The Structure of the (P, Q)-Superlattice and Order Related Properties

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Abstract

In a previous work we have introduced the (P, Q)-superlattice, a hyperstructure of the form $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$. Here $(L, \vee, \wedge)$ is a lattice and the hyperoperations $\overset{P}{\vee}, \overset{Q}{\wedge}$ are defined by $a \overset{P}{\vee} b \doteq a \vee b \vee P$, $a \overset{Q}{\wedge} b \doteq a \wedge b \wedge Q$; when the sets $P, Q \subseteq L$ satisfy appropriate conditions $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice. In this work we continue the investigation of (P, Q)-superlattice and consider the structure of the sets $a \overset{P}{\vee} b$, $a \overset{Q}{\wedge} b$ as well assome "order-like" relationships between such sets.

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1 Introduction

The (P, Q)-superlattice has been introduced in [5], and its properties studied in [7]. Starting from a lattice $(L, \vee, \wedge)$, one can define a hyperstructure of the form $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$; if P, Q are chosen appropriately, then $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice.

In this paper we examine the structure of the sets $a \overset{P}{\vee} b$ and $a \overset{Q}{\wedge} b$ in connection to the properties of P, Q. Furthermore we consider certain "order-like" relationships between such sets.

2 The (P,Q)-Superlattice and Some of Its Properties

Let us first give the definition of a general superlattice, as given in [6]. In what follows $\mathbf{P}(L)$ will denote the power set of a reference set L.

Definition 2.1 *A superlattice is a partially ordered set* $(L, \leq)$ *with two hyperoperations* $\curlyvee, \curlywedge$*, where* $\curlyvee : L \times L \to \mathbf{P}(L)$*,* $\curlywedge : L \times L \to \mathbf{P}(L)$*, and the following properties are satisfied for all* $a, b, c \in L$*.*

S1 $a \in (a \curlyvee a) \cap (a \curlywedge a)$

S2 $a \curlyvee b = b \curlyvee a, a \curlywedge b = b \curlywedge a$

S3 $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c), (a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$

S4 $a \in [(a \curlyvee b) \curlywedge a] \cap [(a \curlywedge b) \curlyvee a]$

S5a $a \leq b \Rightarrow (b \in a \curlyvee b$ *and* $a \in a \curlywedge b)$

S5b $(b \in a \curlyvee b$ *or* $a \in a \curlywedge b) \Rightarrow a \leq b$.

As has been shown in [6], the following definition is equivalent to Definition 2.1,

Definition 2.2 *A superlattice is a hyperstructure* $(L, \curlyvee, \curlywedge)$*, where* $\curlyvee : L \times L \to \mathbf{P}(L)$*,* $\curlywedge : L \times L \to \mathbf{P}(L)$*, and the following properties are satisfied for all* $a, b, c \in L$*.*

S1 $a \in (a \curlyvee a) \cap (a \curlywedge a)$

S2 $a \curlyvee b = b \curlyvee a, a \curlywedge b = b \curlywedge a$

S3 $(a \curlyvee b) \curlyvee c = a \curlyvee (b \curlyvee c), (a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$

S4 $a \in [(a \curlyvee b) \curlywedge a] \cap [(a \curlywedge b) \curlyvee a]$

S6 $b \in a \curlyvee b \Leftrightarrow a \in a \curlywedge b$

S7 $a, b \in a \curlyvee b \Rightarrow a = b$

S8 $b \in a \curlyvee b$ *et* $c \in b \curlyvee c \Rightarrow c \in a \curlyvee c$.

Definition 2.3 *A superlattice* $(L, \curlyvee, \curlywedge)$ *will be called* proper *iff there exist pairs* (a, b)*,* $(c, d) \in L \times L$*, such that* $card(a \curlyvee b) \geq 2$ *and* $card(c \curlywedge d) \geq 2$.

Definition 2.4 *A superlattice* $(L, \curlyvee, \curlywedge)$ *will be called* strictly strong *iff: (a) the corresponding ordered set* $(L, \leq)$ *is a lattice with* sup *operation* $\vee$ *and* inf *operation* $\wedge$ *and (b) for all* $a, b \in L$ *we have:* $a \vee b \in a \curlyvee b$ *and* $a \wedge b \in a \curlywedge b$*.*

Finally, let us mention the manner in which a hyperlattice and a dual hyperlattice[1] can be obtained from a superlattice.

Proposition 2.5 *Suppose* $(L, \curlyvee, \curlywedge)$ *is a strictly strong superlattice with corresponding order* $\leq$ *and that* $(L, \leq)$ *is a lattice with* sup *operation denoted by* $\vee$ *and* inf *operation denoted by* $\wedge$. *Then: (i)* $(L, \curlyvee, \wedge)$ *is a hyperlattice; (ii)* $(L, \vee, \curlywedge)$ *is a dual hyperlattice.*

Proof. (i) Suppose that $(L, \curlyvee, \curlywedge)$ is a superlattice; then S1–S8 hold. Now, the hyperlattice axioms H1-H3 [2] are identical with the superlattice axioms S1-S3. Also, if $(L, \curlyvee, \curlywedge)$ is a strictly strong superlattice then $a \wedge b \in a \curlywedge b$ and $a \vee b \in a \curlyvee b$. Thus

$$a = (a \wedge b) \vee a \in \ (a \wedge b) \curlyvee a \text{ and } a = (a \vee b) \wedge a \in \ (a \curlyvee b) \wedge a$$

which is the hyperlattice axiom H4 (see [2]). Finally, recall that S5 is

$$(a \in a \curlyvee b \text{ or } b \in a \curlywedge b) \Rightarrow b \leq a.$$

and hence: $a \in a \curlyvee b \Rightarrow b \leq a \Rightarrow b = b \wedge a$, which is hyperlattice axiom H5. In short, (S1–S8) $\Rightarrow$ (H1–H5) and so $(L, \curlyvee, \wedge)$ is a hyperlattice.

(ii) is proved dually. ■

We now turn to the study of the (P, Q)-superlattice, which has been introduced in [5]. It is a special kind of superlattice, which can be considered as a generalization of either the P-hyperlattice [3, 4] or the dual Q-hyperlattice. A (P, Q)-superlattice is constructed on a lattice $(L, \vee, \wedge)$ in a manner analogous to the construction of P-hypergroups [1, 9, 10] and P-hyperrings [8].

