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A Note on the Congruences of the Nakano Superlattice and Some Properties of the Associated Quotients

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Abstract

In this paper we explore the *Nakano* superlattice (H, \sqcup, \sqcap) , where \sqcup, \sqcap are the Nakano *hyperoperations* $x \sqcup y = \{z : x \vee z = y \vee z = x \vee y\}$, $x \sqcap y = \{z : x \wedge z = y \wedge z = x \wedge y\}$. In particular, we study the properties of congruences on the Nakano superlattice and the associated quotients. New hyperoperations are introduced on the quotient and their properties studied.

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1 Introduction

Superlattices have been introduced by Mittas and Konstantinidou [6] as a generalization of lattices. A superlattice is a partially ordered set (H, \leq) equipped with two hyperoperations (i.e. operations which assign to every pair of elements of H a *set* of elements of H) satisfying certain properties [6]. An alternative definition of superlattice starts with two hyperoperations on H which satisfy certain properties; these properties are then used to establish a partial order \leq on H . In case (H, \leq) is a lattice, the superlattice is called *strong*.

In [2], Jakubik studies several aspects of the theory of superlattices; in particular he defines *congruences on superlattices* (henceforth called s-congruences) as a generalization of classical congruences (henceforth called l-congruences) and studies the properties of the resulting quotients. Jakubik shows that, in general, the quotient of a superlattice with respect to a s-congruence fails to be a superlattice. A natural question, posed by Jakubik, is: under what conditions is the quotient of a superlattice also a superlattice?

In this paper we answer the above question for a specific superlattice, which we will name *the Nakano superlattice*. This name is chosen because we define the superlattice in terms of a hyperoperation \sqcap first studied by Nakano [8] and its dual hyperoperation \sqcup ; we call these *Nakano hyperoperations*. We define the Nakano hyperoperations on a lattice (H, \leq) assumed to be *modular* for the remainder of the paper. In Section 2 we discuss certain properties of the Nakano superlattice (H, \sqcup, \sqcap) and its quotient H/R with respect to a s-congruence R ; in particular we show that if R is a s-congruence, then one can define certain hyperoperations $\overline{\vee}, \overline{\wedge}$ on the quotient H/R . In Section 3 we find conditions which are necessary and sufficient for $(H/R, \overline{\vee}, \overline{\wedge})$ to be a superlattice; for example, one such condition is that R is convex (i.e. its classes are convex sublattices of H). In Section 4 we examine some order relationships on H/R . In Section 5 we define additional operations and hyperoperations on H/R and, using convexity of R , establish some of their properties.

2 Preliminaries

Consider a modular lattice (H, \leq) , with sup and inf operations denoted by \vee and \wedge respectively. We define the *Nakano hyperoperations* \sqcup, \sqcap on H .

Definition 2.1 For all $x, y \in H$ we define:

$$x \sqcup y \doteq \{z : z \vee x = z \vee y = x \vee y\}; \quad x \sqcap y \doteq \{z : z \wedge x = z \wedge y = x \wedge y\}.$$

Remark. To the best of our knowledge, the \sqcap hyperoperation was first introduced by Nakano in [8], which is an investigation of hyperrings (multirings, in the author's terminology). Evidently, \sqcup is the dual hyperoperation of \sqcap . The \sqcup hyperoperation has also been studied in [4, 7] and it plays a central role in the theory of Boolean hyperrings and Boolean hyperlattices [5].

The hyperstructure (H, \sqcup, \sqcap) is a *superlattice*, i.e. \sqcup, \sqcap satisfy the conditions of the following proposition (note that in this proposition \leq is the order of the original modular lattice). We will call (H, \sqcup, \sqcap) the *Nakano superlattice*, since it makes use of the Nakano hyperoperations.

Proposition 2.2 $(H, \sqcup, \sqcap, \leq)$ satisfies the following for all $x, y, z \in H$.

$$S1 \quad x \in (x \sqcup x), \quad x \in (x \sqcap x).$$

$$S2 \quad x \sqcup y = y \sqcup x, \quad x \sqcap y = y \sqcap x.$$

$$S3 \quad (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z), \quad (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z).$$

$$S4 \quad x \in (x \sqcup y) \sqcap x, \quad x \in (x \sqcap y) \sqcup x.$$

$$S5 \quad x \leq y \Rightarrow y \in x \sqcup y, \quad x \in x \sqcap y.$$

$$S6 \quad \text{If } y \in x \sqcup y \text{ or } x \in x \sqcap y, \text{ then } x \leq y.$$

Proof. $S1, S2$ are obvious. The proof of $S3$ appears in [8] and also in [1].

Regarding $S4$ let us prove that $x \in (x \sqcup y) \sqcap x = \{z : z = u \sqcap x, u \in x \sqcup y\}$. In other words, we must show that there exists some $u \in x \sqcup y$ such that $x \in u \sqcap x$. But this is easy. Taking $u = x \vee y$ one obtains immediately that $x \vee y \in x \sqcup y$; and since $(x \vee y) \wedge x = x \wedge x = (x \vee y) \wedge x$, it follows that $x \in (x \vee y) \sqcap x$. One can prove dually that $x \in (x \sqcap y) \sqcup x$ and this completes the proof of $S4$.

$x \leq y \Rightarrow x \wedge y = x, x \vee y = y$ and these yield $S5$ immediately. Regarding $S6$: $y \in x \sqcup y \Rightarrow x \vee y = y \vee y = x \vee y \Rightarrow x \leq y$; it can be proved dually that $x \in x \sqcap y \Rightarrow x \leq y$. ■

Proposition 2.3 (H, \sqcup, \sqcap) satisfies the following for all $x, y, z \in H$.

$$S6' \quad y \in x \sqcup y \Leftrightarrow x \in x \sqcap y.$$

$$S7' \quad x, y \in x \sqcup y \Leftrightarrow x = y.$$

$$S8' \quad y \in x \sqcup y, z \in y \sqcup z \Rightarrow z \in x \sqcup z.$$

Furthermore, we have: $(S1 - S6) \Leftrightarrow (S1 - S4, S6' - S8')$.

Proof. The proof that $(S1-S6) \Leftrightarrow (S1-S4, S6'-S8')$ appears in [6]. The first part of the proposition then follows from Proposition 2.2. ■

As shown in [6], Proposition 2.3 is true for *any* superlattice, not only the Nakano one. In fact a superlattice can be defined in two alternative ways [6]: one may assume the underlying order \leq and require that the two hyperoperations satisfy properties $S1-S6$; or one may assume that the hyperoperations satisfy $S1-S4, S6'-S8'$ and then define an order on H in terms of the hyperoperations, in which case the resulting order satisfies $S5, S6$.

In the remainder of the paper we will use extensively the concepts of equivalence and congruence. Let us first give the following well known definitions for the sake of completeness.

Definition 2.4 An equivalence on H is a relationship R which satisfies the following $\forall x, y, z \in H$:

$$(i) xRx, \quad (ii) xRy \Rightarrow yRx, \quad (iii) xRy, yRz \Rightarrow xRz.$$

Definition 2.5 Let R be an equivalence on H . For all $x \in H$ the class of x is denoted by \bar{x} and defined by $\bar{x} \doteq \{y : xRy\}$.

Definition 2.6 Let R be an equivalence on H . The quotient of H with respect to R is denoted by H/R and defined by $H/R \doteq \{\bar{x} : x \in H\}$.

Notation. We use the following notation: for all $A \subseteq H$ we write $\bar{A} \doteq \{\bar{x} : x \in A\}$

Remark. It follows from the above notation that for any $A, B \subseteq H$ such that $\bar{A} = \bar{B}$ we have:

$$(i) \forall x \in A \quad \exists y \in B \text{ such that } \bar{x} = \bar{y}, \quad (ii) \forall y \in B \quad \exists x \in A \text{ such that } \bar{x} = \bar{y}.$$

Let us now turn to the concept of congruence. In classical lattice theory, congruence is defined in terms of operations; since here we will make use of both operations and hyperoperations, we will need *two* concepts of congruence. We use the term *l-congruence* to describe what is commonly called “congruence” (with respect to the operations \vee, \wedge) and the term *s-congruence* to describe the analogous property with respect to the hyperoperations \sqcup, \sqcap .

