Some Representation Results for (P, Q)-Superlattices

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Abstract

On the the lattice $(L, \vee, \wedge)$ we construct the *hyperoperations* $\overset{P}{\vee}$, $\overset{Q}{\wedge}$ as follows: $a \overset{P}{\vee} b = a \vee b \vee P$, $a \overset{Q}{\wedge} b = a \wedge b \wedge Q$. If the sets $P, Q \subseteq L$ satisfy appropriate conditions, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a *superlattice*. In this paper we give two *representation* results for (P, Q)-superlattices. The first result is an analog of the [1] representation result. The second result gives necessary and sufficient conditions for a general superlattice to be expressed as a (P, Q)-superlattice; this condition is expressed in terms of associativity of $\overset{P}{\vee}$ with $\vee$ and of $\overset{Q}{\wedge}$ with $\wedge$.

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1 Introduction

In this paper we give two *representation* results for $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ superlattices [4, 7, 8]. The first result concerns the representation of a general (P, Q)-superlattice in terms of an isomorphic (P, Q)-superlattice of sets; it is analogous to a classical theorem about the isomorphism of finite distributive lattices to posets of sets [1]. The second result is related to the following question: what are necessary and sufficient conditions for an arbitrary superlattice $(L, \curlyvee, \curlywedge)$ to be represented as a $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ superlattice? We show that one such condition is expressed in terms of associativity of: (a) $\overset{P}{\vee}$ with $\vee$ and (b) $\overset{Q}{\wedge}$ with $\wedge$. We also give an extended example of a class of superlattices (namely the *Nakano* superlattices) which *cannot* be represented as (P, Q)-superlattices.

2 Preliminaries

2.1 Superlattices

Let us first give the definition of a general superlattice, as given in [6]. In what follows $\mathbf{P}(L)$ will denote the power set of a reference set L.

Definition 2.1 *A superlattice is a partially ordered set* $(L, \leq)$ *with two hyperoperations* $\curlyvee, \curlywedge$*, where* $\curlyvee : L \times L \to \mathbf{P}(L)$*,* $\curlywedge : L \times L \to \mathbf{P}(L)$*, and the following properties are satisfied for all* $a, b, c \in L$*.*

S1 $a \in (a \curlyvee a) \cap (a \curlywedge a)$

S2 $a \curlyvee b = b \curlyvee a, a \curlywedge b = b \curlywedge a$

S3 $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c), (a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$

S4 $a \in [(a \curlyvee b) \curlywedge a] \cap [(a \curlywedge b) \curlyvee a]$

S5a $a \leq b \Rightarrow (b \in a \curlyvee b \text{ and } a \in a \curlywedge b)$

S5b $(b \in a \curlyvee b \text{ or } a \in a \curlywedge b) \Rightarrow a \leq b.$

As has been shown in [6], the following definition is equivalent to Definition 2.1.

Definition 2.2 *A superlattice is a hyperstructure* $(L, \curlyvee, \curlywedge)$*, where* $\curlyvee : L \times L \to \mathbf{P}(L)$*,* $\curlywedge : L \times L \to \mathbf{P}(L)$*, and the following properties are satisfied for all* $a, b, c \in L$*.*

S1 $a \in (a \curlyvee a) \cap (a \curlywedge a)$

S2 $a \curlyvee b = b \curlyvee a, a \curlywedge b = b \curlywedge a$

S3 $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c), (a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$

S4 $a \in [(a \curlyvee b) \curlywedge a] \cap [(a \curlywedge b) \curlyvee a]$

S6 $b \in a \curlyvee b \Leftrightarrow a \in a \curlywedge b$

S7 $a, b \in a \curlyvee b \Rightarrow a = b$

S8 $b \in a \curlyvee b \text{ et } c \in b \curlyvee c \Rightarrow c \in a \curlyvee c.$

The definitions of $\curlyvee$- and $\curlywedge$-distributive $(L_1, \curlyvee, \curlywedge)$ superlattice (regular, feeble and weak) are given in [8] and they are generalizations of $\vee$- and $\wedge$-distributive $(L_1, \vee, \wedge)$ lattice.

We now turn to (P,Q)-superlattices, which were first presented in [4]. To introduce (P,Q)-superlattices, let us select two sets $P, Q \in \mathbf{P}(L)$ and define the following hyperoperations.

Definition 2.3 *For all $a, b \in L$ we define $a \overset{P}{\vee} b \doteq a \vee b \vee P = \{a \vee b \vee p : p \in P\}$.*

Definition 2.4 *For all $a, b \in L$ we define $a \overset{Q}{\wedge} b \doteq a \wedge b \wedge Q = \{a \wedge b \wedge q : q \in Q\}$.*

The necessary and sufficient conditions on P, Q for $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ to be a superlattice are easily stated in terms of the following two collections of sets.

Definition 2.5 $\mathbf{A}(L) \doteq \{A \in \mathbf{P}(L) : \forall x \in L \quad \exists a \in A$ *such that* $a \leq x\}$.

Definition 2.6 $\mathbf{B}(L) \doteq \{B \in \mathbf{P}(L) : \forall y \in L \quad \exists b \in B$ *such that* $y \leq b\}$.

