{M}(\overline{z}) = \overline{M}(\overline{x} \circ \overline{y}) * \overline{M}(\overline{z}) = \overline{M}((\overline{x} \circ \overline{y}) \circ \overline{z}) \tag{3}$$

is equivalent (since $\overline{M}$ is 1-1) to

$$\overline{M}^{-1}\left(\left(\overline{M}(\overline{x}) * \overline{M}(\overline{y})\right) * \overline{M}(\overline{z})\right) = (\overline{x} \circ \overline{y}) \circ \overline{z} \tag{4}$$

Similarly we can show that

$$\overline{M}(\overline{x}) * \left(\overline{M}(\overline{y}) * \overline{M}(\overline{z})\right) = \overline{M}(\overline{x}) * \overline{M}(\overline{y} \circ \overline{z}) = \overline{M}(\overline{x} \circ (\overline{y} \circ \overline{z})) \tag{5}$$

and

$$\overline{M}^{-1}\left(\overline{M}(\overline{x}) * \left(\overline{M}(\overline{y}) * \overline{M}(\overline{z})\right)\right) = \overline{x} \circ (\overline{y} \circ \overline{z}) \tag{6}$$

are equivalent. From **A3**, (3) and (5) follows **B3**; and from **B3**, (4) and (6) follows **A3**. Hence **A3** $\Leftrightarrow$ **B3**. Similarly we can show **A5** $\Leftrightarrow$ **B5** and the proof is complete. ■

Proposition 23 *$(X/M, \circ)$ is a hypergroup (join space) iff $(X, \cdot)$ is a hypergroup (join space).*

Proof. If $(X/M, \circ)$ is a join space, then the conditions **A1** $-$ **A5** presented previously hold for all $\overline{x}, \overline{y}, \overline{z}, \overline{u}, \overline{w} \in X/M$. Also, if $(X, \cdot)$ is a join space, then for all $x, y, z, u, w \in X$ we have

C1 $x \cdot x = x$.

C2 $x \cdot y = y \cdot x$.

C3 $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

C4 $x \cdot X = X$.

C5 $x/y \sim u/w \Rightarrow x \cdot w \sim y \cdot u$.

Now, **A1, A2, A4** are always true; similarly for **C1, C2, C4**. We will next show that **A3** and **C3** are equivalent. On the one hand we have

$$\overline{u} \in (\overline{x} \circ \overline{y}) \circ \overline{z} \Leftrightarrow \left(\exists w : \begin{array}{c} \overline{w} \in \overline{x} \circ \overline{y} \\ \overline{u} \in \overline{w} \circ \overline{z} \end{array}\right) \Leftrightarrow \left(\exists w : \begin{array}{c} w \in x \cdot y \\ u \in w \cdot z \end{array}\right) \Leftrightarrow u \in (x \cdot y) \cdot z. \tag{7}$$

On the other hand

$$\overline{u} \in \overline{x} \circ (\overline{y} \circ \overline{z}) \Leftrightarrow \left(\exists p : \begin{array}{c} \overline{p} \in \overline{y} \circ \overline{z} \\ \overline{u} \in \overline{x} \circ \overline{p} \end{array} \right) \Leftrightarrow \left(\exists p : \begin{array}{c} p \in y \cdot z \\ u \in x \cdot p \end{array} \right) \Leftrightarrow u \in x \cdot (y \cdot z). \tag{8}$$

Now, if **C3** holds, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and from (7), (8) follows that $\overline{x} \circ (\overline{y} \circ \overline{z}) = (\overline{x} \circ \overline{y}) \circ \overline{z}$, i.e. **A3** holds too. The converse is also immediate, hence **A3** $\Leftrightarrow$ **C3**. Similarly we can show **A5** $\Leftrightarrow$ **C5** and we are done. ■

Proposition 24 *If $M(X)$ is a distributive sublattice of $(L, \leq)$ then: $(M(X), *)$, $(X/M, \circ)$ and $(X, \cdot)$ are all join spaces.*

Proof. The proof that $(M(X), *)$ is a join space has been given in [6]; that $(X/M, \circ)$ is a join space follows from Proposition 22; that $(X, \cdot)$ is a join space follows from Proposition 23. ■

Now we can interpret the Corsini result. Corsini takes $L = [0, 1] \subseteq \mathbf{R}$. But then $M(X) \subseteq [0, 1]$ is a chain and so a distributive sublattice of $([0, 1], \leq)$. Hence $(X, \cdot)$ will be a join space by Proposition 23.

We can also give a condition for $(M(X), *)$, $(X/M, \circ)$ and $(X, \cdot)$ to be hypergroups.

Proposition 25 *$(M(X), *)$, $(X/M, \circ)$ and $(X, \cdot)$ are hypergroups iff $\forall p, q, r \in M(X)$, exist $a, b \in M(X)$ such that $r * [p \wedge q, p \vee q] = [a, b]$.*

Proof. In [5] we have shown that the $\cdot$ hyperoperation is associative iff $\forall p, q, r \in M(X)$, exist $a, b \in M(X)$ such that $r * [p \wedge q, p \vee q] = [a, b]$. This, in conjunction with Proposition 24 completes the proof. ■

In the future we plan to extend our investigation in case L and/or M have additional properties. For example, it will be interesting to explore the case where $(L, \leq)$ is a Boolean or deMorgan lattice. It is also interesting to explore the case where X is equipped with an order. For instance, suppose that $(X, \preceq)$ is a lattice. When are $\preceq$ and $\sqsubseteq$ compatible? A more general direction for future research concerns the case where $(M(X), \leq)$ is a lattice but not a sublattice of $(L, \leq)$ and find out if in this case $\cdot$ is a join hyperoperation.

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1 Further Issues

1. Introduce counterexamples: a Tepavcevic counterexample using the Boole^2 as L. Also the [Keh+Kon] nonmodular, nonjoin lattice (to show that $(X, *)$ can fail to be a hypergroup.

2. Properties of Corsini join using the [Keh+Kon] results.

3. Relate to other results about Corsini fuzzy join.

4. Corsini Join on deMorgan L-fuzzy sets, on Boolean L-fuzzy sets. This essentially means to study J_M for special L and M.

5. Extend the Caratheodory result: show that it yields not only hypergroup but also join space.

6. Suppose $(X, \preceq)$ is a lattice. When are $\preceq$ and $\sqsubseteq$ compatible? (Use Tevacevska).

7. What happens if we define $a * b = [a \wedge b, a \vee b]$ (i.e. without the $\cap M(X)$ restriction)?

8. What happens if $(M(X), \leq)$ is a lattice but not a sublattice of $(L, \leq)$?