Definition 2.7 An equivalence R on H is called a *l-congruence* on a lattice (H, \vee, \wedge) iff for all $x, y, z \in H$ we have

$$\bar{x} = \bar{y} \Rightarrow \begin{cases} \overline{x \vee z} = \overline{y \vee z} \\ \overline{x \wedge z} = \overline{y \wedge z} \end{cases}$$

Definition 2.8 An equivalence R on H is called a *s-congruence* on a superlattice (H, \sqcup, \sqcap) iff for all $x, y, z \in H$ we have

$$\bar{x} = \bar{y} \Rightarrow \begin{cases} \overline{x \sqcup z} = \overline{y \sqcup z} \\ \overline{x \sqcap z} = \overline{y \sqcap z} \end{cases}.$$

In the remaining part of this work we will use the expression “s-congruence R on H ” instead of the more correct, but also longer, “s-congruence R on a Nakano superlattice (H, \sqcup, \sqcap) ”.

We now define two new hyperoperations $\overline{\vee}, \overline{\wedge}$ on the quotient H/R .

Definition 2.9 Given a s-congruence R on H we define hyperoperations $\overline{\vee}, \overline{\wedge}$ as follows: for all $x, y \in H$

$$\overline{x \vee y} \doteq \overline{x \sqcup y}; \quad \overline{x \wedge y} \doteq \overline{x \sqcap y}.$$

Remark. The above definition makes sense only if R is a s-congruence. Because, if for some $x, y \in H$ such that $\bar{x} = \bar{y}$ we had some $z \in H$ such that $\bar{x} \sqcup z \neq \bar{y} \sqcup z$, then we would have the following contradiction: $\bar{y} \bar{\vee} \bar{z} = \bar{x} \bar{\vee} \bar{z} = \bar{x} \sqcup z \neq \bar{y} \sqcup z = \bar{y} \bar{\vee} \bar{z}$.

Remark. If the hyperoperations $\bar{\vee}, \bar{\wedge}$ are well defined, then an s-congruence can be equivalently defined as follows.

Definition 2.10 An equivalence R on H is called a s-congruence on H iff for all $x, y, z \in H$ we have

$$\bar{x} = \bar{y} \Rightarrow \begin{cases} \bar{x} \bar{\vee} \bar{z} = \bar{y} \bar{\vee} \bar{z} \\ \bar{x} \bar{\wedge} \bar{z} = \bar{y} \bar{\wedge} \bar{z} \end{cases}.$$

It is worth noting that $(H/R, \bar{\vee}, \bar{\wedge})$ is in general a proper hyperstructure, i.e. the $\bar{\vee}, \bar{\wedge}$ hyperoperations yield non-singleton sets. This is seen from the following proposition.

Proposition 2.11 Let (H, \vee, \wedge) have either a maximum or a minimum element and R be a s-congruence on H . If $\text{card}(H/R) \geq 2$, then $(H/R, \bar{\vee}, \bar{\wedge})$ is a proper hyperstructure.

Proof. Assume H has maximum element denoted by 1. Then $1 \sqcup 1 = \{z : z \leq 1\} = H$ and $\bar{1} \sqcup \bar{1} = \{\bar{z} : z \in H\} = H/R$. Hence $\text{card}(\bar{1} \sqcup \bar{1}) \geq 2$ and so H/R is a proper hyperstructure. The same can be proved dually if H has minimum element denoted by 0. ■

Proposition 2.12 Let R be a s-congruence on H , then $(H/R, \bar{\vee}, \bar{\wedge})$ satisfies for all $x, y, z \in H$ the following:

- T1 $\bar{x} \in (\bar{x} \bar{\vee} \bar{x}), \bar{x} \in (\bar{x} \bar{\wedge} \bar{x})$;
- T2 $\bar{x} \bar{\vee} \bar{y} = \bar{y} \bar{\vee} \bar{x}, \bar{x} \bar{\wedge} \bar{y} = \bar{y} \bar{\wedge} \bar{x}$;
- T3 $(\bar{x} \bar{\vee} \bar{y}) \bar{\vee} \bar{z} = \bar{x} \bar{\vee} (\bar{y} \bar{\vee} \bar{z}), (\bar{x} \bar{\wedge} \bar{y}) \bar{\wedge} \bar{z} = \bar{x} \bar{\wedge} (\bar{y} \bar{\wedge} \bar{z})$;
- T4 $\bar{x} \in (\bar{x} \bar{\vee} \bar{y}) \bar{\wedge} \bar{x}, \bar{x} \in (\bar{x} \bar{\wedge} \bar{y}) \bar{\vee} \bar{x}$.

Proof. It appears in [2]. ■

Remark. In other words, if R is a s-congruence, then $(H/R, \bar{\vee}, \bar{\wedge})$ satisfies the first four properties of a superlattice. A natural question is the following: what are necessary and sufficient conditions for $(H/R, \bar{\vee}, \bar{\wedge})$ to actually be a superlattice? This question can be formulated more precisely in terms of the following properties ($T6' - T8'$).

- $T6'$ $\bar{y} \in (\bar{x} \bar{\vee} \bar{y}) \Leftrightarrow \bar{x} \in (\bar{x} \bar{\wedge} \bar{y})$.
- $T7'$ $\bar{x}, \bar{y} \in (\bar{x} \bar{\vee} \bar{y}) \Rightarrow \bar{x} = \bar{y}$.
- $T8'$ $\bar{y} \in (\bar{x} \bar{\vee} \bar{y}), \bar{z} \in (\bar{y} \bar{\vee} \bar{z}) \Rightarrow \bar{z} \in (\bar{x} \bar{\vee} \bar{z})$.

Question. What conditions must R satisfy so that ($T6' - T8'$) hold?

If ($T6' - T8'$) hold, then $(H/R, \bar{\vee}, \bar{\wedge})$ is a superlattice and we can define the *order* relationship \preceq as follows.

Definition 2.13 Let R be a s-congruence on H , such that ($T1 - T4, T6' - T8'$) hold. We write $\bar{x} \preceq \bar{y}$ iff $\bar{y} \in \bar{x} \bar{\vee} \bar{y}$.

Furthermore, if we prove that ($T1 - T4, T6' - T8'$) hold, then we will know that the following conditions also hold.

- $T5$ $\bar{x} \preceq \bar{y} \Rightarrow \bar{y} \in (\bar{x} \bar{\vee} \bar{y}), \bar{x} \in (\bar{x} \bar{\wedge} \bar{y})$.
- $T6$ $(\bar{y} \in (\bar{x} \bar{\vee} \bar{y}) \text{ or } \bar{x} \in (\bar{x} \bar{\wedge} \bar{y})) \Rightarrow \bar{x} \preceq \bar{y}$.

3 Convexity

We now explore the connection between convexity and conditions $(T6' - T8')$. Let us first give two definitions of convexity on ordered sets.

Definition 3.1 *Given $A \subseteq H$, we say that A is w-convex iff: $\forall x, y \in A$ with $x \leq y$ and $\forall z$ such that $x \leq z \leq y$, we have $z \in A$.*

Definition 3.2 *Given $A \subseteq H$, we say that A is s-convex iff: $\forall x, y \in A$ and $\forall z$ such that $x \wedge y \leq z \leq x \vee y$, we have $z \in A$.*

Remark. In other words, s-convexity is a stronger property than w-convexity: A is s-convex if it is a w-convex sublattice of H .

Let us now define convexity of congruence relationships.

Definition 3.3 *Let R be an s-congruence on H . We say that R is w-convex iff $\forall x \in H$ we have that \bar{x} is w-convex.*

Definition 3.4 *Let R be an s-congruence on H . We say that R is s-convex iff $\forall x \in H$ we have that \bar{x} is s-convex.*

We will now show that: R is w-convex iff $(H/R, \overline{\vee}, \overline{\wedge})$ is a superlattice. To this end we will first prove the auxiliary Propositions 3.5 - 3.8.

Proposition 3.5 *Let R be a w-convex s-congruence on H . Then for all $x, y \in H$ we have:*

$$(i) \bar{x} \in \bar{x} \overline{\vee} \bar{y} \Leftrightarrow \bar{x} = \overline{x \vee y}; \quad (ii) \bar{y} \in \bar{x} \overline{\wedge} \bar{y} \Leftrightarrow \bar{y} = \overline{x \wedge y}.$$

Proof. We only prove (i), since (ii) is proved dually.