In what follows, $(L, \vee, \wedge)$ will always denote a lattice (with $L \neq \emptyset$) and $\leq$ will denote the order of $(L, \vee, \wedge)$. If L possesses a minimum (respectively maximum) element, this will be denoted by 0 (respectively 1).

Given a lattice $(L, \vee, \wedge)$ let us select two sets $P, Q \in \mathbf{P}(L)$ and define the following hyperoperations.

Definition 2.6 *For all* $a, b \in L$ *we define* $a \overset{P}{\vee} b \doteq a \vee b \vee P = \{a \vee b \vee p : p \in P\}$.

Definition 2.7 *For all* $a, b \in L$ *we define* $a \overset{Q}{\wedge} b \doteq a \wedge b \wedge Q = \{a \wedge b \wedge q : q \in Q\}$.

[1] A *dual hyperlattice* can be defined as a hyperstructure $(L, \vee, \curlywedge)$; here $\vee$ is the classical sup operation and $\curlywedge$ is a hyperoperation which generalizes the classical $\wedge$ (inf) operation. As the name suggests, it is dual to the hyperlattice $(L, \curlyvee, \wedge)$.

Remark. A $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ structure (with arbitrary choice of P, Q) is not necessarily a superlattice [7].

The necessary and sufficient conditions on P, Q for $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ to be a superlattice are easily stated in terms of the following two collections of sets.

Definition 2.8 $\mathbf{A}(L) \doteq \{A \in \mathbf{P}(L) : \forall x \in L \quad \exists a \in A$ *such that* $a \leq x\}$.

Definition 2.9 $\mathbf{B}(L) \doteq \{B \in \mathbf{P}(L) : \forall y \in L \quad \exists b \in B$ *such that* $y \leq b\}$.

It is clear that $L \in \mathbf{A}(L) \cap \mathbf{B}(L)$. Also, clearly, if $(L, \vee, \wedge)$ has a 0, then $P \in \mathbf{A}(L) \Leftrightarrow 0 \in P$; if $(L, \vee, \wedge)$ has a 1, then $Q \in \mathbf{B}(L) \Leftrightarrow 1 \in Q$. Furthermore, we have the following:

$(P \in \mathbf{A}(L)$ and P is an interval$) \;\Rightarrow\; L$ has 0;

$(Q \in \mathbf{B}(L)$ and Q is an interval$) \;\Rightarrow\; L$ has 1.

The following proposition yields a necessary and sufficient condition for $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ to be a superlattice.

Proposition 2.10 $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is a superlattice* $\Leftrightarrow (P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$.

Proof. The proof appears in [5]. ■

Remark. Given a lattice $(L, \vee, \wedge)$ and $P \in \mathbf{A}(L)$, , $Q \in \mathbf{B}(L)$ we construct the associated superlattice $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$. Now, the hyperoperations $\overset{P}{\vee}, \overset{Q}{\wedge}$ induce an order $\leqslant$ on L; it is easy to see that $\leqslant$ is identical with the $\leq$ order of the original $(L, \vee, \wedge)$ lattice.

Remark. In the sequel we will assume (unless explicitly stated otherwise) that $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$; hence $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ will be a superlattice.

Proposition 2.11 *For all* $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ *and all* $a, b \in L$ *we have: (i)* $a \vee b = \min(a \overset{P}{\vee} b)$*, (ii)* $a \wedge b = \max(a \overset{Q}{\wedge} b)$*.*

Proof. See [7]. ■

Remark. It follows that for all $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ we have that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a *strictly strong* superlattice.

Let us now introduce the concept of a sub-superlattice of a (P, Q)-superlattice.

Definition 2.12 *Let* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *be a* (P, Q)*-superlattice and* $S \in \mathbf{P}(L)$*. We say that* $(S, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is a* sub-superlattice *of* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *iff* $\forall a, b \in S$ *we have* $a \overset{P}{\vee} b \subseteq S$, $a \overset{Q}{\wedge} b \subseteq S$.

The next proposition gives necessary and sufficient condition for a subset of L to be a sub-superlattice of $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$.

Proposition 2.13 *Let* $(L, \vee, \wedge)$ *be a lattice and* $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ *(hence* $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *is* (P, Q)*-superlattice). Let* S *be such that* $P \cup Q \subseteq S \subseteq L$*. Then*

$$(S, \overset{P}{\vee}, \overset{Q}{\wedge}) \textit{ is subsuperlattice of } (L, \overset{P}{\vee}, \overset{Q}{\wedge}) \Leftrightarrow (S, \vee, \wedge) \textit{ is sublattice of } (L, \vee, \wedge).$$

Proof. (i) Assume that $(S, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a sub-superlattice of $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$. Choose any $a, b \in S$. Then $a \vee b \in a \overset{P}{\vee} b \subseteq S$; similarly we show that $a \wedge b \in S$ and so we conclude that S is a sublattice.

(ii) Assume that $(S, \vee, \wedge)$ is a sublattice of $(L, \vee, \wedge)$. Then for all $a, b \in S$ we have $a \vee b \in S$; since also $P \subseteq S$, we have for all $a, b \in S$ and for all $p \in P$ that $a \vee b \vee p \in S$. Then we have for all $a, b \in S : \cup_{p \in P} a \vee b \vee p \subseteq S$, which implies that for all $a, b \in S : a \overset{P}{\vee} b \subseteq S$. Similarly we prove that for all $a, b \in S : a \overset{Q}{\wedge} b \subseteq S$. Hence we conclude that $(S, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a sub-superlattice of $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$. ■

Proposition 2.14 *For all* $a, b \in L$ *we have: (i)* $a \overset{L}{\vee} b = \{x : a \vee b \leq x\}$*, (ii)* $a \overset{L}{\wedge} b = \{x : x \leq a \wedge b\}$.

Proof. (i) has been proved in [3]; (ii) can be proved dually. ■

Remark. Given $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ with minimum element 0 and maximum element 1, we have $0 \overset{P}{\vee} 0 = P$, $1 \overset{Q}{\wedge} 1 = Q$.

Proposition 2.15 *For all* $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$ *we have:* $P \vee Q \in \mathbf{B}(L)$ *and* $P \wedge Q \in \mathbf{A}(L)$.