It is clear that $L \in \mathbf{A}(L) \cap \mathbf{B}(L)$. Also, clearly, if $(L, \vee, \wedge)$ has a 0, then $P \in \mathbf{A}(L) \Leftrightarrow 0 \in P$; if $(L, \vee, \wedge)$ has a 1, then $Q \in \mathbf{B}(L) \Leftrightarrow 1 \in Q$. Furthermore, the following proposition yields a necessary and sufficient condition for $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ to be a superlattice.

Proposition 2.7 $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is a superlattice* $\Leftrightarrow (P,Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$.

Proof. The proof appears in [4]. ■

The following proposition will be used in later sections.

Proposition 2.8 *For all $(P,Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ and all $a, b \in L$ we have: (i) $a \vee b = \min(a \overset{P}{\vee} b)$, (ii) $a \wedge b = \max(a \overset{Q}{\wedge} b)$.*

Proof. (i) Since $P \in \mathbf{A}(L)$ there will exist a $p \in P$ such that $p \leq a \vee b$. Hence $a \vee b = a \vee b \vee p \in a \overset{P}{\vee} b$. Clearly, for all $x \in a \overset{P}{\vee} b$ we have $a \vee b \leq x$, so we have proved $a \vee b = \min(a \overset{P}{\vee} b)$.

(ii) This is proved dually to (i). ■

2.2 Superlattice Isomorphisms

In this section we generalize homomorphism, isomorphism and related classical concepts to the context of superlattices. In the definitions and propositions of this section we take $(L_1, \curlyvee_1, \curlywedge_1)$ and $(L_2, \curlyvee_2, \curlywedge_2)$ to be superlattices. We denote the respective partial orders by $\leq_1$ and $\leq_2$.

Definition 2.9 *A mapping* $f : L_1 \rightarrow L_2$ *will be called* superlattice homomorphism (from $(L_1, \curlyvee_1, \curlywedge_1)$ to $(L_2, \curlyvee_2, \curlywedge_2)$) *iff for every pair* $(a, b) \in L_1 \times L_1$ *we have*

$$f(a \curlyvee_1 b) \subseteq f(a) \curlyvee_2 f(b), \quad f(a \curlywedge_1 b) \subseteq f(a) \curlywedge_2 f(b).$$

f *will be called a* strong superlattice homomorphism *iff for every pair* $(a, b) \in L_1 \times L_1$ *we have*

$$f(a \curlyvee_1 b) = f(a) \curlyvee_2 f(b), \quad f(a \curlywedge_1 b) = f(a) \curlywedge_2 f(b).$$

An injective, onto strong superlattice homorphism will be called superlattice isomorphism.

Proposition 2.10 *Let f be a superlattice homomorphism from $(L_1, \curlyvee_1, \curlywedge_1)$ to $(L_2, \curlyvee_2, \curlywedge_2)$. Then f is isotone, i.e. for $a, b \in L_1$ such that $a \leq_1 b$, we have $f(a) \leq_2 f(b)$.*

Proof. Indeed, if $a, b \in L_1$ and $a \leq_1 b$, then

$$a \leq_1 b \Rightarrow \left\{ \begin{array}{c} a \in a \curlywedge_1 b \Rightarrow f(a) \in f(a \curlywedge_1 b) \subseteq f(a) \curlywedge_2 f(b) \\ \text{and} \\ b \in a \curlyvee_1 b \Rightarrow f(b) \in f(a \curlyvee_1 b) \subseteq f(a) \curlyvee_2 f(b) \end{array} \right. .$$

Either part of the right hand side above suffices to show (by **S5b**, Definition 2.1) that $f(a) \leq_2 f(b)$. ■

Proposition 2.11 *Let f be a strong superlattice homomorphism from $(L_1, \curlyvee_1, \curlywedge_1)$ to $(L_2, \curlyvee_2, \curlywedge_2)$. Then $(f(L_1), \curlyvee_2, \curlywedge_2)$ is a superlattice. In addition, if $(L_1, \curlyvee_1, \curlywedge_1)$ is $\curlywedge$-distributive, or weakly-$\curlywedge$- distributive, or feebly- $\curlywedge$-distributive, the same is true of $(f(L_1), \curlyvee_2, \curlywedge_2)$. Similarly if $(L_1, \curlyvee_1, \curlywedge_1)$ is $\curlyvee$-distributive, or weakly- $\curlyvee$-distributive, or feebly- $\curlyvee$-distributive, the same is true of $(f(L_1), \curlyvee_2, \curlywedge_2)$.*

Proof. Easy. ■

Proposition 2.12 *If L_1 has minimum element 0_1 (maximum element 1_1) then $f(L_1)$ has minimum element $f(0_1)$ (maximum element $f(1_1)$). If f is onto and L_2 has minimum element 0_2 (maximum element 1_2), then $0_2 = f(0_1)$ ($1_2 = f(1_1)$).*

Proof. Easy. ■

Similarly to classical lattice theory, we have the following definitions.

