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# An example of L-fuzzy Join Space

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## Abstract

On a generalized deMorgan lattice  $(X, \leq, \vee, \wedge, ')$  we introduce a family of join hyperoperations  $*_p$ , parametrized by a parameter  $p \in X$ . As a result we obtain a family of join spaces  $(X, *_p)$ . We show that: for every  $a, b \in X$  the family  $\{a *_p b\}_{p \in X}$  can be considered as the  $p$ -cuts of a  $L$ -fuzzy set  $a * b$ ; in this manner we synthesize a  $L$ -fuzzy hyperoperation  $*$  which takes pairs from  $X$  to  $L$ -fuzzy subsets of  $X$ . We then show that  $(X, *)$  is a  $L$ -fuzzy hypergroup (in the sense of Corsini) and can be considered as a  $L$ -fuzzy join space. Furthermore,  $a * b$  is a  $L$ -fuzzy interval for all  $a, b \in X$ .

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## 1 Introduction

In this paper we present an algebraic structure which is related to several themes from the theories of hypergroups, join spaces and fuzzy sets. Let us first briefly review some work on these themes.

*Hypergroups* have been introduced in [14] and studied extensively by many mathematicians. A comprehensive review of the theory of hypergroups appears in [3].

*Join spaces* have been introduced by Jantosciak and Prenowitz [18]. The topic is covered in considerable detail in [19]. Jantosciak has noted that join spaces are a special case of hypergroups [9].

In recent years there has been considerable activity on *fuzzy algebraic hyperstructures*. Corsini and Zahedi introduced some crisp algebraic hyperstructures related to fuzzy sets [5, 6, 30]. More recently, the concept of *fuzzy hypergroups* has been introduced in [4]; in fact the closely related concept of *fuzzy polygroups* has been introduced somewhat earlier in [28, 29]. *Fuzzy hyperrings* are studied in [7].

*Fuzzy lattices* have been introduced in [27] and studied in more detail in [1]. *L-fuzzy lattices* have been studied in [23] and [24]. *Fuzzy intervals* (in a lattice theoretic sense, as a special case of fuzzy lattices) are studied in [13].

In this paper we do the following. On a generalized deMorgan lattice  $(X, \leq, \vee, \wedge, ')$  we introduce a family of join hyperoperations  $*_p$ , parametrized by a parameter  $p \in X$ . As a result we obtain a family of join spaces  $(X, *_p)$ . We show that: for every  $a, b \in X$  the family  $\{a *_p b\}_{p \in X}$  can be considered as the  $p$ -cuts of a  $L$ -fuzzy set  $a * b$ ; in this manner we synthesize a fuzzy hyperoperation  $*$  which takes pairs from  $X$  to  $L$ -fuzzy subsets of  $X$ . We then show that  $(X, *)$  is a  $L$ -fuzzy hypergroup (in the sense of Corsini) and also a  $L$ -fuzzy join space. Furthermore,  $a * b$  is a  $L$ -fuzzy interval for all  $a, b \in X$ .

## 2 Preliminaries

In what follows,  $(X, \leq, \vee, \wedge, ')$  will be a complete *generalized* de Morgan lattice, i.e. a complete lattice satisfying the following definition.

**Definition 2.1** A generalized deMorgan lattice is a structure  $(X, \leq, \vee, \wedge, ')$ , where  $(X, \leq, \vee, \wedge)$  is a distributive lattice with minimum element 0 and maximum element 1; the symbol  $'$  denotes an unary operation (“negation”); and the following properties are satisfied.

1. For all  $x \in X, Y \subseteq X$  we have  $x \wedge (\vee_{y \in Y} y) = \vee_{y \in Y} (x \wedge y)$ ,  $x \vee (\wedge_{y \in Y} y) = \wedge_{y \in Y} (x \vee y)$ . (Complete distributivity).
2. For all  $x \in X$  we have:  $(x')' = x$ . (Negation is involutory).
3. For all  $x, y \in X$  we have:  $x \leq y \Rightarrow y' \leq x'$ . (Negation is order reversing).
4. For all  $Y \subseteq X$  we have  $(\vee_{y \in Y} y)' = \wedge_{y \in Y} y'$ ,  $(\wedge_{y \in Y} y)' = \vee_{y \in Y} y'$  (Complete deMorgan laws).

A fuzzy set is a mapping from  $X$  to the interval of real numbers  $[0,1]$ ; a  $L$ -fuzzy set is a mapping from  $X$  to a set  $L$ , where  $(L, \sqsubseteq)$  is a complete lattice. In this paper we will be concerned with  $L$ -fuzzy sets which map  $X$  to itself. The following definitions and notation will be used.

1. The collection of all crisp subsets of  $X$  is denoted by  $\mathbf{P}(X)$  (the power set of  $X$ ).
2. The collection of all  $L$ -fuzzy sets (i.e. functions  $M : X \rightarrow X$ ) is denoted by  $\mathbf{F}(X)$ .
3. A (crisp) hyperoperation is a mapping  $\circ : X \times X \rightarrow \mathbf{P}(X)$ ; a  $L$ -fuzzy hyperoperation is a mapping  $\circ : X \times X \rightarrow \mathbf{F}(X)$ .
4. An order is introduced in  $\mathbf{F}(X)$  using the “pointwise” order of  $(X, \leq, \vee, \wedge)$  (the symbols  $\leq, \vee, \wedge$  will be used without danger of confusion), i.e. for  $M, N \in \mathbf{F}(X)$  we write  $M \leq N$  iff for all  $x \in X$  we have:  $M(x) \leq N(x)$ .
5. For all  $M, N \in \mathbf{F}(X)$ : we define the  $L$ -fuzzy set  $M \vee N$  by:  $(M \vee N)(x) = M(x) \vee N(x)$ ; we define the  $L$ -fuzzy set  $M \wedge N$  by:  $(M \wedge N)(x) = M(x) \wedge N(x)$ . It is well known [15] that  $\leq$  is an order on  $\mathbf{F}(X)$  and that  $(\mathbf{F}(X), \leq, \vee, \wedge)$  is a complete and distributive lattice with  $\sup(M, N) = M \vee N$ ,  $\inf(M, N) = M \wedge N$ .
6. For all  $M, N \in \mathbf{P}(X)$  we write  $M \sim N$  iff  $\exists x \in M \cap N$ . For  $M, N \in \mathbf{F}(X)$  and  $p \in X$  we write  $M \sim_p N$  iff  $\exists x \in X : M(x) \wedge N(x) \geq p$ .
7. Given a  $L$ -fuzzy set  $M : X \rightarrow X$ , the  $p$ -cut of  $M$  is denoted by  $M_p$  and defined by  $M_p \doteq \{x : M(x) \geq p\}$ .
8. We use the standard notation of algebraic hyperstructures, whereby for any operation (resp. hyperoperation)  $\circ : X \times X \rightarrow X$  (resp.  $\circ : X \times X \rightarrow \mathbf{P}(X)$ ) and any  $A, B \in \mathbf{P}(X)$ , we define  $A \circ B \doteq \cup_{a \in A, b \in B} a \circ b$ .
9. The collection of crisp intervals of  $X$  is denoted by  $\mathbf{I}(X)$  and defined by  $\mathbf{I}(X) \doteq \{[a, b]\}_{a, b \in X} \subseteq \mathbf{P}(X)$ ; the empty set  $\emptyset$  is also considered a member of  $\mathbf{I}(X)$  and can be symbolized by  $\emptyset = [a, b]$  with any  $a, b$  such that  $a \not\leq b$ .