Suppose that $\bar{x} \in \bar{x} \overline{\vee} \bar{y} = \overline{x \sqcup y}$. This implies that there exists some z such that $z \in x \sqcup y$ and $\bar{z} = \bar{x}$. Since $z \in x \sqcup y$ it follows: $z \vee x = z \vee y = x \vee y \Rightarrow z \vee x = z \vee x \vee y = x \vee x \vee y \Rightarrow x \vee y \in z \sqcup x \Rightarrow \overline{x \vee y} \in \overline{z \sqcup x} = \overline{x \sqcup x}$. Now $\overline{x \vee y} \in \overline{x \sqcup x}$ implies that there exists some w such that $w \in x \sqcup x$ and $\overline{w} = \overline{x \vee y}$. Since $w \in x \sqcup x$ it follows: $x \vee w = x \vee x = x \Rightarrow w \leq x$. Hence we have $w \leq x \leq x \vee y$; since $\overline{w} = \overline{x \vee y}$, by w-convexity of R it follows that $\bar{x} = \overline{x \vee y}$.

Conversely, suppose $\bar{x} = \overline{x \vee y}$; we also have $x \vee y \in x \sqcup y$ and hence $\overline{x \vee y} \in \overline{x \sqcup y} = \bar{x} \overline{\vee} \bar{y}$. It follows that $\bar{x} \in \bar{x} \overline{\vee} \bar{y}$. ■

Proposition 3.6 *Let R be a w-convex s-congruence on H . Then for all $x, y \in H$ we have:*

$$\bar{x} \in \bar{x} \overline{\vee} \bar{y} \Leftrightarrow \bar{y} \in \bar{x} \overline{\wedge} \bar{y}.$$

Proof. By Proposition 3.5 we have: $\bar{x} \in \bar{x} \overline{\vee} \bar{y} \Rightarrow \bar{x} = \overline{x \vee y}$. Now, it is easy to check that for all $u, w \in H$ we have $u \wedge w \in u \sqcap w$. In particular, $y = (x \vee y) \wedge y \in (x \vee y) \sqcap y \Rightarrow \bar{y} \in \overline{(x \vee y) \sqcap y} = \overline{(x \vee y) \wedge y} = \bar{x} \overline{\wedge} \bar{y}$. So we have shown that $\bar{x} \in \bar{x} \overline{\vee} \bar{y} \Rightarrow \bar{y} \in \bar{x} \overline{\wedge} \bar{y}$. It can be shown dually that $\bar{y} \in \bar{x} \overline{\wedge} \bar{y} \Rightarrow \bar{x} \in \bar{x} \overline{\vee} \bar{y}$. ■

Proposition 3.7 *Let R be a w-convex s-congruence on H . Then for all $x, y \in H$ we have:*

$$(i) \bar{x}, \bar{y} \in \bar{x} \overline{\vee} \bar{y} \Rightarrow \bar{x} = \bar{y}; \quad (ii) \bar{x}, \bar{y} \in \bar{x} \overline{\wedge} \bar{y} \Rightarrow \bar{x} = \bar{y}.$$

Proof. We only prove (i), since (ii) is proved dually. From Proposition 3.5 we have

$$\left. \begin{array}{l} \bar{x} \in \bar{x} \bar{\vee} \bar{y} \Rightarrow \bar{x} = \overline{x \vee y} \\ \bar{y} \in \bar{x} \bar{\vee} \bar{y} \Rightarrow \bar{y} = \overline{x \vee y} \end{array} \right\} \Rightarrow \bar{x} = \bar{y}.$$

This completes the proof. ■

Proposition 3.8 *Let R be a w -convex s -congruence on H . Then for all $x, y, z \in H$ we have:*

$$(i) \bar{y} \in \bar{x} \bar{\vee} \bar{y}, \bar{z} \in \bar{y} \bar{\vee} \bar{z} \Rightarrow \bar{z} \in \bar{x} \bar{\vee} \bar{z}; \quad (ii) \bar{x} \in \bar{x} \bar{\wedge} \bar{y}, \bar{y} \in \bar{y} \bar{\wedge} \bar{z} \Rightarrow \bar{x} \in \bar{x} \bar{\wedge} \bar{z}.$$

Proof. We only prove (i), since (ii) is proved dually. From Propositions 3.6 and 3.5 we have: (a) $\bar{y} \in \bar{x} \bar{\vee} \bar{y} \Rightarrow \bar{x} \in \bar{x} \bar{\wedge} \bar{y} \Rightarrow \bar{x} = \overline{x \wedge y}$ and (b) $\bar{z} \in \bar{y} \bar{\vee} \bar{z} \Rightarrow \bar{z} = \overline{y \vee z}$. Now $y \vee z = (x \wedge y) \vee (y \vee z) \in (x \wedge y) \sqcup (y \vee z)$. Using this and (a) and (b) we have $\bar{z} = \overline{y \vee z} \in \overline{(x \wedge y) \sqcup (y \vee z)} = \overline{x \sqcup z} = \bar{x} \bar{\vee} \bar{z}$. ■

When R is a s -congruence, Property $T7'$ implies Properties $T6'$, $T8'$, as can be seen from the next proposition.

Proposition 3.9 *Let R be a s -congruence on H . Then $T7' \Rightarrow T6', T8'$.*

Proof. We will first prove the following: if $T7'$ holds (i.e. for all $x, y \in H$ we have: $\bar{x}, \bar{y} \in \bar{x} \bar{\vee} \bar{y} \Rightarrow \bar{x} = \bar{y}$) then $\forall x \in H$ we have that \bar{x} is w -convex. To show this, take any $x \in H$ and any $y, z \in H$ such that: (a) $x \leq z \leq y$ and (b) $\bar{x} = \bar{y}$. We have: $z = x \vee z \in x \sqcup z \Rightarrow \bar{z} \in \overline{x \sqcup z} = \bar{y} \bar{\sqcup} \bar{z}$. Similarly, $y = y \vee z \in y \sqcup z \Rightarrow \bar{y} \in \overline{y \sqcup z}$. Hence (by $T7'$) $\bar{y} = \bar{z}$ and so \bar{x} is w -convex. Since w -convexity of R implies $T6'$ by Proposition 3.6 and $T8'$ by Proposition 3.8, the proof is complete. ■

We are now ready to prove that: R is w -convex iff $(H/R, \bar{\vee}, \bar{\wedge})$ is a superlattice.

Proposition 3.10 *Let R be a s -congruence on H . Then R is w -convex iff $(H/R, \bar{\vee}, \bar{\wedge})$ is a superlattice.*

Proof. (i) Assume that R is w -convex. Then from Propositions 3.6, 3.7, 3.8 respectively, it follows that properties $T6'$, $T7'$, $T8'$ hold. Furthermore, since R is an s -congruence, $(T1 - T4)$ hold by Proposition 2.12. It follows that $(H/R, \bar{\vee}, \bar{\wedge})$ satisfies $(T1 - T4, T6' - T8')$ and hence is a superlattice.

(ii) Assume $(H/R, \bar{\vee}, \bar{\wedge})$ is a superlattice. Take any $x, y, z \in H$ such that: (a) $\bar{x} = \bar{y}$ and (b) $x \leq z \leq y$. Now $z = x \vee z \in x \sqcup z \Rightarrow \bar{z} \in \overline{x \sqcup z} = \bar{y} \bar{\sqcup} \bar{z}$; similarly $y = z \vee y \in z \sqcup y \Rightarrow \bar{y} \in \overline{z \sqcup y} = \bar{y} \bar{\sqcup} \bar{z}$. Since $(H/R, \bar{\vee}, \bar{\wedge})$ is a superlattice, it follows that $T7'$ holds; from $T7'$ and $\bar{z} \in \bar{y} \bar{\sqcup} \bar{z}$, $\bar{y} \in \bar{z} \bar{\sqcup} \bar{y}$ it follows that $\bar{y} = \bar{z}$. Hence R is w -convex. ■

Before proceeding, let us provide some additional notation and prove an auxiliary proposition.