Definition 2.13 *Let f be a superlattice homomorphism from $(L_1, \curlyvee_1, \curlywedge_1)$ to $(L_2, \curlyvee_2, \curlywedge_2)$. Suppose that L_2 has minimum element 0_2; then the* kernel *of f is denoted by $Ker(f)$ and defined by $Ker(f) \doteq \{x \in L_1 : f(x) = 0_2\}$. Suppose that L_2 has maximum element 1_2; then the* dual kernel *of f is denoted by $DKer(f)$ and defined by $DKer(f) \doteq \{y \in L_1 : f(y) = 1_2\}$.*

3 First Representation Result

Our first result explains how to represent a general (P, Q)-superlattice in terms of an isomorphic (P, Q)-superlattice of sets; it is analogous to a classical theorem about the isomorphism of finite distributive lattices to posets of sets [1].

3.1 Lattice Isomorphisms and Superlattices

In the definitions and propositions of this section we take $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ to be lattices, denoting the respective partial orders by $\leq_1$ and $\leq_2$. We also take $f : L_1 \to L_2$ to be a (classical) lattice homomorphism. Finally we fix sets $(P, Q) \in \mathbf{A}(L_1) \times \mathbf{B}(L_1)$.

Proposition 3.1 $(f(P), f(Q)) \in \mathbf{A}(f(L_1)) \times \mathbf{B}(f(L_1))$.

Proof. Let $a \in f(L_1)$, then exist $b \in L_1$ and $p \in P$ such that $f(b) = a$ and $p \leq_1 b$. Hence, by Proposition 2.10, we have $f(p) \leq_2 f(b) = a$, i.e. for all $a \in f(L_1)$ exists some $p_1 = f(p) \in f(P)$ such that $p_1 \leq_2 a$. Hence $f(P) \in \mathbf{A}(f(L_1))$. We can prove dually that $f(Q) \in \mathbf{B}(f(L_1))$. ■

The hyperstructures $(L_1, \overset{P}{\vee}_1, \overset{Q}{\wedge}_1)$ and $(f(L_1), \overset{f(P)}{\vee_2}, \overset{f(Q)}{\wedge_2})$ are defined in the obvious way. Since $(P, Q) \in \mathbf{A}(L_1) \times \mathbf{B}(L_1)$, it follows that $(L_1, \overset{P}{\vee}_1, \overset{Q}{\wedge}_1)$ is a (P, Q)-superlattice. In addition we have the following.

Proposition 3.2 *The hyperstructure $(f(L_1), \overset{f(P)}{\vee_2}, \overset{f(Q)}{\wedge_2})$ is a $(f(P), f(Q))$-superlattice.*

Proof. This follows from the fact that $(f(L_1), \vee_2, \wedge_2)$ is a lattice, taking into account Proposition 3.1. ■

Corollary 3.3 *If f is onto, then $(L_2, \overset{f(P)}{\vee_2}, \overset{f(Q)}{\wedge_2})$ is a $(f(P), f(Q))$-superlattice.*

Proof. Immediate. ■

Proposition 3.4 *If f is onto, then it is a strong superlattice homorphism from $(L_1, \overset{P}{\vee}_1, \overset{Q}{\wedge}_1)$ to $(L_2, \overset{f(P)}{\vee}_2, \overset{f(Q)}{\wedge}_2)$.*

Proof. In [5] we have shown: for all $a, b \in L_1$ we have $f(a \overset{P}{\vee}_1 b) = f(a) \overset{f(P)}{\vee}_2 f(b)$. It can be proved dually that for every $a, b \in L_1$ we have $f(a \overset{Q}{\wedge}_1 b) = f(a) \overset{f(Q)}{\wedge}_2 f(b)$. ■

Proposition 3.5 *If f is onto, and L_2 has minimum element 0_2 and maximum element 1_2, then the following are equivalent.*

1. *$(L_2, \overset{f(P)}{\vee}_2, \overset{f(Q)}{\wedge}_2)$ reduces to $(L_2, \vee_2, \wedge_2)$.*
2. *$P \subseteq Ker(f)$ and $Q \subseteq DKer(f)$.*

Proof. We have shown in [5] that $(L_2, \overset{f(P)}{\vee}_2, \wedge_2)$ reduces to $(L_2, \vee_2, \wedge_2)$ (i.e. for all $a, b \in L_1$ we have $f(a) \overset{P}{\vee}_2 f(b) = f(a) \vee_2 f(b)$) iff $P \subseteq Ker(f)$.

Similarly, if $Q \subseteq DKer(f)$, then for all $a, b \in L_1$ we will have

$$f(a) \overset{f(Q)}{\wedge}_2 f(b) = f(a) \wedge_2 f(b) \wedge_2 f(Q) = f(a) \wedge_2 f(b) \wedge_2 1_2 = f(a) \wedge_2 f(b).$$

Conversely, suppose that for all $a, b \in L_1$ we have $f(a) \overset{f(Q)}{\wedge}_2 f(b) = f(a) \wedge_2 f(b)$. Choose any $a \in L_1$; we have $f(a) \overset{f(Q)}{\wedge}_2 f(a) = f(a) \wedge_2 f(Q)$ and also $f(a) \overset{f(Q)}{\wedge}_2 f(a) = f(a) \wedge_2 f(a) = f(a)$. It follows that for all $q \in Q$ we have $f(a) \leq_2 f(q)$. In particular, for a such that $f(a) = 1_2$ we get $1_2 \leq_2 f(q) \Rightarrow 1_2 = f(q)$. Hence $f(Q) = 1_2 \Rightarrow Q \in DKer(f)$. ■