Let us also recall a few well known facts about closed intervals.

1. Since  $\subseteq$  is an order on  $\mathbf{P}(X)$ , it is also an order on  $\mathbf{I}(X) \subseteq \mathbf{P}(X)$ .
2. Take any two sets  $A = [a_1, a_2], B = [b_1, b_2] \in \mathbf{I}(X)$ . Note that

$$[a_1, a_2] \subseteq [b_1, b_2] \Leftrightarrow (b_1 \leq a_1 \leq a_2 \leq b_2).$$

3. With  $\cap$  denoting the usual set theoretic intersection we have

$$[a_1, a_2] \cap [b_1, b_2] = [a_1 \vee b_1, a_2 \wedge b_2].$$

4. Also, for any  $A, B \subseteq X$  define the family of sets  $\mathbf{Q}(A, B) \doteq \{C : C \in I(X), A \subseteq C \text{ and } B \subseteq C\}$ ; next define the operation  $\dot{\cup}$  on elements of  $\mathbf{I}(X)$  by

$$A \dot{\cup} B = \cap_{C \in \mathbf{Q}(A, B)} C.$$

5. It is easy to prove that  $[a_1, a_2] \dot{\cup} [b_1, b_2] = [a_1 \wedge b_1, a_2 \vee b_2]$  and that  $(\mathbf{I}(X), \subseteq, \dot{\cup}, \cap)$  is a lattice. If  $(X, \leq, \vee, \wedge)$  is complete (which is assumed throughout this paper) then  $(\mathbf{I}(X), \subseteq, \dot{\cup}, \cap)$  is also complete.

Finally, the following properties of  $\vee$  and  $\wedge$  acting on intervals of a distributive lattice will be used in the sequel.

**Proposition 2.2** *For all  $a, b, x, y \in X$  such that  $x \leq y$ ,  $a \leq b$  we have:*

- (i)  $a \vee [x, y] = [a \vee x, a \vee y];$
- (ii)  $a \wedge [x, y] = [a \wedge x, a \wedge y];$
- (iii)  $[a, b] \vee [x, y] = [a \vee x, b \vee y];$
- (iv)  $[a, b] \wedge [x, y] = [a \wedge x, b \wedge y].$

**Proof.** In [11]. ■

The following propositions describe some properties of  $p$ -cuts. Their proofs can be found in [15].

**Proposition 2.3** *Take any  $M \in \mathbf{F}$  with  $p$ -cuts  $\{M_p\}_{p \in X}$ . Then we have the following.*

- (i) *For all  $p, q \in X$  we have:  $p \leq q \Rightarrow M_q \subseteq M_p$ .*
- (ii) *For all  $P \subseteq X$  we have:  $\cap_{p \in P} M_p = M_{\vee P}$ .*
- (iii)  $M_0 = X$ .

**Proposition 2.4** *Consider a family of sets  $\{\widetilde{M}_p\}_{p \in X}$  which satisfy the following.*

- (i) *For all  $p, q \in X$  we have:  $p \leq q \Rightarrow \widetilde{M}_q \subseteq \widetilde{M}_p$ .*
- (ii) *For all  $P \subseteq X$  we have:  $\cap_{p \in P} \widetilde{M}_p = \widetilde{M}_{\vee P}$ .*
- (iii)  $\widetilde{M}_0 = X$ .

*Define the  $L$ -fuzzy set  $M$  as follows: for all  $x \in X$  define  $M(x) \doteq \vee\{p : x \in \widetilde{M}_p\}$ . Then for all  $p \in X$  we have  $M_p = \widetilde{M}_p$ .*

**Proposition 2.5** *For any fuzzy sets  $M, N \in \mathbf{F}(X)$  we have:  $M = N \Leftrightarrow (\forall p \in X \text{ we have } M_p = N_p)$ .*

### 3 The $p$ -Join and $p$ -Extension Hyperoperations

In this section we define a family of *crisp* join and extension hyperoperations.

### 3.1 The $p$ -Join

For every  $p \in X$  we define a hyperoperation  $*_p : X \times X \rightarrow \mathbf{I}(X)$  as follows.

**Definition 3.1** For every  $p \in X$  and for all  $a, b \in X$ , the  $p$ -join hyperoperation is denoted by  $*_p$  and defined by

$$a *_p b \doteq [a \wedge b \wedge p, a \vee b \vee p'].$$

The following property of the join will be used in the sequel.

**Proposition 3.2** For all  $a, b, c, d, p \in X$  such that  $a \leq b, c \leq d$  we have

$$(i) \quad [a, b] *_p c = [a \wedge c \wedge p, b \vee c \vee p'].$$

$$(ii) \quad [a, b] *_p [c, d] = [a \wedge c \wedge p, b \vee d \vee p'].$$

**Proof.** Choose any  $a, b, c, d, p \in X$  such that  $a \leq b, c \leq d$ .

To prove (i), first choose any  $u \in [a, b] *_p c$ . Then  $\exists x \in [a, b]$  such that  $u \in x *_p c$ . I.e.

$$\exists x : \left. \begin{array}{l} a \leq x \leq b \\ x \wedge c \wedge p \leq u \leq x \vee c \vee p' \end{array} \right\} \Rightarrow$$

$$\exists x : a \wedge c \wedge p \leq x \wedge c \wedge p \leq u \leq x \vee c \vee p' \leq b \vee c \vee p' \Rightarrow u \in [a \wedge c \wedge p, b \vee c \vee p'].$$

Hence  $u \in [a \wedge c \wedge p, b \vee c \vee p']$ . It follows that  $[a, b] *_p c \subseteq [a \wedge c \wedge p, b \vee c \vee p']$ .