Notation. For all $A, B \subseteq H$ we define

$$A \vee B \doteq \{x \vee y : x \in A, y \in B\}, \quad A \wedge B \doteq \{x \wedge y : x \in A, y \in B\}.$$

Proposition 3.11 *Let R be an equivalence relation on H . If for some $z \in H$ we have that \bar{z} is a sub- \vee -semi-lattice of H , then $\bar{z} \vee \bar{z} = \bar{z}$.*

Proof. Clearly $\bar{z} \subseteq \bar{z} \vee \bar{z}$. Now, say $u \in \bar{z} \vee \bar{z}$; then exist $x, y \in \bar{z}$ such that $u = x \vee y$. Since \bar{z} is a sub- \vee -semi-lattice, it follows that $u = x \vee y \in \bar{z}$. So $\bar{z} \vee \bar{z} \subseteq \bar{z}$. ■

Let us now continue the study of properties of s -congruences. We have seen that when R is a s -congruence on H , then w -convexity of R is equivalent to $(H/R, \bar{\vee}, \bar{\wedge})$ being a superlattice. The next proposition shows that a number of other properties are equivalent to the above two.

Proposition 3.12 *Let R be a s -congruence on H . Then the following are equivalent.*

1. For all $x \in H$, \bar{x} is a sub- \vee -semi-lattice of H .
2. For all $x \in H$, \bar{x} is a sublattice of H .
3. For all $x \in H$, \bar{x} is w-convex.
4. For all $x \in H$, \bar{x} is s-convex.
5. $(H/R, \overline{\vee}, \overline{\wedge})$ is a superlattice.

Proof. We will show that $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$; also that $(2) \Leftrightarrow (4)$ and $(3) \Leftrightarrow (5)$.

$(1) \Rightarrow (3)$: Suppose that for all $x \in H$, \bar{x} is a sub- \vee -semi-lattice of H . Take any $A \in H/R$ and any $x, y, z \in H$ such that: (a) $\bar{x} = \bar{y} = A$ and (b) $x \leq z \leq y$. From (a) we have: $\overline{x \sqcup z} = \overline{y \sqcup z}$. From (b) it follows that $z = x \vee z \in x \sqcup z \Rightarrow \bar{z} \in \overline{x \sqcup z} = \overline{y \sqcup z}$. Hence exists some $u \in y \sqcup z$ such that: $\bar{u} = \bar{z}$. But $u \in y \sqcup z \Rightarrow u \vee y = u \vee z = y \vee z = y \Rightarrow \overline{u \vee z} = \bar{y} = \bar{x}$. Also, $u \vee z \in \overline{u \vee z} = \bar{z} \vee \bar{z} = \bar{z}$ (since \bar{z} is a sub- \vee -semi-lattice). Hence $\bar{z} = \overline{u \vee z} = \bar{x} = A$. So for all $x \in H$, \bar{x} is w-convex.

$(3) \Rightarrow (2)$: Suppose that for all $x \in H$, \bar{x} is w-convex. Take any $A \in H/R$ and any x, y such that $\bar{x} = \bar{y} = A$. Then $\overline{x \sqcup x} = \overline{x \sqcup y}$. Now $x \vee y \in x \sqcup y \Rightarrow \overline{x \vee y} \in \overline{x \sqcup y} = \overline{x \sqcup x}$. Hence exists some z such that: (a) $z \in x \sqcup x$ and (b) $\bar{z} = \overline{x \vee y}$. We have $z \in x \sqcup x \Rightarrow x \vee z = x \vee x = x \Rightarrow z \leq x$. So we have $z \leq x \leq x \vee y$ and $\bar{z} = \overline{x \vee y}$; by w-convexity of R it follows that $x \vee y \in \bar{x} = A$. We show dually that $x \wedge y \in A$ and so we have established that for all $x \in H$, \bar{x} is a sublattice of H .

$(2) \Rightarrow (1)$: This is obvious.

$(4) \Rightarrow (2)$: This is obvious.

$(2) \Rightarrow (4)$: Assume that for all $x \in H$ we have that \bar{x} is a sublattice. Then, we have already established that, for all $x \in H$, \bar{x} is also w-convex. Since for all $x \in H$, \bar{x} is a w-convex sublattice, it is also s-convex.

$(3) \Leftrightarrow (5)$: This was proved in Proposition 3.10. ■

4 Order on $(H/R, \overline{\vee}, \overline{\wedge})$

We have already noted that: if $(H/R, \overline{\vee}, \overline{\wedge})$ is a superlattice (equivalently, if R is a s-convex s-congruence on H) the order relationship \preceq can be defined on H/R in terms of either $\overline{\vee}$ or $\overline{\wedge}$. We now introduce two additional relationships: \preceq and \sqsubseteq . Then we show that \preceq , \preceq and \sqsubseteq are equivalent when $(H/R, \overline{\vee}, \overline{\wedge})$ is a superlattice.

Definition 4.1 Let R be an equivalence on H . For all $A, B \in H/R$ we write $A \preceq B$ iff: $\forall x \in A, y \in B$ we have $x \wedge y \in A$ and $x \vee y \in B$.

Proposition 4.2 If R is a s-convex equivalence on H , then \preceq is an order relationship on H/R .

Proof. We will show that for all $A, B, C \in H/R$ we have: (i) $A \preceq A$, (ii) $A \preceq B$ and $B \preceq A \Rightarrow A = B$ and (iii) $A \preceq B$ and $B \preceq C \Rightarrow A \preceq C$.

(i) Take any $x, y \in A$. Since A is s-convex, $x \vee y \in A$ and $x \wedge y \in A$.

(ii) Take any $x \in A$ and any $y \in B$. Now $A \preceq B \Rightarrow x \vee y \in B$; and $B \preceq A \Rightarrow x \wedge y \in A$. Since A, B are classes and $A \cap B \neq \emptyset$, it follows that $A = B$.

(iii) Take any $x \in A$, any $y \in B$ and any $z \in C$. Now $x \wedge y \in A$, $x \vee y \in B$, $y \wedge z \in B$, $y \vee z \in C$. Since $x \wedge y \in A$, $y \wedge z \in B$ and $A \preceq B$, it follows that $x \wedge y \wedge z \in A$. Also, by assumption $x \in A$. Since $x \wedge y \wedge z \leq x \wedge z \leq x$, by s-convexity it follows that $x \wedge z \in A$. We can prove similarly that $x \vee z \in C$. Hence $A \preceq C$. ■

Proposition 4.3 *Let R be an equivalence on H . If \preceq is an order relationship on H/R , then for all $A \in H/R$ we have that A is a sublattice.*

Proof. Take any $A \in H/R$ and any $x, y \in A$. Since $A \preceq A$, it follows that $x \wedge y, x \vee y \in A$. ■

Definition 4.4 *Let R be an equivalence on H . For all $A, B \in H/R$ we write $A \sqsubseteq B$ iff:*

$$(i) \forall x \in A \quad \exists y \in B \text{ such that } x \leq y; \quad (ii) \forall y \in B \quad \exists x \in A \text{ such that } x \leq y.$$

Proposition 4.5 *Let R be an equivalence on H . For all $A, B, C \in H/R$ we have:*

$$(i) A \sqsubseteq A; \quad (ii) \text{ if } A \sqsubseteq B \text{ and } B \sqsubseteq C, \text{ then } A \sqsubseteq C.$$

Proof. (i) For all $x \in A$ we have $x \leq x$, so $A \sqsubseteq A$.

(ii) Take any $x \in A$; then exists $y \in B$ such that $x \leq y$. For this y exists $z \in C$ such that $y \leq z$. Hence for every $x \in A$ exists some $z \in C$ such that $x \leq z$. We can similarly prove that for all $z \in C$ exists some $x \in A$ such that $x \leq z$. ■

Proposition 4.6 *Let R be a w -convex equivalence on H and $A, B \in H/R$. Then*

$$(A \neq B \text{ and } A \sqsubseteq B) \Rightarrow \nexists (x, y) \in A \times B \text{ such that } y < x.$$

Proof. Assume that there exists some $x \in A, y \in B$ such that $y < x$. There also exists some $y_1 \in B$ such that $x \leq y_1$. Hence we have $y < x \leq y_1 \Rightarrow x \in B$ (by convexity) which implies that $A \cap B \neq \emptyset$. But A, B were assumed to be distinct classes, so we have reached a contradiction. ■

We will now prove that: if $(H/R, \overline{\vee}, \overline{\wedge})$ is a superlattice then $\preceq, \preceq, \sqsubseteq$ are equivalent. To this end we first need an auxiliary proposition.