Proposition 3.6 *Let f be an* isomorphism *from $(L_1, \vee_1, \wedge_1)$ to $(L_2, \vee_2, \wedge_2)$ and suppose that L_1 has 0_1 and 1_1. Then: $(L_1, \overset{P}{\vee}_1, \overset{Q}{\wedge}_1)$ is a proper (P, Q)-superlattice iff $(L_2, \overset{f(P)}{\vee}_2, \overset{f(Q)}{\wedge}_2)$ is a proper $(f(P), f(Q))$-superlattice.*

Proof. In [7] we have shown that: if $(L_1, \overset{P}{\vee}_1, \overset{Q}{\wedge}_1)$ is a proper (P,Q)-superlattice, then card$(P) \geq 2$, card$(Q) \geq 2$. Also, f is a lattice isomorphism from L_1 to L_2 and L_1 possesses 0_1 and 1_1; hence L_2 possesses 0_2 and 1_2. Finally, $Ker(f) = \{0_1\} \subset P$ and $DKer(f) = \{1_1\} \subset Q$. Then, by Proposition 3.5 $(L_2, \overset{f(P)}{\vee}_2, \overset{f(Q)}{\wedge}_2)$ is a proper $(f(P), f(Q))$-superlattice.

Conversely, if $(L_2, \overset{f(P)}{\vee}_2, \overset{f(Q)}{\wedge}_2)$ is a proper $(f(P), f(Q))$-superlattice then card$(f(P)) \geq 2$, card$(f(Q)) \geq 2$, which implies (since f is an isomorphism) that card$(P) \geq 2$, card$(Q) \geq 2$, which in turn implies (as shown in [7]) that $(L_1, \overset{P}{\vee}_1, \overset{Q}{\wedge}_1)$ is a proper (P,Q)-superlattice. ■

Corollary 3.7 *Let L_1 and L_2 be finite, and let f be an* isomorphism *from $(L_1, \vee_1, \wedge_1)$ to $(L_2, \vee_2, \wedge_2)$. If $(L_1, \overset{P}{\vee}_1, \overset{Q}{\wedge}_1)$ is a proper (P,Q)-superlattice, then $(L_2, \overset{f(P)}{\vee}_2, \overset{f(Q)}{\wedge}_2)$ is a proper $(f(P), f(Q))$-superlattice.*

Proof. Immediate. ■

3.2 Isomorphism of an Arbitrary (P,Q)-superlattice to a (P,Q)-Superlattice of Sets

In this section we assume $(L, \vee, \wedge)$ to be finite and distributive. Since $(L, \vee, \wedge)$ is finite, it posseses minimum element 0 and maximum element 1.

Definition 3.8 *An element $a \in L$ is called* $\vee$-irreducible *iff*

$$\forall b, c \in L: \quad a = b \vee c \Rightarrow (a = b \text{ or } a = c).$$

Definition 3.9 *The set of all nonzero $\vee$-irreducible elements of L is denoted by J_L, i.e.*

$$J_L \doteq \{x : x \text{ is } \vee\text{-irreducible}, x \neq 0\}.$$

For all $a \in L$ we define

$$\Phi(a) \doteq \{x : x \text{ is } \vee\text{-irreducible}, 0 < x \leq a\}. \qquad (1)$$

Proposition 3.10 *(i)For all $a, b \in L$: $a \leq b \Rightarrow \Phi(a) \subseteq \Phi(b)$;*
(ii) $\Phi(0) = \emptyset$;
(iii) $\Phi(1) = \cup_{a \in L} \Phi(a) = J_L$.

Proof. Obvious. ■

Definition 3.11 *A set* $X \subseteq J_L$ *is called* hereditary *iff for every* $x \in X$ *and* $y \in J_L$ *we have*

$$y \leq x \Rightarrow y \in X.$$

The set of all hereditary subsets of J_L *is denoted by* $H(J_L)$*.*

Proposition 3.12 $(H(J_L), \cup, \cap)$ *is a lattice with the set inclusion ordering order, and minimum element* $\emptyset$*, maximum element* J_L*.*

Proof. See [1]. ■

We now present the main representation result of this section.

Proposition 3.13 *Let* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *be a proper* (P, Q)*-superlattice. Define*

$$P' \doteq \{\{x \in J_L : x \leq p\}\}_{p \in P}, \quad Q' \doteq \{\{y \in J_L : y \leq q\}\}_{q \in Q};$$

then Φ *(as given by eq.(1)) is a superlattice isomorphism:*

$$(L, \overset{P}{\vee}, \overset{Q}{\wedge}) \xrightarrow{\Phi} \left(H(J_L), \overset{P'}{\cup}, \overset{Q'}{\cap} \right).$$

Proof. Since $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice, we will have $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$, which implies: $0 \in P$ and $1 \in Q$. In [5] we have shown that $\emptyset \in P'$and consequently $P' \in \mathbf{A}(H(J_L))$. Similarly,

$$J_L = \{x \in J_L : x \leq 1\} \in \{\{y \in J_L : y \leq q\}\}_{q \in Q} = Q'$$

and $J_L \in Q' \Rightarrow Q' \in \mathbf{B}(H(J_L))$.