Second, choose any  $v \in [a \wedge c \wedge p, b \vee c \vee p']$ . Set  $z_v = (v \vee a) \wedge b = (v \wedge b) \vee a$ . So  $a \leq (v \wedge b) \vee a = z_v = (v \vee a) \wedge b \leq b \Rightarrow$

$$z_v \in [a, b]. \quad (1)$$

Also,  $z_v \wedge c \wedge p = (v \vee a) \wedge b \wedge c \wedge p = (v \wedge b \wedge c \wedge p) \vee (a \wedge b \wedge c \wedge p)$ . But  $v \wedge b \wedge c \wedge p \leq v$  and  $a \wedge b \wedge c \wedge p = a \wedge c \wedge p \leq v$ . Hence

$$z_v \wedge c \wedge p \leq v \quad (2)$$

Similarly,  $z_v \vee c \vee p' = (v \wedge b) \vee a \vee c \vee p' = (v \vee a \vee c \vee p') \wedge (b \vee a \vee c \vee p')$ . But  $v \vee a \vee c \vee p' \geq v$  and  $b \vee a \vee c \vee p' = b \vee c \vee p' \geq v$ . Hence

$$v \leq z_v \vee c \vee p'. \quad (3)$$

From (2)-(3) we see that  $v \in z_v *_p c$  which, together with (1) implies that  $v \in [a, b] *_p c$ . Hence  $[a \wedge c \wedge p, b \vee c \vee p'] \subseteq [a, b] *_p c$ .

We conclude that  $[a \wedge c \wedge p, b \vee c \vee p'] = [a, b] *_p c$ , which proves (i); (ii) is proved similarly. ■

We are now ready to present the basic properties of the  $p$ -join hyperoperation.

**Proposition 3.3** The following properties hold for any  $a, b, c, p, q \in X$ .

$$(i) \quad a \in a *_p a; \quad a, b \in a *_p b.$$

$$(ii) \quad a *_p b = b *_p a.$$

$$(iii) \quad (a *_p b) *_p c = a *_p (b *_p c).$$

$$(iv) \quad a *_p X = X.$$

**Proof.** Choose any  $a, b, c, p, q \in X$ .

(i) and (ii) are obvious.

For (iii), note that  $(a *_p b) *_p c = [a \wedge b \wedge p, a \vee b \vee p'] *_p c = [a \wedge b \wedge c \wedge p, a \vee b \vee c \vee p']$  (where we have used Proposition 3.2). Similarly  $a *_p (b *_p c) = a *_p [b \wedge c \wedge p, b \vee c \vee p'] = [a \wedge b \wedge c \wedge p, a \vee b \vee c \vee p']$  and we are done.

For (iv), clearly we have  $a *_p X \subseteq X$ . On the other hand, taking any  $x \in X$  we have  $x \in a *_p x$ . Hence  $X = \cup_{x \in X} \{x\} \subseteq \cup_{x \in X} a *_p x = a *_p X$  and we are done. ■

**Proposition 3.4** (i) For every  $p \in X$ ,  $(X, *_p)$  is a commutative hypergroup  
(ii) For every  $a, b \in X$ ,  $(a *_p b, *_p)$  is a commutative sub-hypergroup of  $X$ .

**Proof.** Choose any  $p \in X$ . The proof of (i) is immediate by Proposition 3.3 and the definition of hypergroup [3].

For (ii), choose any  $a, b \in X$ . We need to prove: (ii.1)  $x, y \in a *_p b \Rightarrow x *_p y \subseteq a *_p b$  and (ii.2)  $x \in a *_p b \Rightarrow x *_p a *_p b = a *_p b$ . Regarding (ii.1) we have

$$\left. \begin{array}{l} a \wedge b \wedge p \leq x \leq a \vee b \vee p' \\ a \wedge b \wedge p \leq y \leq a \vee b \vee p' \end{array} \right\} \Rightarrow a \wedge b \wedge p \leq x \wedge y \wedge p \leq x \vee y \vee p' \leq a \vee b \vee p' \Rightarrow$$

$$x *_p y \subseteq a *_p b.$$

Regarding (ii.2) we have  $a \wedge b \wedge p \leq x \leq a \vee b \vee p'$ , which implies  $x *_p a *_p b = [x \wedge a \wedge b \wedge p, x \vee a \vee b \vee p'] = [a \wedge b \wedge p, a \vee b \vee p'] = a *_p b$ . ■

**Proposition 3.5** For all  $a, b, c, p \in X$ , we have  $a *_p (b *_q c) = (a *_p b) *_q c = a *_p \wedge q b *_p \wedge q c$ .

**Proof.** Choose any  $a, b, c, p \in X$ . We have

$$a *_p (b *_q c) = a *_p [b \wedge c \wedge q, b \vee c \vee q'] = [a \wedge b \wedge c \wedge p \wedge q, a \vee b \vee c \vee p' \vee q'] = a *_p \wedge q b *_p \wedge q c$$

(where we have used  $p' \vee q' = (p \wedge q)'$ ); similarly

$$(a *_p b) *_q c = [a \wedge b \wedge p, a \vee b \vee p'] *_q c = [a \wedge b \wedge c \wedge p \wedge q, a \vee b \vee c \vee p' \vee q'] = a *_p \wedge q b *_p \wedge q c.$$

■

### 3.2 The $p$ -Extension

We will now introduce the  $p$ -extension hyperoperation which is derived, in the manner of Prenowitz [19], from the  $p$ -join hyperoperation.

**Definition 3.6** For every  $p \in X$  and for all  $a, b \in X$ , the  $p$ -extension hyperoperation is denoted by  $a/_p b$  and is defined by:

$$a/_p b \doteq \{x : a \in x *_p b\} = \{x : x \wedge b \wedge p \leq a \leq x \vee b \vee p'\}.$$

It is seen immediately that for any  $a, b \in X$  we have  $a \in a/_p b$ . In addition, the extension hyperoperation enjoys the *join* property [19].