Proof. For every $a \in L$ there is some $q \in Q$ such that $a \leq q$. But $Q \subseteq P \vee Q \Rightarrow q \in P \vee Q$. In short, for every $a \in L$ there is some $q \in P \vee Q$ such that $a \leq q$ and so $P \vee Q \in \mathbf{B}(L)$; $P \wedge Q \in \mathbf{A}(L)$ is proved dually. ■

3 The Structure of $a \overset{P}{\vee} b$ and $a \overset{Q}{\wedge} b$

In this section we examine the structure of $a \overset{P}{\vee} b$ and $a \overset{Q}{\wedge} b$ under various conditions on P and Q. For the rest of the paper we *always* assume that $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$, unless explicitly stated otherwise.

Proposition 3.1 *For all $a, b \in L$ we have: $x \in a \overset{P}{\vee} b$, $y \in a \overset{Q}{\wedge} b \Rightarrow y \leq x$.*

Proof. $x \in a \overset{P}{\vee} b \Rightarrow x = a \vee b \vee p$; $y \in a \overset{Q}{\wedge} b \Rightarrow y = a \wedge b \wedge q$ (for approriate $p \in P$, $q \in Q$). Obviously $y = a \wedge b \wedge q \leq a \vee b \vee p = x$. ∎

Proposition 3.2 *For all $a, b \in L$ we have: $(a \overset{P}{\vee} b) \cap (a \overset{Q}{\wedge} b\ ) \neq \emptyset \Rightarrow a = b$.*

Proof. Suppose there exists $z \in (a \overset{P}{\vee} b) \cap (a \overset{Q}{\wedge} b)$. Then exists $p \in P$, $q \in Q$ such that $z = a \vee b \vee p = a \wedge b \wedge q$. Then we have

$$z = a \wedge b \wedge q \leq a \wedge b \leq a \vee b \leq a \vee b \vee p = z$$

from which follows immediately that $a = b$. ∎

Proposition 3.3 *(i) If $(L, \vee, \wedge)$ is distributive, then:*

$$(P \textit{ is a sublattice}) \Rightarrow \left(\forall a, b \in L \quad a \overset{P}{\vee} b \textit{ is a sublattice}\right).$$

(ii) If $(L, \vee, \wedge)$ is distributive, then:

$$(Q \textit{ is a sublattice}) \Rightarrow \left(\forall a, b \in L \quad a \overset{Q}{\wedge} b \textit{ is a sublattice}\right).$$

(iii) If $(L, \vee, \wedge)$ has a minimum element 0, then:

$$\left(\forall a, b \in L \quad a \overset{P}{\vee} b \textit{ is a sublattice}\right) \Rightarrow (P \textit{ is a sublattice}).$$

(iv) If $(L, \vee, \wedge)$ has a maximum element 1, then:

$$\left(\forall a, b \in L \quad a \overset{Q}{\wedge} b \textit{ is a sublattice}\right) \Rightarrow (Q \textit{ is a sublattice}).$$

Proof. For (i) assume that P is a sublattice of L. Take any $a, b \in L$. For any $x_1, x_2 \in a \overset{P}{\vee} b$ there exist $p_1, p_2 \in P$ such that $x_1 = a \vee b \vee p_1$ and $x_2 = a \vee b \vee p_2$. Furthermore, $p_1 \vee p_2 = p_3 \in P$, $p_1 \wedge p_2 = p_4 \in P$. Hence $x_1 \vee x_2 = a \vee b \vee p_3 \in a \overset{P}{\vee} b$ and $x_1 \wedge x_2 = (a \vee b \vee p_1) \wedge (a \vee b \vee p_2) = (a \vee b) \vee (p_1 \wedge p_2) = (a \vee b) \vee p_4 \in a \overset{P}{\vee} b$. Part (ii) is proved duallly to (i). Part (iii) is obvious, since $0 \overset{P}{\vee} 0 = 0 \vee P = P$; (iv) is proved dually to (iii). ■

Proposition 3.4 *(i) If* $(L, \vee, \wedge)$ *is distributive, then:* (P *is convex sublattice*) $\Rightarrow$ ($\forall a, b \in L : a \overset{P}{\vee} b$ *is convex sublattice*).

(ii) If $(L, \vee, \wedge)$ *is distributive, then:* (Q *is convex sublattice*) $\Rightarrow$ ($\forall a, b \in L : a \overset{Q}{\wedge} b$ *is convex sublattice*).

(iii) If $(L, \vee, \wedge)$ *has minimum element 0, then:* ($\forall a, b \in L : a \overset{P}{\vee} b$ *is convex sublattice*) $\Rightarrow$ (P *is convex sublattice*).

(iv) If $(L, \vee, \wedge)$ *has maximum element 1, then* ($\forall a, b \in L : a \overset{Q}{\wedge} b$ *is convex sublattice*) $\Rightarrow$ (Q *is convex sublattice*).

Proof. For part (i) assume P is a convex sublattice. Choose any $a, b \in L$ and any $x, y \in a \overset{P}{\vee} b$; i.e. exist p_1 and p_2 such that $x = a \vee b \vee p_1$ and $y = a \vee b \vee p_2$. By Proposition 3.3, $x \vee y \in a \overset{P}{\vee} b$ and $x \wedge y \in a \overset{P}{\vee} b$. Now take any $z \in [x \wedge y, x \vee y] = a \vee b \vee [p_1 \wedge p_2, p_1 \vee p_2] \subseteq a \overset{P}{\vee} b$ (the equality holds because of distributivity, and the inclusion because $[p_1 \wedge p_2, p_1 \vee p_2] \subseteq P$, since P is a convex sublattice). Now part (ii) is proved dually; for (iii) just take $a = b = 0$; for (iv) just take $a = b = 1$. ■

Proposition 3.5 *(i) If* $(L, \vee, \wedge)$ *is distributive, then:*

$$(P \textit{ is an ideal}) \Rightarrow (\forall a, b \in L : a \overset{P}{\vee} b \textit{ is a convex sublattice}).$$

(ii) If $(L, \vee, \wedge)$ *is distributive, then:*

$$(Q \textit{ is a filter}) \Rightarrow (\forall a, b \in L : a \overset{Q}{\wedge} b \textit{ is a convex sublattice}).$$

(iii) If $(L, \vee, \wedge)$ *has a minimum element 0, then:*

$$\left(\forall a, b \in L : a \overset{P}{\vee} b \textit{ is an ideal}\right) \Rightarrow (P \textit{ is an ideal}).$$

(iv) If $(L, \vee, \wedge)$ *has a maximum element 1, then:*

$$\left(\forall a, b \in L : a \overset{Q}{\wedge} b \textit{ is a filter}\right) \Rightarrow Q \textit{ is a filter}).$$

Proof. Part (i) is proved using the fact that an ideal is a convex sublattice and Proposition 3.4; part (ii) is proved using the fact that a filter is a convex sublattice and Proposition 3.4; part (iii) is proved setting $a = b = 0$; part (iv) is proved setting $a = b = 1$. ■

Remark. If $(L, \vee, \wedge)$ is not distributive, then parts (ii), (iv) of Proposition 3.4 do not necessarily hold. Consider the lattice of Figure 1 and take $P = \{c, d\}$. Then $a \overset{P}{\vee} c = a \vee P = a \vee \{c, d\} = \{a, e\}$ which is not convex.