According to a classical theorem [1] the mapping Φ is a lattice isomorphism:

$$(L, \vee, \wedge) \xrightarrow{\Phi} \left(H(J(L)), \cup, \cap \right).$$

It follows immediately that Φ is also a superlattice isomorphism:

$$(L, \overset{P}{\vee}, \overset{Q}{\wedge}) \xrightarrow{\Phi} \left(H(J(L)), \overset{\Phi(P)}{\cup}, \overset{\Phi(Q)}{\cap} \right).$$

But we have

$$\Phi(P) = \{\Phi(p)\}_{p\in P} = \{\{z \in J_L : z \le p\}\}_{p\in P} = P'$$
$$\Phi(Q) = \{\Phi(q)\}_{q\in Q} = \{\{w \in J_L : w \le q\}\}_{q\in Q} = Q'$$

which yield the required result. ∎

Remark. Proposition 3.13 is a superlattice version of Theorem 7.9 of [1]. For this classical result it is required that $(L, \vee, \wedge)$ is a distributive finite lattice. The same assumption is required for the "carrier" lattice of Proposition 3.13. However, this does not mean that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is $\overset{P}{\vee}$- and $\overset{Q}{\wedge}$-distributive. In fact, in [8] we have shown that:

1. proper $\overset{P}{\vee}$-distributive and -$\overset{Q}{\wedge}$-distributive (P, Q)-superlattices *do not exist*;
2. proper *feebly* $\overset{P}{\vee}$-distributive and -$\overset{Q}{\wedge}$-distributive (P, Q)-superlattices *exist* under appropriate conditions;
3. *every* (P, Q)-superlattice obtained from a distributive lattice is *weakly*-$\overset{P}{\vee}$-distributive and *weakly*-$\overset{Q}{\wedge}$-distributive.

Hence, in comparing the classical and the superlattice result, it is interesting to note that both require $\vee$- and $\wedge$-distributivity of $(L, \vee, \wedge)$, but the superlattice result does not require (regular) $\overset{P}{\vee}$- and $\overset{Q}{\wedge}$-distributivity of $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$.

Corollary 3.14 *Let $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ be a finite proper (P, Q)-superlattice such that for every $a, b \in L$ we have*

$$\left(a \overset{P}{\vee} x = b \overset{P}{\vee} x \text{ and } a \overset{Q}{\wedge} x = b \overset{Q}{\wedge} x \right) \Rightarrow a = b. \tag{2}$$

Then, for every *pair $(P_1, Q_1) \in \mathbf{A}(L) \times \mathbf{B}(L)$ with card(P_1)≥ 2 and card(Q_1)≥ 2 we have that Φ (as given by eq.(1)) is a superlattice isomorphism:*

$$(L, \overset{P_1}{\vee}, \overset{Q_1}{\wedge}) \overset{\Phi}{\to} \left(H(J(L)), \overset{\Phi(P_1)}{\cup}, \overset{\Phi(Q_1)}{\cap} \right). \tag{3}$$

Proof. In [7] we have shown that (2) implies that the carrier lattice $(L, \vee, \wedge)$ is distributive. Then (3) follows from Proposition 3.13. ∎

Remark. Hence, if Φ is a superlattice isomorphism for *some* $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$, it will be (by the above proposition) a superlattice isomorphism for *every* $(P_1, Q_1) \in \mathbf{A}(L) \times \mathbf{B}(L)$.

4 Second Representation Result

Our second result is related to the following question: what are necessary and sufficient conditions for an arbitrary superlattice $(L, \curlyvee, \curlywedge)$ to be represented as a $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ superlattice? We show that one such condition is expressed in terms of associativity of: (a) $\overset{P}{\vee}$ with $\vee$ and (b) $\overset{Q}{\wedge}$ with $\wedge$.

4.1 Representation of an Arbitrary Superlattice as a (P, Q) superlattice

In this section we consider whether an arbitrary superlattice $(L, \curlyvee, \curlywedge)$ is a (P, Q)-superlattice, i.e. if there are sets P and Q such that $\curlyvee = \overset{P}{\vee}$ and $\curlywedge = \overset{Q}{\wedge}$ and, furthermore, P and Q yield a superlattice.

Certain necessary requirements are obvious. First, the ordered set $(L, \leq$ $)$ (where $\leq$ is the order obtained from $\curlyvee, \curlywedge$) must be a lattice $(L, \vee, \wedge)$. Second, if one can identify the set P, it must be $P \in \mathbf{A}(L)$; similarly $Q \in \mathbf{B}(L)$.

In the following propositions we examine the case where $(L, \leq)$ has minimum element 0 and maximum element 1.