**Proposition 3.7** For all  $a, b, c, d \in X$  we have:  $(a/_p b) \sim (c/_p d) \Rightarrow a *_p d \sim b *_p c$ .

**Proof.** Suppose we have  $(a/_p b) \sim (c/_p d)$ . Then there exists some  $x$  such that

$$x \in a/_p b \cap c/_p d \Rightarrow \left\{ \begin{array}{l} a \in x *_p b \\ c \in x *_p d \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x \wedge b \wedge p \leq a \leq x \vee b \vee p' \\ x \wedge d \wedge p \leq c \leq x \vee d \vee p' \end{array} \right\}.$$

Now, we have

$$\begin{aligned} a \leq x \vee b \vee p' &\Rightarrow a \wedge d \leq (x \vee b \vee p') \wedge d = (x \wedge d) \vee ((b \vee p') \wedge d) \Rightarrow a \wedge d \leq (x \wedge d) \vee (b \vee p') \Rightarrow \\ &a \wedge d \wedge p \leq ((x \wedge d) \vee (b \vee p')) \wedge p. \end{aligned} \quad (4)$$

On the other hand,

$$x \wedge d \wedge p \leq c \Rightarrow x \wedge d \wedge p \leq c \wedge p \leq (c \vee p') \wedge p \quad (5)$$

From (4) and (5) we obtain

$$\begin{aligned} a \wedge d \wedge p &\leq (x \wedge d \wedge p) \vee ((b \vee p') \wedge p) \Rightarrow a \wedge d \wedge p \leq ((c \vee p') \wedge p) \vee ((b \vee p') \wedge p) \Rightarrow \\ &a \wedge d \wedge p \leq ((c \vee p') \vee (b \vee p')) \wedge p \leq b \vee c \vee p'. \end{aligned} \quad (6)$$

In a similar manner we have

$$\begin{aligned} c \leq x \vee d \vee p' &\Rightarrow c \wedge b \leq (x \vee d \vee p') \wedge b = (x \wedge b) \vee ((d \vee p') \wedge b) \Rightarrow \\ &c \wedge b \leq (x \wedge b) \vee (d \vee p') \Rightarrow c \wedge b \wedge p \leq ((x \wedge b) \vee (d \vee p')) \wedge p. \end{aligned} \quad (7)$$

On the other hand,

$$x \wedge b \wedge p \leq a \Rightarrow x \wedge b \wedge p \leq a \wedge p \leq (a \vee p') \wedge p \quad (8)$$

From (7) and (8) we obtain

$$\begin{aligned} c \wedge b \wedge p &\leq (x \wedge b \wedge p) \vee ((d \vee p') \wedge p) \Rightarrow c \wedge b \wedge p \leq ((d \vee p') \wedge p) \vee ((a \vee p') \wedge p) \Rightarrow \\ &c \wedge b \wedge p \leq ((d \vee p') \vee (a \vee p')) \wedge p \leq a \vee d \vee p'. \end{aligned} \quad (9)$$

From (6) and (9) we have that

$$(a \wedge d \wedge p) \vee (c \wedge b \wedge p) \leq (a \vee d \vee p') \wedge (b \vee c \vee p')$$

and so the interval

$$a *_p d \cap b *_p c = [(a \wedge d \wedge p) \vee (c \wedge b \wedge p), (a \vee d \vee p') \wedge (b \vee c \vee p')]$$

is nonempty, i.e.  $b *_p c \sim a *_p d$ . ■

**Proposition 3.8** For all  $p \in X$ ,  $(X, \leq, *_p)$  is a join space.

**Proof.** This follows from the definition of join space [19] and from Propositions 3.3, 3.7. ■

By its definition,  $a *_p b$  is always a closed interval. The same is true of  $a/_p b$  as seen in the next proposition. Note the critical use of complete distributivity.

**Proposition 3.9** For all  $p \in X$ , for all  $a, b \in X$ , the set  $a/_p b$  is a closed interval. In particular

$$a/_p b = [p_1, p_2]$$

where

$$p_1 = \wedge \{x : (b \vee p') \vee x \geq a\}, \quad p_2 = \vee \{x : (b \wedge p') \wedge x \leq a\}.$$

**Proof.** Choose any  $a, b, p \in X$ . It is well known [2] that a completely distributive lattice is *Brouwerian* (i.e. that for any  $u, w \in X$  the set  $\{z : u \wedge z \leq w\}$  has a maximum element) and *dually Brouwerian* (i.e. that for any  $u, w \in X$  the set  $\{z : u \vee z \geq w\}$  has a minimum element). Since  $(X, \leq)$  has been assumed completely distributive, it is Brouwerian and dually Brouwerian.

Define  $Q^{ab} = \{x : (b \vee p') \vee x \geq a\}$ , then (according to the previous remarks)  $Q^{ab} = [p_1, 1]$ . Similarly, define  $Q_{ab} = \{x : (b \wedge p) \wedge x \leq a\}$ , then, according to the previous remarks  $Q_{ab} = [0, p_2]$ . Now  $a/_p b = \{x : b \wedge x \wedge p' \leq a \leq b \vee x \vee p'\} = Q^{ab} \cap Q_{ab} = [p_1, 1] \cap [0, p_2] = [p_1, p_2]$ . ■

A special case of interest is the following.

**Proposition 3.10** *If  $(X, \leq, \wedge, \vee, ')$  is Boolean, then  $a/_p b = a *_p b'$ .*

**Proof.** Choose any  $a, b, p \in X$ .

Take any  $x \in a/_p b$ , i.e.  $x \wedge b \wedge p \leq a \leq x \vee b \vee p'$ . Now  $a \leq x \vee b \vee p' \Rightarrow a \wedge b' \wedge p \leq (x \vee b \vee p') \wedge b' \wedge p = (x \vee b \vee p') \wedge (b \vee p')' = (x \wedge (b \vee p')') \vee ((b \vee p') \wedge (b \vee p')') = (x \wedge (b \vee p')') \vee 0 = x \wedge b' \wedge p$ . In short

$$a \wedge b' \wedge p \leq x \wedge b' \wedge p. \quad (10)$$

Now from (10) we get:  $0 \leq a \wedge b' \wedge p \leq x \wedge b' \wedge p \Rightarrow 0 = 0 \wedge x' \leq a \wedge b' \wedge p \wedge x' \leq x \wedge b' \wedge p \wedge x' \leq x \wedge x' = 0$ . Hence  $a \wedge b' \wedge p \wedge x' = 0 \Rightarrow x' \leq (a \wedge b' \wedge p)'$

$$a \wedge b' \wedge p \leq x. \quad (11)$$

Similarly we get

$$x \leq a \vee b' \vee p'. \quad (12)$$

From (11) and (12) we get  $x \in a *_p b'$ . Hence  $a/_p b \subseteq a *_p b'$ .