Figure 1

Proposition 3.6 *(i) If* $(L, \vee, \wedge)$ *is distributive, then we have:* (P *is an interval*) $\Rightarrow$ ($\forall a, b \in L : a \overset{P}{\vee} b$ *is an interval*);

(ii) If $(L, \vee, \wedge)$ *is distributive, then we have:* (Q *is an interval*) $\Rightarrow$ ($\forall a, b \in L : a \overset{Q}{\wedge} b$ *is an interval*).

(iii) If $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *has minimum element 0, then:* ($\forall a, b \in L : a \overset{P}{\vee} b$ *is an interval*) $\Rightarrow$ (P *is an interval*);

(iv) If $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ *has maximum element 1, then:* ($\forall a, b \in L : a \overset{Q}{\wedge} b$ *is an interval*) $\Rightarrow$ (Q *is an interval*).

Proof. For part (i) assume $P = [x, y]$, then (using distributivity) $a \overset{P}{\vee} b = a \vee b \vee [x, y] = [a \vee b \vee x, a \vee b \vee y] = [a \vee b, a \vee b \vee y]$, since $\min(a \overset{P}{\vee} b) = a \vee b$. Part (ii) is proved dually; for part (iii) take $a = b = 0$; for part (iv) take $a = b = 1$. ■

Proposition 3.7 *If $(L, \vee, \wedge)$ is a distributive lattice then:*

(i) (P is an interval) $\Rightarrow$ $(\forall a, b \in L$ such that $a \leq b :$ $(a \overset{P}{\vee} c) \vee (b \overset{P}{\vee} c) = b \overset{P}{\vee} c)$;

(ii) (P is an interval) $\Rightarrow$ $(\forall a, b \in L$ such that $a \leq b :$ $(a \overset{P}{\vee} c) \overset{P}{\vee} (b \overset{P}{\vee} c) = b \overset{P}{\vee} c)$;

(iii) (Q is an interval) $\Rightarrow$ $(\forall a, b \in L$ such that $a \leq b :$ $(a \overset{Q}{\wedge} c) \wedge (b \overset{Q}{\wedge} c) = a \overset{Q}{\wedge} c)$;

(iv) (Q is an interval) $\Rightarrow$ $(\forall a, b \in L$ such that $a \leq b :$ $(a \overset{Q}{\wedge} c) \overset{Q}{\wedge} (b \overset{Q}{\wedge} c) = a \overset{Q}{\wedge} c)$.

Proof. (i) Assume $P = [x, y]$, then, since L is distributive, $a \overset{P}{\vee} c = [a \vee c \vee x, a \vee c \vee y]$ and $b \overset{P}{\vee} c = [b \vee c \vee x, b \vee c \vee y]$. Again by distributivity, we have

$$(a \overset{P}{\vee} c) \vee (b \overset{P}{\vee} c) = [a \vee c \vee x, a \vee c \vee y] \vee [b \vee c \vee x, b \vee c \vee y] = [a \vee b \vee c \vee x, a \vee b \vee c \vee y] =$$

$$[b \vee c \vee x, b \vee c \vee y] = b \vee c \vee [x, y] = b \overset{P}{\vee} c.$$

(ii) is proved similarly.

(iii) is proved dually to (i) and (iv) is proved dually to (ii). ■

The next two propositions give additional information on the structure of $\overset{P}{\vee}$ and $\overset{Q}{\wedge}$.

Proposition 3.8 *If $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is a superlattice, then*

$$\left(\forall a \in L : a \overset{P}{\vee} x = a \overset{P}{\vee} a \right) \Leftrightarrow x \leq a; \quad \left(\forall a \in L : a \overset{Q}{\wedge} x = a \overset{Q}{\wedge} a \right) \Leftrightarrow a \leq x.$$

Proof. Pick any $a \in L$. Pick some $x \in L$ such that $a \overset{P}{\vee} x = a \overset{P}{\vee} a \Rightarrow \min(a \overset{P}{\vee} x) = \min(a \overset{P}{\vee} a) \Rightarrow a \vee x = a \vee a = a \Rightarrow x \leq a$. Conversely, assume

$x \le a$; then $a \overset{P}{\vee} x = \cup_{p\in P} a \vee x \vee p = \cup_{p\in P} a \vee p = \cup_{p\in P} a \vee a \vee p = a \overset{P}{\vee} a$. So we have proved the first equivalence; the second equivalence is proved dually. ■

Proposition 3.9 *($(L, \vee, \wedge)$ is distributive)* $\Leftrightarrow$

$$\left(\text{For all } a, x, y \in L \text{ we have: } \left. \begin{array}{c} a \overset{P}{\vee} x = a \overset{P}{\vee} y \\ a \overset{Q}{\wedge} x = a \overset{Q}{\wedge} y \end{array} \right\} \Rightarrow x = y \right). \qquad (1)$$

Proof. See [7]. ■

Remark. Note that if (1) holds for *some* pair $(P_1, Q_1) \in \mathbf{A}(L) \times \mathbf{B}(L)$, then $(L, \vee, \wedge)$ is distributive and so (1) holds for *every* pair $(P, Q) \in \mathbf{A}(L) \times \mathbf{B}(L)$.

Remark. One would expect that the above relations imply that $(L, \overset{P}{\vee}, \overset{Q}{\wedge})$ is distributive. However, we have shown in [7] that this is not the case.

4 Properties Related to Order

4.1 Some Order-like Relationships

We now introduce the relations $\preceq$, $\precsim$, $\sqsubseteq$ between elements of $\mathbf{P}(L)$.