Proposition 4.1 *Consider a superlattice* $(L, \curlyvee, \curlywedge)$ *such that* $(L, \leq)$ *is a lattice* $(L, \vee, \wedge)$ *with 0 and 1.* $(L, \curlyvee, \curlywedge)$ *is a* (P, Q) *superlattice iff*

(i) $0 \in 0 \curlyvee 0$ *and* $1 \in 1 \curlywedge 1$*;*

(ii) For all $a, b \in L$ *we have* $a \curlyvee b = a \vee b \vee (0 \curlyvee 0)$ *and* $a \curlywedge b = a \wedge b \wedge (1 \curlywedge 1)$.

Proof. (a) Suppose $(L, \curlyvee, \curlywedge)$ is a (P, Q) superlattice. This implies that exist $P, Q \in \mathbf{P}(L)$ such that:

$$\forall a, b \in L : a \curlyvee b = a \overset{P}{\vee} b = a \vee b \vee P, \tag{4}$$

$$\forall a, b \in L : a \curlywedge b = a \overset{Q}{\wedge} b = a \wedge b \wedge Q. \tag{5}$$

$$P \in \mathbf{A}(L), Q \in \mathbf{B}(L). \tag{6}$$

Now, taking in (4) $a = b = 0$ we obtain $0 \curlyvee 0 = P$; taking in (5) $a = b = 1$ we obtain $1 \curlywedge 1 = Q$. Substituting back in (4) we obtain (ii). Also, it is clear that $P \in \mathbf{A}(L)$ iff $0 \in P = 0 \curlyvee 0$; similarly $Q \in \mathbf{B}(L)$ iff $1 \in Q = 1 \curlywedge 1$; so we obtain (i).

(b) Conversely, suppose that (i) and (ii) hold. Clearly then, $(L, \curlyvee, \curlywedge)$ is a $(L, \overset{0 \curlyvee 0}{\vee}, \overset{1 \curlywedge 1}{\wedge})$ hyperstructure and, for it to be a $(0 \curlyvee 0, 1 \curlywedge 1)$-superlattice, we must have: $0 \curlyvee 0 \in \mathbf{A}(L)$ (which is equivalent to $0 \in 0 \curlyvee 0$) and $1 \curlywedge 1 \in \mathbf{B}(L)$ (which is equivalent to $1 \in 1 \curlywedge 1$). ■

Remark. To apply Proposition 4.1 we must check both (i) and (ii). In particular, we must check (ii) for all pairs $(a, b) \in L \times L$, which may be inconvenient. A simpler criterion can be applied in some special cases, as explained in the next proposition.

Proposition 4.2 *Consider a superlattice $(L, \curlyvee, \curlywedge)$ such that $(L, \leq)$ is a distributive lattice $(L, \vee, \wedge)$ with 0 and 1. If for all $a, b \in L$ we have that $a \curlyvee b$ and $a \curlywedge b$ are intervals, then $(L, \curlyvee, \curlywedge)$ is a (P, Q) superlattice iff*

(i) $0 \curlyvee 0 = [0, p]$ *and* $1 \curlywedge 1 = [q, 1]$*;*

(ii) For all $a, b \in L$ we have $a \curlyvee b = [a \vee b, a \vee b \vee p]$ and $a \curlywedge b = [a \wedge b \wedge q, a \wedge b]$.

Proof. (a) Suppose that $(L, \curlyvee, \curlywedge)$ is a (P, Q) superlattice. Then Proposition 4.1 holds and so $0 \in 0 \curlyvee 0$; but, by assumption, $0 \curlyvee 0$ is an interval, so it must be of the form $[0, p]$; similarly we obtain $1 \curlywedge 1 = [q, 1]$ and so we have shown (i). Again by Proposition 4.1 we have $a \curlyvee b = a \vee b \vee [0, p] = [a \vee b, a \vee b \vee p]$ by distributivity; similarly we obtain $a \curlywedge b = [a \wedge b \wedge q, a \wedge b]$ and we have shown (ii).

(b) Conversely, suppose that (i) and (ii) hold; then it is easy to show that $(L, \curlyvee, \curlywedge)$ is a $(L, \overset{[0,p]}{\vee}, \overset{[q,1]}{\wedge})$ superlattice. ■

However, the above proposition applies only to the special case where P and Q are intervals. The next proposition gives a criterion of a more *fundamental* nature. This criterion is related to some *associativity* properties between the $\vee$, $\curlyvee$ and $\wedge$, $\curlywedge$ operations.

Proposition 4.3 *Consider a superlattice $(L, \curlyvee, \curlywedge)$ such that $(L, \leq)$ is a lattice $(L, \vee, \wedge)$ with 0 and 1. $(L, \curlyvee, \curlywedge)$ is a (P, Q) superlattice iff*

(i) $0 \in 0 \curlyvee 0$ *and* $1 \in 1 \curlywedge 1$*;*

(ii) For all $a, b \in L$ we have $a \vee (b \curlyvee c) = (a \vee b) \curlyvee c$ and $a \wedge (b \curlywedge c) = (a \wedge b) \curlywedge c$.