Now take any  $x \in a *_p b'$ . Then  $a \wedge b' \wedge p \leq x \Rightarrow (a \wedge b' \wedge p) \vee (b \vee p') \leq x \vee b \vee p' \Rightarrow (a \vee b \vee p') \wedge ((b' \wedge p) \vee (b \vee p')) = (a \vee b \vee p') \wedge ((b' \wedge p) \vee (b' \wedge p)') \leq x \vee b \vee p' \Rightarrow (a \vee b \vee p') \wedge 1 \leq x \vee b \vee p' \Rightarrow$

$$a \leq x \vee b \vee p' \quad (13)$$

Similarly  $x \leq a \vee b' \vee p' \Rightarrow x \wedge b \wedge p \leq (a \vee b' \vee p') \wedge (b \wedge p) \Rightarrow x \wedge b \wedge p \leq (a \wedge b \wedge p) \vee ((b' \vee p') \wedge (b \wedge p)) = (a \wedge b \wedge p) \vee ((b \wedge p)' \wedge (b \wedge p)) \Rightarrow x \wedge b \wedge p \leq (a \wedge b \wedge p) \vee 0 \Rightarrow$

$$x \wedge b \wedge p \leq a. \quad (14)$$

From (13) and (14) follows that  $a \in x *_p b$  and so  $x \in a/_p b$ . Hence  $a *_p b' \subseteq a/_p b$ . In conjunction with the previously established  $a/_p b \subseteq a *_p b'$ , it follows that  $a/_p b = a *_p b'$ . ■

## 4 The $L$ -fuzzy Join and Extension Hyperoperations

In this section we introduce the fuzzy join and the  $L$ -fuzzy extension hyperoperations, which yield a *L-fuzzy join space*. To this end we will use the family of (crisp) hyperoperations  $\{*_p\}_{p \in X}$  and the family of (crisp) hyperoperations  $\{/_p\}_{p \in X}$ .

We first show that  $a *_p b$  and  $a/_p b$  viewed as functions of  $p$  behave like  $p$ -cuts of a  $L$ -fuzzy set.

**Proposition 4.1** *For all  $a, b \in X$  we have:*

- (i) *For all  $p, q \in X : p \leq q \Rightarrow a *_q b \subseteq a *_p b$ ;*
- (ii) *For all  $P \subseteq X : \bigcap_{p \in P} a *_p b = a *_P b$ ;*
- (iii)  *$a *_0 b = X$ .*

**Proof.** Choose any  $a, b \in X$ .

(i) Choose any  $p, q \in X$  with  $p \leq q$ . Then

$$p \leq q \Rightarrow p \wedge a \wedge b \leq q \wedge a \wedge b. \quad (15)$$

Also, by the order inversion property of negation we have

$$p \leq q \Rightarrow q' \leq p' \Rightarrow q' \vee a \vee b \leq p' \vee a \vee b. \quad (16)$$

Now (15) and (16) yield  $[q \wedge a \wedge b, q' \vee a \vee b] \subseteq [p \wedge a \wedge b, p' \vee a \vee b]$ , which is exactly the required result.

(ii) Choose any  $P \subseteq X$ . Then

$$\begin{aligned} \cap_{p \in P} a *_p b &= \cap_{p \in P} [a \wedge b \wedge p, a \vee b \vee p'] = [\vee_{p \in P} (a \wedge b \wedge p), \wedge_{p \in P} (a \vee b \vee p')] \\ &= [a \wedge b \wedge (\vee_{p \in P} p), a \vee b \vee (\wedge_{p \in P} p')] = [a \wedge b \wedge (\vee_{p \in P} p), a \vee b \vee (\vee_{p \in P} p)'] \\ &= [a \wedge b \wedge (\vee P), a \vee b \vee (\vee P)'] = a *_\vee P b. \end{aligned}$$

(In the above derivation note the crucial use of the generalized deMorgan properties.)

(iii)  $a *_0 b = [a \wedge b \wedge 0, a \vee b \vee 1] = [0, 1] = X$ . ■

**Proposition 4.2** For all  $a, b \in X$  we have:

- (i) For all  $p, q \in X : p \leq q \Rightarrow a/_p b \subseteq a/_q b$ ;
- (ii) For all  $P \subseteq X : \cap_{p \in P} a/_p b = a/_\vee P b$ ;
- (iii)  $a/_0 b = X$ .

**Proof.** Choose any  $a, b \in X$ .

(i) Choose any  $p, q \in X$  with  $p \leq q$ . Choose any  $x \in a/_q b$ . Then

$$b \wedge x \wedge q \leq a \leq b \vee x \vee q'. \quad (17)$$

Also

$$p \leq q \Rightarrow \begin{cases} b \wedge x \wedge p \leq b \wedge x \wedge q \\ b \vee x \vee q' \leq b \vee x \vee p' \end{cases}. \quad (18)$$

From (17) and (18) we get

$$b \wedge x \wedge p \leq b \wedge x \wedge q \leq a \leq b \vee x \vee q' \leq b \vee x \vee p'$$

which implies  $x \in a/_p b$ . Hence  $a/_q b \subseteq a/_p b$ .

(ii) Choose any  $P \subseteq X$ . Then:

$$\forall p \in P : p \leq \vee P \Rightarrow \forall p \in P : a/_\vee P b \subseteq a/_p b \Rightarrow a/_\vee P b \subseteq \cap_{p \in P} a/_p b.$$

On the other hand, take any  $x \in \cap_{p \in P} a/_p b$ . We have:  $\forall p \in P : x \in a/_p b \Rightarrow \forall p \in P : a \in x *_p b \Rightarrow \forall p \in P : x \wedge b \wedge p \leq a \leq x \vee b \vee p' \Rightarrow \vee_{p \in P} (x \wedge b \wedge p) \leq a \leq \wedge_{p \in P} (x \vee b \vee p') \Rightarrow x \wedge b \wedge (\vee_{p \in P} p) \leq a \leq x \vee b \vee (\wedge_{p \in P} p') \Rightarrow x \wedge b \wedge (\vee P) \leq a \leq x \vee b \vee (\vee P)' \Rightarrow a \in x *_\vee P b \Rightarrow x \in a/_\vee P b$ . Hence  $\cap_{p \in P} a/_p b \subseteq a/_\vee P b$ . (In these derivations note the crucial use of the generalized deMorgan properties.) So, from  $a/_\vee P b \subseteq \cap_{p \in P} a/_p b$  and  $\cap_{p \in P} a/_p b \subseteq a/_\vee P b$  we obtain  $\cap_{p \in P} a/_p b = a/_\vee P b$ .

(iii)  $a/_0 b = \{x : a \in x *_0 b\} = \{x : a \in [x \wedge b \wedge 0, x \vee b \vee 1]\} = \{x : a \in [0, 1]\} = X$ . ■

Analogously to [4], we define a  $L$ -fuzzy hyperoperation to be an operation that takes pairs of  $X$  elements to  $L$ -fuzzy subsets of  $X$ . We will now construct two  $L$ -fuzzy hyperoperations (namely  $*$  and  $/$ ) by associating appropriate  $L$ -fuzzy sets with every pair  $a, b \in X$ .