Proposition 4.7 *Let R be an equivalence on H . If R is s -convex, then for all $A \in H/R$ we have:*

$$(i) A \vee A = A, \quad (ii) A \wedge A = A.$$

Proof. We will only prove (i), since (ii) is proved similarly. Clearly $A \subseteq A \vee A$. Now take any $x \in A \vee A$, then there exist $y, z \in A$ such that $x = y \vee z$. But, $y \vee z \in A$ by s -convexity of A . So $A \vee A \subseteq A$. ■

Proposition 4.8 *Let R be a s -convex s -congruence on H . Then for all $A, B, C \in H/R$ the following are equivalent: (i) $A \preceq B$, (ii) $A \preceq B$, (iii) $A \sqsubseteq B$.*

Proof. We will prove (i) \Leftrightarrow (ii), (ii) \Leftrightarrow (iii).

(i) \Rightarrow (ii). Assume that $A \preceq B$. Take any $x \in A, y \in B$. Equivalently $\overline{y} \in \overline{x} \overline{\vee} \overline{y}$ and from Proposition 3.5 we have that $\overline{y} = \overline{x} \overline{\vee} \overline{y}$, i.e. that $x \vee y \in \overline{y}$. Similarly we prove that $x \wedge y \in \overline{x}$. Hence $\overline{x} \preceq \overline{y}$, i.e. $A \preceq B$.

(ii) \Rightarrow (i). Assume that $A \preceq B$. Take any $x \in A, y \in B$. Then $x \wedge y \in A = \overline{x} \Rightarrow \overline{x \wedge y} = \overline{x}$. This, by Proposition 3.5, implies $\overline{x} \in \overline{x} \overline{\wedge} \overline{y}$ and so $\overline{x} \preceq \overline{y}$, i.e. $A \preceq B$.

(ii) \Rightarrow (iii). Assume that $A \preceq B$. Take any $x \in A, y \in B$. Then $x \wedge y \in A$; so for any $y \in B$ exists $x_1 = x \wedge y \in A$ such that $x_1 \leq y$. Similarly we can show that for any $x \in A$ exists $y_1 = x \vee y \in B$ such that $x \leq y_1$. Hence $A \sqsubseteq B$.

(iii) \Rightarrow (ii). Assume that $A \sqsubseteq B$. Take any $x \in A, y \in B$. We will consider three cases for the relationship between x and y and we will show that in every case $x \wedge y \in A, x \vee y \in B$.

- (a) If $x \leq y$, then $x \wedge y = x \in A$, $x \vee y = y \in B$.
- (b) Assume $y < x$. Since $A \sqsubseteq B$, we must have $A = B$; because if $A \neq B$ then, by Proposition 4.6, we cannot have $y < x$. Furthermore A is a sublattice, so $x \wedge y \in A$ and $x \vee y \in A = B$.
- (c) Assume $x \parallel y$. Since $A \sqsubseteq B$, exists some $x_1 \in A$ such that $x_1 \leq y$. So $x \wedge x_1 \leq x \wedge y \leq x$ and $x \wedge x_1, x \in A$; then by convexity $x \wedge y \in A$. Similarly it can be shown that $x \vee y \in B$.

So we have shown that for all $x \in A$, $y \in B$ we have $x \wedge y \in A, x \vee y \in B$, i.e. that $A \preceq B$. ■

Remark. Hence when R is a s-convex s-congruence, the \sqsubseteq relationship is an order.

In the case of main interest to us, namely when $(H/R, \overline{\vee}, \overline{\wedge})$ is a superlattice, the three relationships $\preceq, \preceq, \sqsubseteq$ are equivalent and they can be denoted by a single symbol. In such a case we will use the symbol \preceq to denote this *order* on the superlattice $(H/R, \overline{\vee}, \overline{\wedge})$. Let us now establish further properties of the \preceq order.

Proposition 4.9 *Let R be an s-convex s-congruence on H . Then for all $x, y \in H$ we have:*

$$x \leq y \Rightarrow \overline{x} \preceq \overline{y}.$$

Proof. We have $x \leq y \Rightarrow x = x \wedge y \Rightarrow \overline{x} = \overline{x \wedge y}$. But then $\overline{x} \preceq \overline{y}$ by Proposition 3.5. ■

Corollary 4.10 *Let R be an s-convex s-congruence on H . Then for all $x, y, z \in H$ we have:*

$$(i) x \leq y \Rightarrow \overline{x \vee z} \preceq \overline{y \vee z} \quad (ii) x \leq y \Rightarrow \overline{x \wedge z} \preceq \overline{y \wedge z}.$$

Proof. This follows immediately from Proposition 4.9. ■

Proposition 4.11 *Let R be an s-convex s-congruence on H . Then R is a l-congruence on H , i.e. for all $x, y, z \in H$ we have:*

$$(i) \overline{x} = \overline{y} \Rightarrow \overline{x \vee z} = \overline{y \vee z}, \quad (ii) \overline{x} = \overline{y} \Rightarrow \overline{x \wedge z} = \overline{y \wedge z}.$$

Proof. We only prove (i), since (ii) is proved similarly. $\overline{x} = \overline{y} \Rightarrow \overline{x \sqcup z} = \overline{y \sqcup z}$. Since $x \vee z \in x \sqcup z$, exists some $u \in y \sqcup z$ such that $\overline{x \vee z} = \overline{u}$. Since $u \in y \sqcup z$ it follows that $u \leq y \vee z$ and then (by Proposition 4.9) $\overline{x \vee z} = \overline{u} \preceq \overline{y \vee z}$. Similarly we prove $\overline{y \vee z} \preceq \overline{x \vee z}$. Hence $\overline{y \vee z} = \overline{x \vee z}$. ■

Remark. The converse is **not** true, i.e. an l-congruence (which is necessarily s-convex) is not necessarily an s-congruence. This can be seen in the next example.

Example 4.12 *Take the lattice of Figure 1 and let $R = \{A, B\}$, $A = \{0, x_1\}$, $B = \{x_2, 1\}$. It is easy to check that R is a l-congruence, but it is not a s-congruence. For instance $0 \sqcap x_1 = \{0\}$, $x_1 \sqcap x_1 = \{x_1, x_2, 1\}$, $\overline{0 \sqcap x_1} = \{\overline{0}\} = \{A\}$, $\overline{x_1 \sqcap x_1} = \{\overline{x_1}, \overline{x_2}, \overline{1}\} = \{A, B\}$; so $\overline{0} = \overline{x_1}$, but $\overline{0 \sqcap x_1} \neq \overline{x_1 \sqcap x_1}$.*

Figure 1 to be placed here

In fact, the above example is also related to the following proposition.

Proposition 4.13 *Let (H, \vee, \wedge) be a chain and R be a s-congruence on H . If there is some $x \in H$ such that $\text{card}(\overline{x}) \geq 2$, then $H/R = \{H\}$.*

Proof. Given x such that $\text{card}(\bar{x}) \geq 2$, choose some $y \neq x$ such that $\bar{x} = \bar{y}$. Since (H, \vee, \wedge) is a chain, we will have either $x < y$ or $y < x$. Without loss of generality, take $x < y$. Then it is easy to check that $\overline{x \sqcup x} = \overline{x \sqcup y} = \bar{y}$. Define $A = \{z : z \leq x\}$; then it is easy to check that $A = x \sqcup x$. Now take any $z \in A = x \sqcup x$; then $\bar{z} \in \overline{x \sqcup x} = \bar{y}$, i.e. $z \in \bar{z} = \bar{y}$. Hence $A \subseteq \bar{y}$. Defining $B = \{z : x \leq z\}$, one can show similarly that $B \subseteq \bar{y}$. Hence $H = A \cup B \subseteq \bar{y} \subseteq H$ and so the only class of H/R is $\bar{y} = H$. ■

Remark. Hence, if (H, \vee, \wedge) is a chain, the only s-congruences on H are $R_1 = \{H\}$ and $R_2 = \{\{x\}\}_{x \in H}$. The connection to Example 4.12 is now obvious.

Proposition 4.14 *Let R be an s-convex s-congruence on H . Then for all $x, y \in H$ we have:*

$$(i) \bar{x} \preceq \bar{y} \Rightarrow \overline{x \vee z} \preceq \overline{y \vee z}, \quad (ii) \bar{x} \preceq \bar{y} \Rightarrow \overline{x \wedge z} \preceq \overline{y \wedge z}.$$

Proof. We only prove (i), since (ii) is proved similarly. Assuming $\bar{x} \preceq \bar{y}$, it follows that there exists some $y_1 \in \bar{y}$ such that $x \leq y_1$. Then (from Corollary 4.10) we have $\overline{x \vee z} \preceq \overline{y_1 \vee z}$. But, from Corollary 4.11 we have $\overline{y_1 \vee z} = \overline{y \vee z}$ and the proof is complete. ■

5 Additional Operations on Classes

We now define new operations $\bar{\vee}, \bar{\wedge}$ and new hyperoperations $\bar{\sqcup}, \bar{\sqcap}$ between classes in H/R .