Proof. (a) Suppose that $(L, \curlyvee, \curlywedge)$ is a (P, Q) superlattice. From Proposition 4.1 we obtain (i). Also we obtain that: for all $b, c \in L$ we have $b \curlyvee c = b \vee c \vee (0 \curlyvee 0)$. Then we also have

$$a \vee (b \curlyvee c) = a \vee b \vee c \vee (0 \curlyvee 0) = (a \vee b) \vee c \vee (0 \curlyvee 0) = (a \vee b) \curlyvee c.$$

We can prove similarly that $a \wedge (b \curlywedge c) = (a \wedge b) \curlywedge c$; so we have shown (ii).

(b) Conversely, suppose that (i) and (ii) hold. Then from (ii), for all $a, b \in L$ we have $a \vee b \vee (0 \curlyvee 0) = a \vee (b \vee (0 \curlyvee 0)) = a \vee ((b \vee 0) \curlyvee 0) =$

$a \vee (b \curlyvee 0) = a \vee (0 \curlyvee b) = (a \vee 0) \curlyvee b = a \curlyvee b$. Similarly we can show that for all $a, b \in L$ we have $a \curlywedge b = a \wedge b \wedge (1 \curlywedge 1)$; and so we have shown the second condition of Proposition 4.1; the first condition is the same as (i) of the current proposition. Hence the conditions of the current proposition imply the two conditions of Proposition 4.1, which in turn imply that $(L, \curlyvee, \curlywedge)$ is a (P, Q) superlattice. ■

Remark. In effect, the above proposition says that the *defining* property of a (P, Q) superlattice is given by the "associative" identities $a \vee (b \curlyvee c) = (a \vee b) \curlyvee c$ and $a \wedge (b \curlywedge c) = (a \wedge b) \curlywedge c$. In the next section we will elaborate this point by considering in some detail the differences between (P, Q)-superlatices and *Nakano superlattices*.

4.2 Comparison of (P, Q)-superlatices and Nakano superlattices

Recall the following from [2].

Definition 4.4 *Given a modular lattice* $(L, \vee, \wedge)$*, define the following hyperoperations:* $\sqcup : L \times L \to \mathbf{P}(L)$ *is given for all* $a, b \in L$ *by* $a \sqcup b \doteq \{x : a \vee b = a \vee x = b \vee x\}$ *and* $\sqcap : L \times L \to \mathbf{P}(L)$ *is given for all* $a, b \in L$ *by* $a \sqcap b \doteq \{x : a \wedge b = a \wedge x = b \wedge x\}$

Proposition 4.5 *The hyperctructure* $(L, \sqcup, \sqcap)$ *is a superlattice (the so-called* Nakano superlattice*).*

Now consider the Nakano superlattice $(L, \sqcup, \sqcap)$, built on the lattice of Figure 1. Is there a choice of P, Q such that $(L, \sqcup, \sqcap)$ is identical to a $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$? The answer is negative. For, by Proposition 4.3 we should then have $1 \vee (a \sqcup a) = (1 \vee a) \sqcup a$. But we have $1 \vee (a \sqcup a) = 1 \vee \{0, a\} = \{1\}$; while $(1 \vee a) \sqcup a = 1 \sqcup a = \{b, 1\}$.

Figure 1 here

Let us generalize the above remark by proving the following proposition.

Proposition 4.6 *Let* $(L, \vee, \wedge)$ *be a modular lattice with 0 and 1, and denote the associated Nakano superlattice by* $(L, \sqcup, \sqcap)$*. Then:*

(i) If exist $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ *such that for all* $a, b \in L$ *we have* $a \sqcup b = a \overset{P}{\vee} b$ *or* $a \sqcap b = a \overset{Q}{\wedge} b$*, then card(L) = card(P) = card(Q) = 1.*

(ii) If exist $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ *such that for all* $a, b \in L$ *we have* $a \sqcup b = a \overset{Q}{\wedge} b$ *or* $a \sqcap b = a \overset{P}{\vee} b$ *, then card(L) = card(P) = card(Q) =1.*

Proof. (i) We know that $a \sqcup a = \{x : a \vee a = x \vee a\} = \{x : x \leq a\}$. Hence $\max(a \sqcup a) = a$. We also know from Proposition 2.8 that $\min(a \overset{P}{\vee} a) = a$. Since $a \sqcup a = a \overset{P}{\vee} a$, it follows that (for all $a \in L$) $a \sqcup a = a \overset{P}{\vee} a = \{a\}$. Fix some $a \in L$ and choose any $b \in L$. We have $a \wedge b \leq a \vee b$. Hence $a \wedge b \in (a \vee b) \sqcup (a \vee b) = \{a \vee b\}$, i.e. $a \wedge b = a \vee b$, which means $a = b$. In short: $b \in L \Rightarrow b = a$, which yields the required result immediately. Note that we could have started with $a \sqcap b = a \overset{Q}{\wedge} b$ and proceed dually to obtain the same result.