Definition 4.1 *Take any $A, B \in \mathbf{P}(L)$; we write $A \preceq B$ iff*

(i) $\forall a \in A \quad \exists b_1 \in B : a \le b_1$, (ii) $\forall b \in B \quad \exists a_1 \in A : a_1 \le b$.

Definition 4.2 *Take any $A, B \in \mathbf{P}(L)$; we write $A \sqsubseteq B$ iff*

(i) $\exists b_1 \in B$: $\forall a \in A : a \le b_1$, (ii) $\exists a_1 \in A$: $\forall b \in B \quad : a_1 \le b$.

Definition 4.3 *Take any $A, B \in \mathbf{P}(L)$; we write $A \precsim B$ iff $\forall a \in A, \forall b \in B$: $a \wedge b \in A, a \vee b \in B$.*

Proposition 4.4 *For all $A, B \in \mathbf{P}(L)$ we have: $A \precsim B \Rightarrow A \preceq B$.*

Proof. Choose any $a \in A$ and any $b \in B$. Then $a \wedge b \in A$ and also $a \wedge b \le b$. Similarly, $a \vee b \in B$ and also $a \le a \vee b$. ■

Remark. The converse is not necessarily true. Consider the lattice of Figure 2 with $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Here for all $a \in A$ exists some $b \in B$ such that $a \le b$, and for all $b \in B$ exists some $a \in A$ such that $a \le b$. Hence $A \preceq B$. However, $a_2 \wedge b_1 = b_1 \in B$, so $A \not\precsim B$.

Figure 2

Proposition 4.5 *For all* $A, B \in \mathbf{P}(L)$ *we have:* $A \sqsubseteq B \Rightarrow A \preceq B$.

Proof. This is obvious. ■

Remark. The converse is not necessarily true. Consider the lattice of of integers, with the natural order. Take $A = \{..., -1, 1, 3, ...\}$ and $B = \{..., 0, 2, 4, ...\}$; clearly $A \preceq B$ but $A \not\sqsubseteq B$.

Remark. The relations $\preceq$, $\precsim$, $\sqsubseteq$ defined above, generally are *not* order relations on $\mathbf{P}(L)$. We now explore situations where each of the above is an order relationship. This generally happens if we restrict ourselves to a subset of $\mathbf{P}(L)$.

Proposition 4.6 *If* $\mathbf{S}$ *is a collection of intervals of* $(L, \vee, \wedge)$*, then* $\preceq$, $\precsim$, $\sqsubseteq$ *are orders on* $\mathbf{S}$.

Proof. (i) Let us first show that $\preceq$ is an order on $\mathbf{S}$.

(i.1) Obviously, for all $A \in \mathbf{S}$ we have $A \preceq A$.

(i.2) Choose any $A = [a_1, a_2]$, $B = [b_1, b_2] \in \mathbf{S}$ such that $A \preceq B$ and $B \preceq A$. From $A \preceq B$ we have that $\exists b_3 \in B$ such that $a_2 \leq b_3 \leq b_2$ and $\exists a_3 \in A$ such that $a_1 \leq a_3 \leq b_1$. From $B \preceq A$ we have that $\exists b_4 \in B$ such that $b_1 \leq b_4 \leq a_1$ and $\exists a_4 \in A$ such that $b_2 \leq a_4 \leq a_2$. From these follows that $a_1 \leq b_1 \leq a_1 \Rightarrow a_1 = b_1$ and $b_2 \leq a_2 \leq b_2 \Rightarrow a_2 = b_2$; hence $A = B$.

(i.3) Choose any $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2] \in \mathbf{S}$ such that $A \preceq B$ and $B \preceq C$. Now we have $a_1 \leq b_1$ and $b_1 \leq c_1$, so $a_1 \leq c_1$; and $a_2 \leq b_2$ and $b_2 \leq c_2$, so $a_2 \leq c_2$. This means that: for all $a \in A$ we have $a \leq a_2 \leq c_2 \in C$; and for all $c \in C$ we have $A \ni a_1 \leq c_1 \leq c \in C$. Hence $A \preceq C$.

From (i.1), (i.2), (i.3) follows that $\preceq$ is an order on $\mathbf{S}$.

(ii) Next we show that $\sqsubseteq$ is an order on $\mathbf{S}$. For this it suffices to show that: when $\mathbf{S}$ is a class of intervals we have $A \preceq B \Leftrightarrow A \sqsubseteq B$. Obviously, we have $A \sqsubseteq B \Rightarrow A \preceq B$. To show the converse, recall that $[a_1, a_2] = A \preceq B = [b_1, b_2] \Rightarrow\ (a_1 \leq b_1$ and $a_2 \leq b_2)$. From this follows immediately that: $\forall b \in B$ we have $a_1 \leq b_1 \leq b$; and $\forall a \in A$ we have $a \leq a_2 \leq b_2$. Hence $A \sqsubseteq B$. In short, we have shown that $\preceq$ and $\sqsubseteq$ are equivalent on $\mathbf{S}$ and, since $\preceq$ is an order, so is $\sqsubseteq$.

From (ii.1), (ii.2), (ii.3) follows that $\sqsubseteq$ is an order on $\mathbf{S}$.

(iii) Last we show that $\precsim$ is an order on $\mathbf{S}$.

(iii.1) Choose any $A = [a_1, a_2] \in \mathbf{S}$ and any $x, y \in [a_1, a_2]$. I.e.

$$\left.\begin{array}{l} a_1 \leq x \leq a_2 \\ a_1 \leq y \leq a_2 \end{array}\right\} \Rightarrow a_1 \leq \left\{\begin{array}{c} x \vee y \\ x \wedge y \end{array}\right\} \leq a_2.$$

Hence $x \wedge y, x \vee y \in A$ and so $A \precsim A$.

(iii.2) Choose any $A = [a_1, a_2]$, $B = [b_1, b_2] \in \mathbf{S}$ such that $A \precsim B$ and $B \precsim A$. Then we have $a_1 \wedge b_1 \in A$ and $a_1 \wedge b_1 \in B$. But then $a_1 = a_1 \wedge b_1 = b_1$. Similarly $a_2 = a_2 \vee b_2 = b_2$ and so $A = B$.