**Definition 4.3** For all  $a, b \in X$  define the  $L$ -fuzzy set  $a * b$  by defining (for any  $x \in X$ )

$$(a * b)(x) \doteq \vee \{q : x \in a *_q b\}. \quad (19)$$

**Proposition 4.4** For all  $a, b \in X$  and  $p \in X$  we have  $(a * b)_p = a *_p b$ .

**Proof.** From Proposition 4.1 we see that the family  $\{a *_p b\}_{p \in X}$  has the  $p$ -cut properties. Then, the required result follows from Definition 4.3 and Proposition 2.4. ■

**Definition 4.5** For all  $a, b \in X$  define the  $L$ -fuzzy set  $a/b$  by defining (for any  $x \in X$ )

$$(a/b)(x) = \vee \{p : x \in a/_p b\}.$$

**Proposition 4.6** For all  $p \in X$  and  $a, b \in X$  we have  $(a/b)_p = a/_p b$ .

**Proof.** From Proposition 4.2 we see that the family  $\{a/_p b\}_{p \in X}$  has the  $p$ -cut properties. Then, the required result follows from Definition 4.3 and Proposition 2.4. ■

We now proceed to establish that  $(X, *)$  is a  $L$ -fuzzy hypergroup. To this end we need some auxiliary definitions, which regard the manner in which  $L$ -fuzzy sets are combined using a  $L$ -fuzzy hyperoperation  $\circ$ . These definitions can be considered as extensions of the one given in Section 2, where for a *crisp* hyperoperation  $\circ$  and *crisp* sets  $A, B \in \mathbf{P}(X)$  we defined  $A \circ B \doteq \cup_{a \in A, b \in B} a \circ b$ .

**Definition 4.7** Given a  $L$ -fuzzy hyperoperation  $\circ : X \times X \rightarrow \mathbf{F}(X)$ , for all  $a \in X$ ,  $B \in \mathbf{F}(X)$  define the  $L$ -fuzzy set  $a \circ B$  by

$$(a \circ B)(x) \doteq \vee_{b: B(b) > 0} (a \circ b)(x)$$

and the *crisp* set  $a \hat{\circ} B$  by

$$a \hat{\circ} B \doteq \cup_{b: B(b) > 0} \{x : (a \circ b)(x) = 1\}.$$

**Definition 4.8** Given a  $L$ -fuzzy hyperoperation  $\circ : X \times X \rightarrow \mathbf{F}(X)$ , for all  $A, B \in \mathbf{F}(X)$  define the  $L$ -fuzzy set  $A \circ B$  by

$$(A \circ B)(x) \doteq \vee_{a: A(a) > 0, b: B(b) > 0} (a \circ b)(x).$$

and the *crisp* set  $A \hat{\circ} B$  by

$$A \hat{\circ} B \doteq \cup_{a: A(a) > 0, b: B(b) > 0} \{x : (a \circ b)(x) = 1\}.$$

**Remark.** Note that if  $A$  and  $B$  are *crisp* sets then the above definition reduces to  $(A \circ B)(x) = \vee_{a \in A, b \in B} (a \circ b)(x)$ . This emphasizes the similarity with the previously mentioned definition  $A \circ B \doteq \cup_{a \in A, b \in B} a \circ b$  (where  $\circ$  is a *crisp* hyperoperation  $\circ$  and  $A, B$  are *crisp* sets).

**Proposition 4.9** For all  $a \in X$ , for all  $B \in \mathbf{P}(X)$ , for all  $p \in X$  we have:  $a *_p B \subseteq (a * B)_p$ .

**Proof.** Choose any  $a \in X$ , any  $B \in \mathbf{P}(X)$ , any  $p \in X$ . If  $p = 0$  then  $a *_p B = X = (a * B)_p$ . If  $p > 0$ , then choose any  $x \in a *_p B$ . For this  $x$ , there exists some  $b \in B$  such that  $x \in a *_p b = (a * b)_p$ . Hence  $(a * b)(x) \geq p > 0$ . Now

$$(a * B)(x) = \vee_{z: z \in B} (a * z)(x) \geq (a * b)(x) \geq p$$

which implies  $x \in (a * B)_p$ . Hence  $a *_p B \subseteq (a * B)_p$ . ■

We now establish some properties of  $*$  and  $/$ .

**Proposition 4.10** For all  $a, b, c, p \in X$  we have

- (i)  $(a * b)(a) = (a * b)(b) = 1, (a * a)(a) = 1.$
- (ii)  $a * b = b * a.$
- (iii)  $(a * b) * c \sim_p a * (b * c).$
- (iv.1)  $a \hat{*} X = X$  (i.e. for all  $x \in X, \exists y \in X : (a * y)(x) = 1$ ).
- (iv.2)  $a * X = X$  (i.e. for all  $x \in X : (a * X)(x) = \bigvee_{y \in X} (a * y)(x) = 1$ ).

**Proof.** Choose any  $a, b, c \in X$ .

(i) We know that  $a \in a *_1 b = [a \wedge b, a \vee b]$ . Hence  $1 \in \{p : a \in a * _p b\}$  and so  $(a * b)(a) = \bigvee \{p : a \in a * _p b\} = 1$ . Similarly  $(a * b)(b) = 1$ . To show  $(a * a)(a) = 1$ , just take  $a = b$ .

(ii) For all  $p \in X$  we have  $(a * b)_p = a * _p b = b * _p a = (b * a)_p$ . Since  $a * b$  and  $b * a$  have the same cuts, they are identical (by Proposition 2.5).

(iii) Take any  $p \in X$ .

(iii.1) If  $p = 0$ , then it is easy to see that  $(a * (b * c))_0 = X = ((a * b) * c)_0$  and so  $(a * b) * c \sim_0 a * (b * c)$ .