(ii) Fix some $a \in L$ and choose any $b \in L$. We have: $x \in a \overset{P}{\vee} b = \{a \vee b \vee p\}_{p \in P}$, hence $x = a \vee b \vee p$ (for some $p \in P$). Also, $x \in a \overset{P}{\vee} b = a \sqcap b \Rightarrow a \wedge b = a \wedge x = b \wedge x$. This implies two things: (a) $a \wedge b = a \wedge (a \vee b \vee p) = a$; (b) $a \wedge b = b \wedge (a \vee b \vee p) = b$. In short, $a = b$, i.e. $L = \{a\}$, which yields the required result immediately. Note that we could have started with $a \sqcup b = a \overset{Q}{\wedge} b$ and proceed dually to obtain the same result. ■

Further differences in the associativity behavior of the Nakano and (P, Q)-superlattices are given by Proposition 4.7.

Proposition 4.7 *Given a Nakano superlattice* $(L, \sqcup, \sqcap)$ *we have for all* $a, b, c \in L$*:*

$(i) \quad a \vee (b \sqcup c) \subseteq (a \vee b) \sqcup c \subseteq (a \vee b) \sqcup (a \vee c);$

$(ii) \quad a \wedge (b \sqcap c) \subseteq (a \wedge b) \sqcap c \subseteq (a \wedge b) \sqcap (a \wedge c).$

Proof. Choose any $a, b, c \in L$ and any $y \in a \vee (b \sqcup c)$. Then exists some $x \in b \sqcup c$ such that $y = a \vee x$. Since $x \in b \sqcup c$, we must have

$$x \vee b = x \vee c = b \vee c. \tag{7}$$

Now for (i), note that (7) implies: $(x \vee a) \vee (b \vee a) = (x \vee a) \vee c = (a \vee b) \vee c \Rightarrow y = \ x \vee a \in (a \vee b) \sqcup c \Rightarrow (a \vee b) \sqcup c \subseteq (a \vee b) \sqcup c$.

Further: take any $z \in (a \vee b) \sqcup c \Rightarrow z \vee (a \vee b) = z \vee c = c \vee (a \vee b) \Rightarrow z \vee (a \vee b) = z \vee (a \vee c) = (a \vee c) \vee (a \vee b) \Rightarrow z \in (a \vee b) \sqcup (a \vee c) \Rightarrow (a \vee b) \sqcup c \subseteq \ (a \vee b) \sqcup (a \vee c)$.

(ii) is proved dually. ■

Remark. Note that in a (P, Q) superlattice (i) and (ii) hold with equality rather than inclusion. On the other hand, in a Nakano superlattice the inclusions in (i) and (ii) can be proper. To see this, consider the lattice of Figure 2.

Figure 2 here

In the lattice above, we see that $a \vee (b \sqcup c) = a \vee \{1, a\} = \{1, a\}$; on the other hand $(a \vee b) \sqcup c = 1 \sqcup c = \{a, b, 1\}$; finally, $(a \vee b) \sqcup (a \vee c) = 1 \sqcup 1 = \{0, a, b, c, 1\}$. Hence we see an example where both inclusions in (i) are proper. Similar results can be obtained for (ii).

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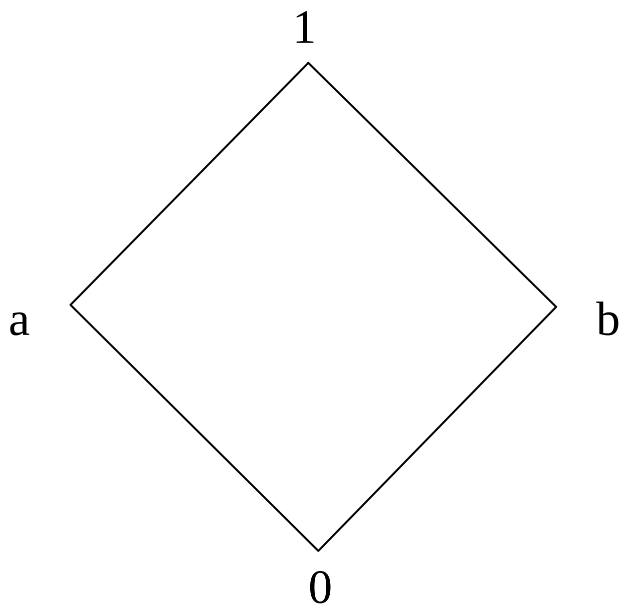

Figure 1

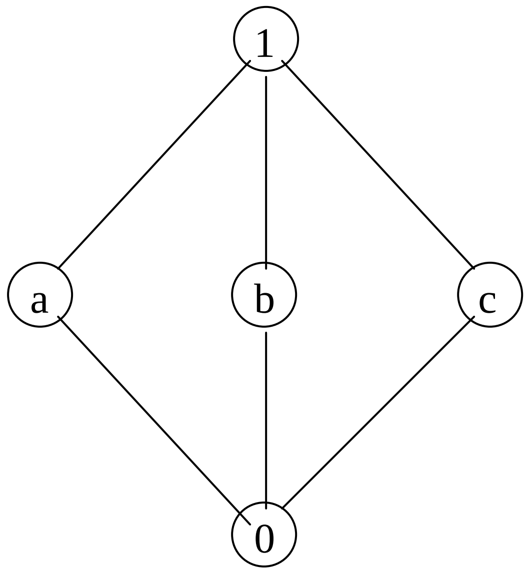

Figure 2