(iii.3) Choose any $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2] \in \mathbf{S}$ such that $A \precsim B$ and $B \precsim C$. Now we have $a_1 \wedge b_1 \in A = [a_1, a_2]$ and so $a_1 \wedge b_1 = a_1 \Rightarrow a_1 \leq b_1$; similarly $b_1 \wedge c_1 \in B \Rightarrow b_1 \leq c_1$; and so we get that $a_1 \leq c_1$. Similarly we get $a_2 \leq c_2$. Now choose any $a \in A$ and any $c \in C$; then we have

$$\left.\begin{array}{c} a_1 \leq a \leq a_2 \\ c_1 \leq c \leq c_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a_1 = a_1 \wedge c_1 \leq a \wedge c \leq a_2 \wedge c_2 \leq a_2 \\ c_1 \leq a_1 \vee c_1 \leq a \vee c \leq a_2 \vee c_2 = c_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \wedge c \in A \\ a \vee c \in C \end{array}\right. .$$

Hence $A \precsim C$.

From (iii.1), (iii.2), (iii.3) follows that $\precsim$ is an order on $\mathbf{S}$ and the proof of the proposition is complete. ■

Proposition 4.7 *If* $\mathbf{S}$ *is a collection of convex sublattices of* $(L, \vee, \wedge)$*, then* $\precsim$ *is an order on* $\mathbf{S}$.

Proof. (i) Choose any $A \in \mathbf{S}$. Obviously, for all $x, y \in A$ we have $x \wedge y \in A$ and $x \vee y \in A$. Hence $A \precsim A$.

(ii) Choose any A, $B \in \mathbf{S}$ such that $A \precsim B$ and $B \precsim A$; choose any $a \in A$, $b \in B$. Then, $a \wedge b \in A$ but also $a \wedge b \in B$; similarly, $a \vee b \in A$ but also $a \vee b \in B$. From:

$$a \wedge b, a \vee b \in B, \quad a \wedge b \leq a \leq a \vee b$$

and convexity, we obtain $a \in B$. Hence $A \subseteq B$. But similarly we get $B \subseteq A$ and so $A = B$.

(iii) Choose any A, B, $C \in \mathbf{S}$ such that $A \precsim B$ and $B \precsim C$. Choose any $a \in A$, $b \in B$, $c \in C$. Then $a \vee b \in B$ and so $a \vee b \vee c \in C$. Then $a \vee c \in [c, a \vee b \vee c] \subseteq C$. Similarly, $b \wedge c \in B$ and so $a \wedge b \wedge c \in A$. Then $a \wedge c \in [a \wedge b \wedge c, a] \subseteq A$. Hence $A \precsim C$.

From (i), (ii), (iii) follows that $\precsim$ is an order on $\mathbf{S}$ and the proof of the proposition is complete ■

Corollary 4.8 *(i) If $\mathbf{S}$ is a collection of ideals of $(L, \vee, \wedge)$, then $\precsim$ is an order on $\mathbf{S}$.*

(ii) If $\mathbf{S}$ is a collection of filters of $(L, \vee, \wedge)$, then $\precsim$ is an order on $\mathbf{S}$.

Proof. (i) This follows from the fact that every ideal is a convex sublattice and from Proposition 4.7.

(ii) This follows from the fact that every filter is a convex sublattice and from Proposition 4.7. ■

From Propositions 4.6 and 4.7 one infers the following propositions.

Proposition 4.9 *Let $(L, \vee, \wedge)$ be distributive. Then:*

(i) if P is an interval, then $\preceq$, $\precsim$, $\sqsubseteq$ are orders on $\{a \overset{P}{\vee} b\}_{a,b \in L}$;

(ii) if Q is an interval, then $\preceq$, $\precsim$, $\sqsubseteq$ are orders on $\{a \overset{Q}{\wedge} b\}_{a,b \in L}$.

Proof. (i) follows immediately from Propositions 3.6 and 4.6; (ii) is proved dually. ■

Proposition 4.10 *Let $(L, \vee, \wedge)$ be distributive. Then:*

(i) if P is an ideal, then $\precsim$ is an order on $\{a \overset{P}{\vee} b\}_{a,b \in L}$;

(ii) if Q is a filter, then $\precsim$ is an order on $\{a \overset{Q}{\wedge} b\}_{a,b \in L}$.

Proof. (i) follows immediately from Propositions 3.5 and 4.7; (ii) is proved dually. ■

4.2 Properties of $\preceq$

Proposition 4.11 *For all $a, b \in L$ we have: (i) $P \preceq a \overset{P}{\vee} b$, (ii) $a \overset{Q}{\wedge} b \preceq Q$.*

Proof. Take any $p \in P$; there exists some $x = a \vee b \vee p \in a \overset{P}{\vee} b$ and we obviously have $p \leq x$. Take any $x \in a \overset{P}{\vee} b$; there exists some $p \in P$ such that $x = a \vee b \vee p \in a \overset{P}{\vee} b$ and we obviously have $p \leq x$. (ii) is proved dually. ■

Proposition 4.12 *(i) If $A, B \in \mathbf{P}(L)$ have minimum elements, then: $A \preceq B \Rightarrow \min(A) \leq \min(B)$.*

(ii) If $A, B \in \mathbf{P}(L)$ have maximum elements and $A \preceq B$, then $\max(A) \leq \max(B)$.

Proof. (i) Since $\underline{b} = \min(B) \in B$, then exists some $a \in A$ such that $a \leq \underline{b}$; also $\underline{a} = \min(A) \leq a$; hence $\underline{a} \leq \underline{b}$. (ii) is proved dually. ■

Proposition 4.13 *For all $a, b \in L$ the following conditions are equivalent.*

(i) $a \leq b$.

(ii) For all $c \in L$ we have: $a \overset{P}{\vee} c \preceq b \overset{P}{\vee} c$

(iii) For all $c \in L$ we have: $a \overset{Q}{\wedge} c \preceq b \overset{Q}{\wedge} c$.

Proof. We will show (i)⇒(ii), (i)⇒(iii), (ii)⇒(i), (iii)⇒(i)

<u>(i)⇒(ii)</u> If $x \in a \overset{P}{\vee} c$ then exists some $p_1 \in P$ such that $x = a \vee c \vee p_1 \leq b \vee c \vee p_1 = y \in b \overset{P}{\vee} c$. Similarly, if $z \in b \overset{P}{\vee} c$ then exists some $p_2 \in P$ such that $z = b \vee c \vee p_2 \geq a \vee c \vee p_2 = w \in a \overset{P}{\vee} c$. Hence $a \overset{P}{\vee} c \preceq b \overset{P}{\vee} c$.