(iii.2) If  $p > 0$ , take any  $x \in a * _p b * _p c = a * _p (b * _p c)$ , then there exists some  $z \in b * _p c$  such that  $x \in a * _p z$ . For this  $z$  we have:

$$z \in b * _p c = (b * c)_p \Rightarrow (b * c)(z) \geq p > 0 \quad (20)$$

and

$$x \in a * _p z = (a * z)_p \Rightarrow (a * z)(x) \geq p > 0. \quad (21)$$

From (20) follows that

$$z \in \{u : (b * c)(u) > 0\} \quad (22)$$

and from (21), (22) follows that

$$p \leq (a * z)(x) \leq \bigvee_{\{u : (b * c)(u) > 0\}} (a * u)(x) = (a * (b * c))(x) \Rightarrow$$

$$x \in (a * (b * c))_p \Rightarrow a * _p b * _p c \subseteq (a * (b * c))_p$$

Hence  $a * _p b * _p c \subseteq (a * (b * c))_p$ . Similarly we show that  $a * _p b * _p c = (a * _p b) * _p c \subseteq ((a * b) * c)_p$  and so we have

$$x \in a * _p b * _p c \Rightarrow (a * (b * c))(x) \wedge ((a * b) * c)(x) \geq p \geq 0.$$

Since  $a * _p b * _p c$  is nonempty, it follows that  $(a * b) * c \sim_p a * (b * c)$ .

(iv) We prove (iv.1), i.e. that for all  $x \in X, \exists y \in X : (a * y)(x) = 1$ . Indeed, take any  $x \in X$ , and set  $y = x$ ; then  $(a * x)(x) = 1$ , since  $x \in [a \wedge x, a \vee x] = a * _1 x = (a * x)_1$ . Now (iv.2) follows immediately. ■

**Proposition 4.11** For all  $a, b, c, d, p \in X$  we have: (i)  $a * b \sim_p c * d \Leftrightarrow a * _p b \sim c * _p c$ , (ii)  $a/b \sim_p c/d \Leftrightarrow a/_p b \sim_p c/_p d$ .

**Proof.** Choose any  $a, b, c, d, p \in X$ . For (i) we have:

$$a * b \sim_p c * d \Leftrightarrow \left( \exists x : \begin{array}{l} (a * b)(x) \geq p \\ (c * d)(x) \geq p \end{array} \right) \Leftrightarrow \left( \exists x : \begin{array}{l} x \in (a * b)_p = a * _p b \\ x \in (c * d)_p = c * _p d \end{array} \right) \Leftrightarrow a * _p b \sim c * _p d$$

and (ii) is proved similarly. ■

**Proposition 4.12** For all  $a, b, c, d, p \in X$ , for all  $p \in X$  we have:  $a/b \sim_p c/d \Rightarrow a * d \sim_p b * c$ .

**Proof.** Choose any  $a, b, c, d, p \in X$ . Then  $a/b \sim_p c/d \Rightarrow a/_p b \sim c/_p d \Rightarrow a *_p d \sim b *_p c \Rightarrow a * d \sim_p b * c$ . ■

In [4] Corsini and Tofan introduce four different types of reproducibility, which they denote by  $R_1, R_2, R_3$  and  $R_4$ . They proceed to define accordingly four different variants of fuzzy hypergroup. In this paper we restrict ourselves to  $R_3$ -type reproducibility. This is the strongest type, i.e. it also implies  $R_1, R_2$  and  $R_4$  reproducibility (see [4]). We now present some definitions which are essentially the ones presented in [4], but here they are adapted to the  $L$ -fuzzy context.

**Definition 4.13** *Given a  $L$ -fuzzy hyperoperation  $\circ : X \times X \rightarrow \mathbf{F}(X)$ , the hyperstructure  $(X, \circ)$  is called  $L$ -fuzzy<sub>3</sub>-hypergroup if it satisfies the following conditions.*

- (i) *For all  $a, b, c, p \in X$  we have  $(a \circ b) \circ c = a \circ (b \circ c)$  (associativity).*
- (ii) *For all  $a \in X$  we have  $a \hat{\circ} X = X$  ( $R_3$  reproducibility).*

**Definition 4.14** *Given a  $L$ -fuzzy hyperoperation  $\circ : X \times X \rightarrow \mathbf{F}(X)$ , the hyperstructure  $(X, \circ)$  is called  $L$ -fuzzy<sub>3</sub>- $p$ -hypergroup if it satisfies the following conditions.*

- (i) *For all  $a, b, c, p \in X$  we have  $(a \circ_p b) \circ_p c = a \circ_p (b \circ_p c)$  ( $p$ -associativity).*
- (ii) *For all  $a \in X$  we have  $a \hat{\circ} X = X$  ( $R_3$  reproducibility).*

We now introduce an additional definition which makes use of  $H_v$  associativity (see Spartalis [20, 21] and Vougiouklis [22, 25, 26]).

**Definition 4.15** *Given a hyperoperation  $\circ : X \times X \rightarrow \mathbf{F}(X)$ , the hyperstructure  $(X, \circ)$  is called  $L$ -fuzzy<sub>3</sub>- $H_v$ -hypergroup if it satisfies the following conditions.*

- (i) *For all  $a, b, c, p \in X$  we have  $(a \circ b) \circ c \sim_p a \circ (b \circ c)$  (fuzzy  $H_v$  associativity).*
- (ii) *For all  $a \in X$  we have  $a \hat{\circ} X = X$  ( $R_3$  reproducibility).*

In view of the above definitions and Proposition 4.10, we have the following.

**Proposition 4.16** *The hyperstructure  $(X, *)$  is a  $L$ -fuzzy<sub>3</sub>- $H_v$ -hypergroup and also a  $L$ -fuzzy<sub>3</sub>- $p$ -hypergroup.*

**Proof.** Easy. ■

**Remark.** As already remarked,  $R_3$  reproducibility implies  $R_i$  reproducibility, with  $i = 1, 2, 4$ . Hence  $(X, *)$  is a  $L$ -fuzzy <sub>$i$</sub> - $H_v$ -hypergroup and also a  $L$ -fuzzy <sub>$i$</sub> - $p$ -hypergroup, with  $i = 1, 2, 4$ .

Now we introduce some definitions of  $L$ -fuzzy join spaces.

**Definition 4.17** *Given a  $L$ -fuzzy hyperoperation  $\circ : X \times X \rightarrow \mathbf{F}(X)$ , the hyperstructure  $(X, \circ)$  is called  $L$ -fuzzy<sub>3</sub>- $p$ -join space if it is a commutative  $L$ -fuzzy<sub>3</sub>- $p$ -hypergroup and also satisfies (with  $x \wr y \doteq \{z : x \in z \circ y\}$ ):*

$$\text{For all } a, b, c, d, p \in X \text{ we have: } a \wr b \sim_p c \wr d \Rightarrow a \circ d \sim_p b \circ c.$$

**Definition 4.18** *Given a  $L$ -fuzzy hyperoperation  $\circ : X \times X \rightarrow \mathbf{F}(X)$ , the hyperstructure  $(X, \circ)$  is called  $L$ -fuzzy<sub>3</sub>- $H_v$ -join space if it is a commutative  $L$ -fuzzy<sub>3</sub>- $H_v$ -hypergroup and also satisfies (with  $x \wr y \doteq \{z : x \in z \circ y\}$ ):*

$$\text{For all } a, b, c, d, p \in X \text{ we have: } a \wr b \sim_p c \wr d \Rightarrow a \circ d \sim_p b \circ c.$$

Hence we have the following.