Definition 5.1 *Let R be a s-convex s-congruence on H . For all $x, y \in H$ we define*

$$\bar{x} \bar{\vee} \bar{y} \doteq \overline{x \vee y}, \quad \bar{x} \bar{\wedge} \bar{y} \doteq \overline{x \wedge y}.$$

Remark. The above operations are well defined in light of Proposition 4.11.

Proposition 5.2 *If R is a s-convex s-congruence on H , then $(H/R, \bar{\vee}, \bar{\wedge})$ is a modular lattice and $\bar{x} \bar{\vee} \bar{y} = \bar{y} \Leftrightarrow \bar{x} \bar{\wedge} \bar{y} = \bar{x} \Leftrightarrow \bar{x} \preceq \bar{y}$.*

Proof. Since R is an s-convex congruence, in view of Proposition 4.11 it is also an l-congruence. Hence $(H/R, \bar{\vee}, \bar{\wedge})$ is a lattice and the mapping $\rho : H \rightarrow H/R$ is a homomorphism [9].

The $\bar{\vee}, \bar{\wedge}$ hyperoperations are compatible with the \preceq order, i.e. $\bar{x} \bar{\vee} \bar{y} = \bar{y} \Leftrightarrow \bar{x} \preceq \bar{y}$. To see this, suppose that $\bar{x} \bar{\vee} \bar{y} = \bar{y}$, i.e. $\overline{x \vee y} = \bar{y}$; then, using Proposition 4.9, we have $x \leq x \vee y \Rightarrow \bar{x} \preceq \overline{x \vee y} = \bar{y}$. Conversely, from Proposition 4.14 we have $\bar{x} \preceq \bar{y} \Rightarrow \overline{x \vee y} \preceq \overline{y \vee y} = \bar{y}$; since also $y \leq x \vee y \Rightarrow \bar{y} \preceq \overline{x \vee y}$ it follows that $\bar{y} = \overline{x \vee y}$.

Furthermore, $(H/R, \bar{\vee}, \bar{\wedge})$ is a homomorphic image of (H, \vee, \wedge) and, since we have assumed (H, \vee, \wedge) to be modular, $(H/R, \bar{\vee}, \bar{\wedge})$ is also modular [9]. ■

By the above proposition, given a s-convex s-congruence R on H , $(H/R, \bar{\vee}, \bar{\wedge})$ is a modular lattice. Hence we can define the $\bar{\sqcup}, \bar{\sqcap}$ hyperoperations on elements of H/R , in complete analogy to the definition of \sqcup, \sqcap on elements of H .

Definition 5.3 *Let R be a s-convex s-congruence on H . For all $x, y \in H$ we define*

$$\bar{x} \bar{\sqcup} \bar{y} \doteq \{\bar{z} : \bar{x} \bar{\vee} \bar{y} = \bar{x} \bar{\vee} \bar{z} = \bar{y} \bar{\vee} \bar{z}\}$$

$$\bar{x} \bar{\sqcap} \bar{y} \doteq \{\bar{z} : \bar{x} \bar{\wedge} \bar{y} = \bar{x} \bar{\wedge} \bar{z} = \bar{y} \bar{\wedge} \bar{z}\}$$

Proposition 5.4 *Let R be a s-convex s-congruence on H . Then $(H/R, \bar{\sqcup}, \bar{\sqcap})$ is a superlattice.*

Proof. This is proved in exactly the same manner as Proposition 2.2. ■

We have at this point obtained two superlattices on H/R , namely $(H/R, \overline{\vee}, \overline{\wedge})$ and $(H/R, \overline{\sqcup}, \overline{\sqcap})$. A natural question to ask is what is the relationship between these. We will show that in certain cases $(H/R, \overline{\vee}, \overline{\wedge})$ and $(H/R, \overline{\sqcup}, \overline{\sqcap})$ are identical. To this end we first establish the auxiliary propositions 5.5 - 5.9.

Proposition 5.5 *If (H, \vee, \wedge) is a distributive lattice, then for all x_1, x_2, y_1, y_2 (where $x_1 \leq x_2$ and $y_1 \leq y_2$) we have*

$$[x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2].$$

$$[x_1, x_2] \wedge [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2].$$

Proof. This can be found in [3]. ■

Proposition 5.6 *Let R be a s -convex s -congruence on H . If $A \in H/R$ and $x, y \in A$ with $x < y$, then:*

- (i) *there does not exist (nonempty) $B \in H/R$ such that: $\forall z \in B$ we have $z \leq x$;*
- (ii) *there does not exist (nonempty) $C \in H/R$ such that: $\forall u \in C$ we have $y \leq u$.*

Proof. We only prove (i); (ii) is proved dually.

Assume that there exists some $B \in H/R$ such that: $\forall z \in B$ we have $z \leq x$. Then $z \in B \Rightarrow z \vee x = x \vee x \Rightarrow z \in x \sqcup x \Rightarrow B = \overline{z} \in \overline{x \sqcup x}$. In short:

$$z \in B \Rightarrow B \in \overline{x \sqcup x}. \quad (1)$$

But also: $z \in B \Rightarrow x \vee z = x$, and $y \vee z = y \neq x$ and $x \vee y = y$; these mean $z \notin x \sqcup y$. In short,

$$\forall z \in B : z \notin x \sqcup y. \quad (2)$$

Now, (2) implies that $B \notin \overline{x \sqcup y}$. Indeed, assume the contrary, i.e. that $B \in \overline{x \sqcup y}$. Then exists some $w \in B$ such that $w \in x \sqcup y$; but this contradicts (2). In short we have proved:

$$z \in B \Rightarrow B \notin \overline{x \sqcup y}. \quad (3)$$

But, since $\overline{x \sqcup x} = \overline{x \sqcup y}$, (1) and (3) are in contradiction and we have completed the proof of the proposition. ■

Proposition 5.7 *Let R be a s -convex s -congruence on H and $\overline{x} = [x_1, x_2]$, $\overline{y} = [y_1, y_2]$. Then:*

- (i) *If $\overline{x \vee y} = [p, q]$, then $p = x_1 \vee y_1$ and $q = x_2 \vee y_2$.*
- (ii) *If $\overline{x \wedge y} = [r, s]$, then $r = x_1 \wedge y_1$ and $s = x_2 \wedge y_2$.*

Proof. We will only prove (i), since (ii) is proved dually.

(a) Clearly $\overline{x_1 \vee y_1} = \overline{x \vee y} = [p, q] \Rightarrow x_1 \vee y_1 \in [p, q] \Rightarrow p \leq x_1 \vee y_1$. Also, $\overline{x} \preceq \overline{x \vee y} = [p, q] \Rightarrow \exists x_0 \in \overline{x}$ such that $x_1 \leq x_0 \leq p$; similarly $\exists y_0 \in \overline{y}$ such that $y_1 \leq y_0 \leq p$; hence $x_1 \vee y_1 \leq p$. Hence $x_1 \vee y_1 = p$.

(b) We consider two cases.

(b.1) $\overline{x} = \overline{y}$. Then $[p, q] = \overline{x \vee y} = \overline{x \vee x} = \overline{x} = [x_1, x_2] \Rightarrow q = x_2 = x_2 \vee y_2$.