<u>(i)⇒(iii)</u> It is proved dually to the previous.

<u>(ii)⇒(i)</u> Set $c = b$. Then $a \overset{P}{\vee} b \preceq b \overset{P}{\vee} b$. From Proposition 4.12 we get $a \vee b = \min(a \overset{P}{\vee} b) \leq \min(b \overset{P}{\vee} b) = b \vee b = b$. Hence $a \vee b \leq b \Rightarrow b = a \vee b \Rightarrow a \leq b$.

<u>(iii)⇒(i)</u> It is proved dually to the previous. ■

Remark. Note that in general $a \overset{P}{\vee} c$ and $b \overset{P}{\vee} c$ will *not* be intervals, hence it is *not* necessary that $a \overset{P}{\vee} c \preceq b \overset{P}{\vee} c$ denotes an order relationship.

4.3 Properties of $\sqsubseteq$

Proposition 4.14 *(i) If $(L, \vee, \wedge)$ is distributive and P is an interval, then for all $a, b \in L$ we have: $P \sqsubseteq a \overset{P}{\vee} b$.*

(ii) If $(L, \vee, \wedge)$ is distributive and Q is an interval, then for all $a, b \in L$ we have: $a \overset{Q}{\wedge} b \sqsubseteq Q$.

Proof. (i) In the proof of Proposition 4.6 we have seen that, for A, B intervals, we have $A \sqsubseteq B \Leftrightarrow A \preceq B$. Also, from Proposition 3.6 we know that when P is an interval, $a \overset{P}{\vee} b$ is also an interval. So $P \sqsubseteq a \overset{P}{\vee} b \Leftrightarrow P \preceq a \overset{P}{\vee} b$, which is true by Proposition 4.11.

(ii) is proved dually. ■

Proposition 4.15 *(i) If $A, B \in \mathbf{P}(L)$ have minimum elements, then: $A \sqsubseteq B \Rightarrow \min(A) \leq \min(B)$,*

(ii) If $A, B \in \mathbf{P}(L)$ have maximum elements, then: $A \sqsubseteq B \Rightarrow \max(A) \leq \max(B)$.

Proof. (i) There exists some $\underline{a}$ such that for all $b \in B$ we have $\underline{a} \leq b$. Also, $\min(A) \leq \underline{a}$. Hence, for all $b \in B$ we have $\min(A) \leq b$; since $\min(B) \in B$, we then get $\min(A) \leq \min(B)$; (ii) is proved dually. ■

Proposition 4.16 *Assume $(L, \vee, \wedge)$ is distributive and P, Q are intervals. Then the following conditions are equivalent.*

(i) $a \leq b$.

(ii) For all $c \in L$ we have: $a \overset{P}{\vee} c \sqsubseteq b \overset{P}{\vee} c$.

(iii) For all $c \in L$ we have: $a \overset{Q}{\wedge} c \sqsubseteq b \overset{Q}{\wedge} c$.

Proof. We will show (i)⇒(ii), (i)⇒(iii), (ii)⇒(i), (iii)⇒(i)

<u>(i)⇒(ii)</u> This is obvious if we use the fact that for A, B intervals we have $A \sqsubseteq B \Leftrightarrow A \preceq B$ and then use Proposition 4.13.

<u>(i)⇒(iii)</u> It is proved dually to the previous.

<u>(ii)⇒(i)</u> Set $c = b$. Then $a \overset{P}{\vee} b \preceq b \overset{P}{\vee} b$. From Proposition 4.15 we get $a \vee b = \min(a \overset{P}{\vee} b) \leq \min(b \overset{P}{\vee} b) = b \vee b = b$. Hence $a \leq a \vee b = b$.

<u>(iii)⇒(i)</u> It is proved dually to the previous. ■

Remark. In the above proposition note that, since $(L, \vee, \wedge)$ has been assumed distributive and P, Q have been assumed intervals, $a \overset{P}{\vee} c$ and $b \overset{P}{\vee} c$ will also be intervals (by Proposition 3.6). Hence $a \overset{P}{\vee} c \sqsubseteq b \overset{P}{\vee} c$ will be an order relationship.

4.4 Properties of $\precsim$

Proposition 4.17 *(i) If P is an ideal, then for all $a, b \in L$ we have: $P \precsim a \overset{P}{\vee} b$.*

(ii) If Q is a filter, then for all $a, b \in L$ we have: $a \overset{Q}{\wedge} b \precsim Q$.

Proof. (i) Take $x \in a \overset{P}{\vee} b$, i.e. $x = a \vee b \vee p_1$, for some $p_1 \in P$. Take any $p \in P$. Then $x \vee p = a \vee b \vee (p_1 \vee p)$. Since $p_1 \vee p \in P$, it follows that $x \vee p \in a \overset{P}{\vee} b$. On the other hand, $x \wedge p = (a \vee b \vee p_1) \wedge p \leq p$. Since $p \in P$, it follows that $x \wedge p \in P$. Hence $P \precsim a \overset{P}{\vee} b$.

(ii) is proved dually to (i). ■

Proposition 4.18 *(i) If $A, B \in \mathbf{P}(L)$ have minimum elements, then: $A \precsim B \Rightarrow \min(A) \leq \min(B)$.*

(ii) If $A, B \in \mathbf{P}(L)$ have maximum elements, then: $A \precsim B \Rightarrow \max(A) \leq \max(B)$.

Proof. (i) Set $\underline{a} = \min(A)$, $\underline{b} = \min(B)$. Now $A \precsim B \Rightarrow c = \underline{a} \wedge \underline{b} \in A$. But, since $\underline{a} = \min(A)$ it follows that $\underline{a} \leq c = \underline{a} \wedge \underline{b} \leq \underline{a}$. In short, $\underline{a} = \underline{a} \wedge \underline{b} \Rightarrow \underline{a} \leq \underline{b}$ and the proof of (i) is complete; (ii) is proved dually. ■

Proposition 4.19 . *The following are true.*

(i) Assume $(L, \vee, \wedge)$ is distributive, P is an ideal and $a, b \in L$ satisfy $a \leq b$; then for all $c \in L$ we have: $a \overset{P}{\vee} c \precsim b \overset{P}{\vee} c$;

(ii) Assume $(L, \vee, \wedge)$ is distributive, Q is a filter and $a, b \in L$ satisfy $a \leq b$; then for all $c \in L$ we have: $a \overset{Q}{\wedge} c \precsim b \overset{Q}{\wedge} c$.

(iii) Assume $a, b \in L$ are such that for all $c \in L$ we have $a \overset{P}{\vee} c \precsim b \overset{P}{\vee} c$; then $a \leq b$.

(iv) Assume $a, b \in L$ are such that for all $c \in L$ we have $a \overset{Q}{\wedge} c \precsim b \overset{Q}{\wedge} c$; then $a \leq b$.

Proof. (i) Take any $x \in a \overset{P}{\vee} c$, and any $y \in b \overset{P}{\vee} c$; i.e. $x = a \vee c \vee p_1$, $p_1 \in P$ and $y = b \vee c \vee p_2$, $p_2 \in P$. Now we have $x \vee y = a \vee c \vee p_1 \vee b \vee c \vee p_2 = (a \vee b) \vee c \vee (p_1 \vee p_2) = b \vee c \vee (p_1 \vee p_2) \in b \overset{P}{\vee} c$. Also, $x \wedge y = (a \vee c \vee p_1) \wedge (b \vee c \vee p_2) = ((a \vee p_1) \wedge (b \vee p_2)) \vee c = ((a \wedge b) \vee (p_1 \wedge b) \vee (a \wedge p_2) \vee (p_1 \wedge p_2)) \vee c = a \vee (p_1 \wedge b) \vee (p_2 \wedge a) \vee (p_1 \wedge p_2) \vee c = a \vee (p_1 \wedge b) \vee (p_1 \wedge p_2) \vee c = a \vee c \vee p,$

where $p = (p_1 \wedge b) \vee (p_1 \wedge p_2) \in P$. Hence $x \wedge y = a \vee c \vee p \in a \overset{P}{\vee} c$. In short we have shown $a \overset{P}{\vee} c \precsim b \overset{P}{\vee} c$; (ii) is proved dually.

(iii) From Proposition 4.4 we get $a \overset{P}{\vee} c \precsim b \overset{P}{\vee} c \Rightarrow a \overset{P}{\vee} c \preceq b \overset{P}{\vee} c$. Hence, for all $c \in L$ we have $a \overset{P}{\vee} c \preceq b \overset{P}{\vee} c$, which by Proposition 4.13 implies $a \leq b$; (iv) is proved dually. ∎

Corollary 4.20 *Assume $(L, \vee, \wedge)$ is distributive, P is an ideal and Q is a filter. Then the following conditions are equivalent.*

(i) $a \leq b$.

(ii) For all $c \in L$ we have: $a \overset{P}{\vee} c \precsim b \overset{P}{\vee} c$.

(iii) For all $c \in L$ we have: $a \overset{Q}{\wedge} c \precsim b \overset{Q}{\wedge} c$.

Proof. Follows immediately from Proposition 4.19. ∎

Remark. Note that in the above proposition $(L, \vee, \wedge)$ has been assumed distributive, P has been assumed an ideal and Q has been assumed a filter. Hence, by Proposition 4.10 we have that $\precsim$ is an order on both $\{a \overset{P}{\vee} b\}_{a,b \in L}$ and $\{a \overset{Q}{\wedge} b\}_{a,b \in L}$. Hence both $a \overset{P}{\vee} c \precsim b \overset{P}{\vee} c$ and $a \overset{Q}{\wedge} c \precsim b \overset{Q}{\wedge} c$ are order relationships.

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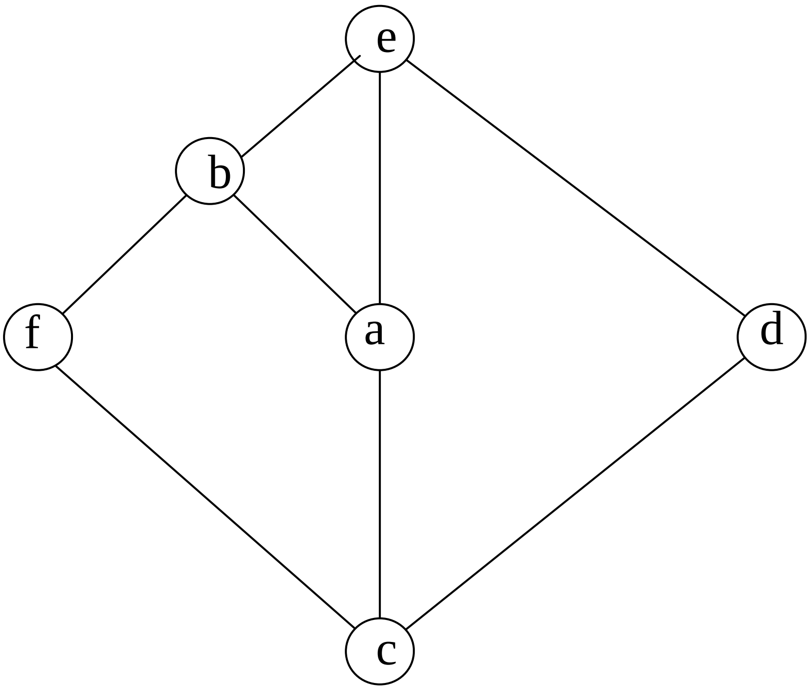

Figure 1

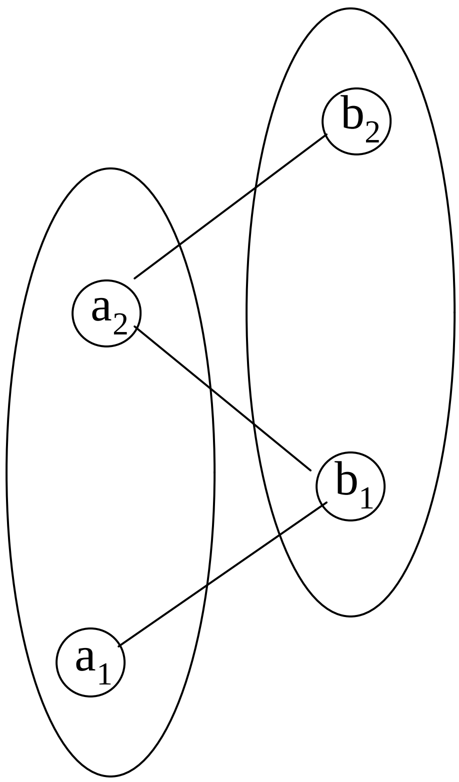

Figure 2