(b.2) $\overline{x} \neq \overline{y}$. Then either $\overline{x} \neq \overline{x \vee y}$ or $\overline{y} \neq \overline{x \vee y}$ or both. Assume $\overline{x} \neq \overline{x \vee y}$. Now $\overline{x_2 \vee y_2} = \overline{x \vee y} = [p, q]$. Hence $x_2 \vee y_2 \in [p, q] \Rightarrow x_2 \vee y_2 \leq q$. On the other hand, since $x_2 \in \overline{x}$, $x_2 \vee y_2 \in \overline{x \vee y}$ and $\overline{x} \neq \overline{x \vee y}$, it follows that $x_2 \neq x_2 \vee y_2$ and this means $x_2 < x_2 \vee y_2$. Now assume that $x_2 \vee y_2 < q$. Then for all $z \in \overline{x}$ we have $z \leq x_2 < x_2 \vee y_2 < q$. In other words, for all $z \in \overline{x}$ we have $z < x_2 \vee y_2 < q$. But this contradicts Proposition 5.6 (since $x_2 \vee y_2 \in \overline{x \vee y}$, $q \in \overline{x \vee y}$). So we cannot have $x_2 \vee y_2 < q$ – hence we must have $x_2 \vee y_2 = q$. ■

Proposition 5.8 *Let R be a s -convex s -congruence on H . If (H, \vee, \wedge) is a distributive lattice and for all $Q \in H/R$ there exist $q_1, q_2 \in H$ with $Q = [q_1, q_2]$, then for all $x, y \in H$ we have:*

$$(i) \ \overline{x \vee y} = \overline{x} \vee \overline{y}, \quad (ii) \ \overline{x \wedge y} = \overline{x} \wedge \overline{y}.$$

Proof. We will only prove (i), since (ii) is proved dually. Take any $x, y \in H$. By assumption we have $\overline{x \vee y} = \overline{x} \vee \overline{y} = [p, q]$, $\overline{x} = [x_1, x_2]$ and $\overline{y} = [y_1, y_2]$. By Proposition 5.7 we have $p = x_1 \vee y_1$ and $q = x_2 \vee y_2$. Hence $\overline{x \vee y} = [x_1 \vee y_1, x_2 \vee y_2] = [x_1, x_2] \vee [y_1, y_2] = \overline{x} \vee \overline{y}$ (note the use of Proposition 5.5). ■

Proposition 5.9 *Let R be a s -convex s -congruence on H .*

(i) *If for all $x, y \in H$ we have $\overline{x \vee y} = \overline{x} \vee \overline{y}$, then for all $x, y \in H$ we also have $\overline{x \sqcup y} = \overline{x} \sqcup \overline{y}$.*

(ii) *If for all $x, y \in H$ we have $\overline{x \wedge y} = \overline{x} \wedge \overline{y}$, then for all $x, y \in H$ we also have $\overline{x \sqcap y} = \overline{x} \sqcap \overline{y}$.*

Proof. We will only prove (i), since (ii) is proved dually.

(i.1) Take any $A \in \overline{x \sqcup y}$. Then exists $z_0 \in A$ such that $z_0 \vee x = z_0 \vee y = x \vee y \Rightarrow \overline{z_0 \vee x} = \overline{z_0 \vee y} = \overline{x \vee y} \Rightarrow \overline{z_0 \vee x} = \overline{z_0} \vee \overline{x} = \overline{z_0} \vee \overline{y} = \overline{x \sqcup y}$.

(i.2) Take any $\overline{z_0} \in \overline{x \sqcup y}$. This implies that $\overline{z_0 \vee x} = \overline{z_0} \vee \overline{x} = \overline{x \sqcup y}$. This implies (using the assumption of (i)) that $\overline{z_0 \vee x} = \overline{z_0} \vee \overline{x} = \overline{x \vee y}$. Now $z_0 \vee x \in \overline{z_0 \vee x}$ and so exist $z_1 \in \overline{z_0}$, $y_1 \in \overline{y}$ such that $z_0 \vee x = z_1 \vee y_1$. Similarly, exist $x_2 \in \overline{x}$, $y_2 \in \overline{y}$ such that $z_0 \vee y = x_2 \vee y_2$. Now:

$$z_0 \vee x = z_1 \vee y_1 \Rightarrow z_0 \vee x \vee z_1 = z_1 \vee y_1 \vee z_1 = z_1 \vee y_1 = x_2 \vee y_2 \quad (4)$$

and

$$z_0 \vee x = z_1 \vee y_1 \Rightarrow z_0 \vee x \vee z_0 = z_1 \vee y_1 \vee z_0 = z_0 \vee x_2 \vee y_2 = x_2 \vee y_2 \quad (5)$$

(in the last step we have used that $z_0 \leq z_0 \vee x = x_2 \vee y_2$.) Now, (4) implies

$$z_0 \vee x \vee z_1 = x_2 \vee y_2 \Rightarrow z_0 \vee x \vee z_1 \vee x_2 = x_2 \vee y_2 \quad (6)$$

and (5) implies

$$z_0 \vee y_1 \vee z_1 = x_2 \vee y_2 \Rightarrow z_0 \vee x \vee z_1 \vee y_2 = x_2 \vee y_2. \quad (7)$$

But $x \leq z_0 \vee x = x_2 \vee y_2$ and $y_1 \leq z_1 \vee y_1 = x_2 \vee y_2$ imply

$$x \vee y_1 \leq x_2 \vee y_2 \Rightarrow x_2 \vee y_2 = x \vee y_1 \vee x_2 \vee y_2. \quad (8)$$

Hence (6) and (8) imply

$$\begin{aligned} z_0 \vee x \vee z_1 \vee x_2 &= x \vee y_1 \vee x_2 \vee y_2 \Rightarrow \\ (z_0 \vee z_1) \vee (x \vee x_2) &= (x \vee x_2) \vee (y_1 \vee y_2); \end{aligned} \quad (9)$$

similarly (7) and (8) imply

$$\begin{aligned} z_0 \vee y_1 \vee z_1 \vee y_2 &= x \vee y_1 \vee x_2 \vee y_2 \Rightarrow \\ (z_0 \vee z_1) \vee (y_1 \vee y_2) &= (x \vee x_2) \vee (y_1 \vee y_2). \end{aligned} \quad (10)$$

From (9) and (10) we see that $z_0 \vee z_1 \in (x \vee x_2) \sqcup (y_1 \vee y_2)$. Then it follows that

$$\overline{z_0 \vee z_1} \in \overline{(x \vee x_2) \sqcup (y_1 \vee y_2)} = \overline{(x \vee x_2)} \overline{\vee} \overline{(y_1 \vee y_2)}. \quad (11)$$

Finally

$$z_0, z_1 \in \overline{z_0} \Rightarrow z_0 \vee z_1 \in \overline{z_0} \Rightarrow \overline{z_0 \vee z_1} = \overline{z_0} \quad (12)$$

$$x, x_2 \in \bar{x} \Rightarrow x \vee x_2 \in \bar{x} \Rightarrow \overline{x \vee x_2} = \bar{x} \quad (13)$$

$$y_1, y_2 \in \bar{y} \Rightarrow y_1 \vee y_2 \in \bar{y} \Rightarrow \overline{y_1 \vee y_2} = \bar{y} \quad (14)$$

and (12), (13), (14) in conjunction with (11) imply that $\bar{z}_0 \in \bar{x} \bar{y}$ and hence $\bar{x} \bar{y} \subseteq \bar{x} \bar{y}$. This, in conjunction with the conclusion of (i.1) means that $\bar{x} \bar{y} = \bar{x} \bar{y}$. ■

We are now ready to give a condition for $(H/R, \bar{\vee}, \bar{\wedge})$ and $(H/R, \bar{\sqcup}, \bar{\sqcap})$ to be identical.

Proposition 5.10 *Let R be a s -convex s -congruence on H . If for all $x, y \in H$ we have $\bar{x} \bar{y} = \bar{x} \vee \bar{y}$, $\bar{x} \bar{y} = \bar{x} \wedge \bar{y}$ then $(H/R, \bar{\vee}, \bar{\wedge})$ and $(H/R, \bar{\sqcup}, \bar{\sqcap})$ are identical hyperstructures.*

Proof. From Proposition 5.9 follows that $\bar{\sqcup}, \bar{\vee}$ give identical results; the same is true of $\bar{\sqcap}, \bar{\wedge}$. ■

A special case of interest is the following.

Proposition 5.11 *Let R be a s -convex s -congruence on H . If (H, \vee, \wedge) is a finite distributive lattice, then $(H/R, \bar{\vee}, \bar{\wedge})$ and $(H/R, \bar{\sqcup}, \bar{\sqcap})$ are identical hyperstructures.*

Proof. Since (H, \vee, \wedge) is finite, every convex sublattice is an interval. Hence all the classes of H/R are intervals. Since (H, \vee, \wedge) is also distributive, we can apply Proposition 5.8 to obtain the conditions of Proposition 5.9, which in turn allow application of Proposition 5.10. ■

Finally, we present two examples where l -congruences fail to be s -congruences and an example of a l -congruence which is also a s -congruence.

Example 5.12 *In the lattice of Figure 2, take $R = \{A, B, C, D\}$, $A = \{0\}$, $B = \{x_1\}$, $C = \{x_2\}$, $D = \{x_3, 1\}$. It is easy to check that R is a l -congruence but it is not a s -congruence (for instance check that $\overline{x_3 \sqcup x_3} = \{A, B, C, D\} \neq \{D\} = \overline{x_3 \sqcup 1}$). In addition, note that, for example, $B \bar{\vee} C = D = \{x_3, 1\} \neq \{x_3\} = B \vee C$. Also note that in class D we have $x_3 < 1$ and that x_3 is greater than any element of class B (compare with Proposition 5.6).*

Figure 2 to appear here

Example 5.13 *In the lattice of Figure 3, take $R = \{A, B, C, D\}$, $A = \{0\}$, $B = \{x_1, x_3\}$, $C = \{x_2, x_5\}$, $D = \{x_4, x_6, x_7, 1\}$. It is easy to check that R is a l -congruence but it is not a s -congruence (for instance check that $\overline{x_4 \sqcup x_4} = \{A, B, C\} \neq \{D\} = \overline{x_4 \sqcup 1}$). However, note that, for example, $C \bar{\vee} D = D = \{x_4, x_6, x_7, 1\} = C \vee D$. Note also that for the pair $(x_6, 1)$ we have $x_6 < 1$ and yet x_6 is greater than any element of class B (compare with Proposition 5.6).*

Figure 3 to appear here

Example 5.14 *Finally, in the Boolean lattice of Figure 4, $A = \{0, x_2\}$, $B = \{x_1, x_4\}$, $C = \{x_3, x_6\}$, $D = \{x_5, 1\}$. It is easy to check that R is both an l -congruence and a s -congruence.*

Figure 4 to appear here

6 Discussion

We have introduced the Nakano superlattice (H, \sqcup, \sqcap) and studied the properties of s-congruences R on (H, \sqcup, \sqcap) . When R is s-convex, the resulting hyperstructure $(H/R, \overline{\vee}, \overline{\wedge})$ possesses many interesting properties. The following questions arise naturally.

Question 1 Is there an example of a relationship R which is a s-congruence with respect to the Nakano superlattice, but is not s-convex?

Question 2 What are *convenient* necessary and/or sufficient conditions to check if an equivalence is s-congruence with respect to the Nakano superlattice?

Furthermore, the study of the Nakano superlattice appears to be of special interest when the underlying lattice (H, \leq) is either Boolean or deMorgan (in the latter case connections to the theory of fuzzy sets are possible). These issues will be the subject of further research.

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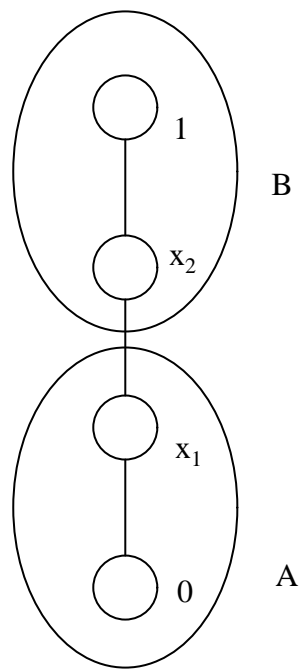


Figure 1

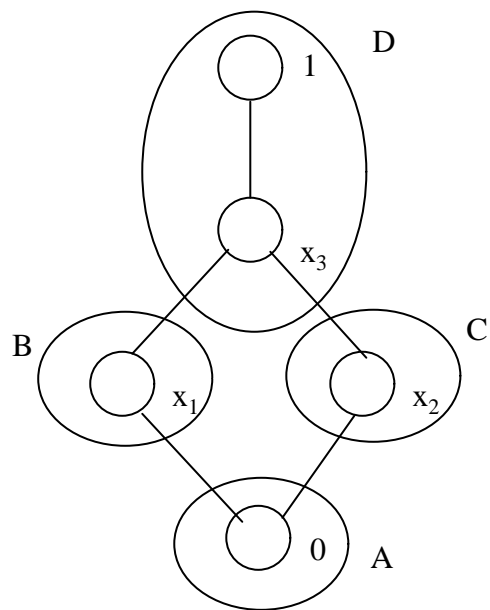


Figure 2

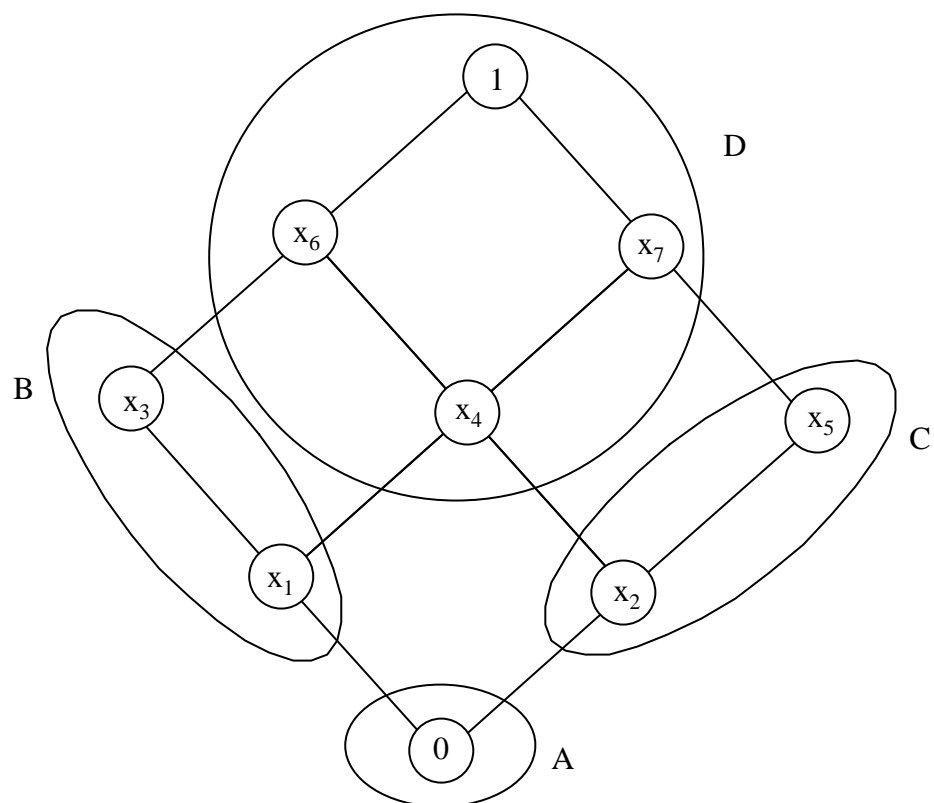


Figure 3

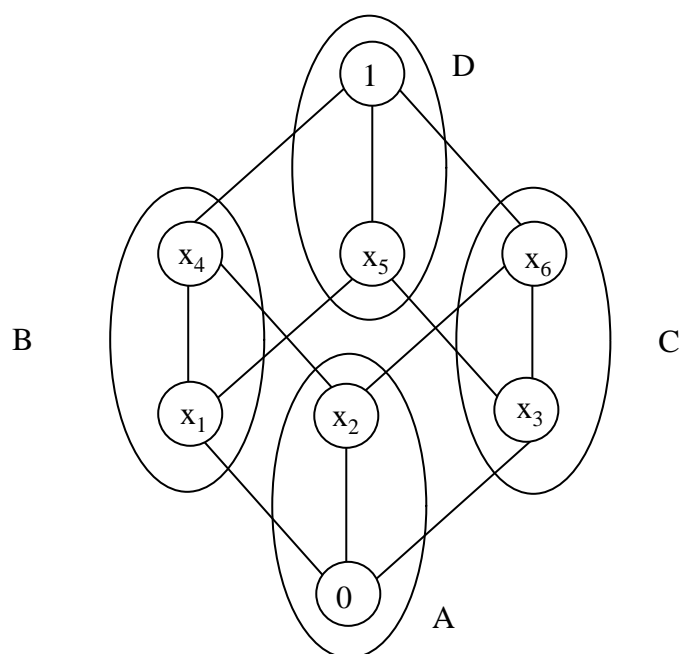


Figure 4