**Proposition 4.19** *The hyperstructure  $(X, *)$  is a  $L$ -fuzzy<sub>3</sub>- $H_v$ -join space and also a  $L$ -fuzzy<sub>3</sub>- $p$ -join space.*

**Proof.** Easy. ■

The hyperstructure  $(X, *)$  has an additional interesting property which is related to the theory of *L-fuzzy sublattices*. Notice that for any fixed  $a, b \in X$ , the family  $\{(a * b)_p\}_{p \in X}$  is a family of (crisp) closed intervals. The same is true of the family  $\{(a/b)_p\}_{p \in X}$ . Hence  $a * b$  is a *L-fuzzy interval* and, *a fortiori*, a *L-fuzzy convex sublattice*. The same is true of  $a/b$ . For the definition and some properties of *L-fuzzy intervals*, see the Appendix. In this section we simply present the following.

**Definition 4.20** We say  $M : X \rightarrow X$  is a *L-fuzzy interval* of  $(X, \leq)$  iff

$$\forall p \in X : M_p \text{ is a closed interval of } (X, \leq).$$

**Proposition 4.21** For all  $a, b \in X$ , the *L-fuzzy sets*  $a * b$  and  $a/b$  are *L-fuzzy intervals*.

**Proof.** This is obvious from the definitions of the  $*$ ,  $/$  hyperoperations. ■

## 5 The Lattice of $p$ -cuts of $a * b$

Let us pick any  $a, b \in X$  and keep them fixed for the rest of this section. We will now study some properties of the family of  $p$ -cuts  $\{a * b\}_p$ . To this end we introduce the following.

**Definition 5.1** For all  $a, b \in X$ , define  $\mathbf{I}_{a,b}(X) \doteq \{(a * b)_p\}_{p \in X}$ .

We obviously have the inclusions:  $\mathbf{I}_{a,b}(X) \subseteq \mathbf{I}(X) \subseteq \mathbf{P}(X)$ . This section is devoted to clarifying the connection between  $(\mathbf{I}_{a,b}(X), \subseteq)$  and  $(X, \leq)$

**Proposition 5.2** For all  $a, b \in X$ , for all  $p, q \in X$  we have

$$a * p \ b \cap a * q \ b = a *_{p \vee q} \ b, \quad a * p \ b \dot{\cup} a * q \ b = a *_{p \wedge q} \ b. \quad (23)$$

$$(a * b)_p \cap (a * b)_q = (a * b)_{p \vee q}, \quad (a * b)_p \dot{\cup} (a * b)_q = (a * b)_{p \wedge q}. \quad (24)$$

**Proof.** Choose any  $a, b, p \in X$ . We have

$$\begin{aligned} a * p \ b \cap a * q \ b &= [a \wedge b \wedge p, a \vee b \vee p'] \cap [a \wedge b \wedge q, a \vee b \vee q'] \\ &= [(a \wedge b \wedge p) \vee (a \wedge b \wedge q), (a \vee b \vee p') \wedge (a \vee b \vee q')] \\ &= [a \wedge b \wedge (p \vee q), a \vee b \vee (p' \wedge q')] \\ &= [a \wedge b \wedge (p \vee q), a \vee b \vee (p \vee q)'] = a *_{p \vee q} \ b. \end{aligned}$$

and

$$\begin{aligned} a * p \ b \dot{\cup} a * q \ b &= [a \wedge b \wedge p, a \vee b \vee p'] \dot{\cup} [a \wedge b \wedge q, a \vee b \vee q'] \\ &= [(a \wedge b \wedge p) \wedge (a \wedge b \wedge q), (a \vee b \vee p') \vee (a \vee b \vee q')] \\ &= [a \wedge b \wedge (p \wedge q), a \vee b \vee (p' \vee q')] \\ &= [a \wedge b \wedge (p \wedge q), a \vee b \vee (p \wedge q)'] = a *_{p \wedge q} \ b. \end{aligned}$$

Hence we have proved (23). Now (24) follows from the equality (for any  $r \in X$ ) between  $a * r \ b$  and  $(a * b)_r$ . ■

**Proposition 5.3** For all  $a, b \in X$ ,  $(\mathbf{I}_{a,b}(X), \subseteq, \dot{\cup}, \cap)$  is a lattice.

**Proof.** Since  $(\mathbf{P}(X), \subseteq)$  is an ordered set and  $\mathbf{I}_{a,b} \subseteq \mathbf{P}(X)$ , it follows that  $(\mathbf{I}_{a,b}(X), \subseteq)$  is an ordered set. It remains to show that for every  $a, b \in X$ , we have  $\sup(a *_p b, a *_q b) = a *_p b \dot{\cup} a *_q b$  and  $\inf(a *_p b, a *_q b) = a *_p b \cap a *_q b$  (both the sup and inf being with respect to  $\mathbf{I}_{a,b}(X)$ ). Choose any  $a, b \in X$  and consider them fixed. Choose any  $p, q \in X$ .

(i) We have already seen that  $a *_p b \dot{\cup} a *_q b = a *_{p \wedge q} b \in \mathbf{I}_{a,b}(X)$ . Also we have

$$\begin{aligned} p \wedge q \leq p &\Rightarrow a *_p b \subseteq a *_{p \wedge q} b; \\ p \wedge q \leq q &\Rightarrow a *_q b \subseteq a *_{p \wedge q} b. \end{aligned}$$

Also, take any  $r \in X$  such that

$$\left. \begin{array}{l} a *_p b \subseteq a *_r b \\ a *_q b \subseteq a *_r b \end{array} \right\} \Rightarrow (a *_p b) \dot{\cup} (a *_q b) \subseteq a *_r b \Rightarrow a *_{p \wedge q} b \subseteq a *_r b.$$

Hence  $\sup(a *_p b, a *_q b) = a *_{p \wedge q} b = a *_p b \dot{\cup} a *_q b$  (the sup being with respect to  $\mathbf{I}_{a,b}(X)$ ).

(ii) We have already seen that  $a *_p b \cap a *_q b = a *_{p \vee q} b \in \mathbf{I}_{a,b}(X)$ . Also we have

$$\begin{aligned} p \leq p \vee q &\Rightarrow a *_{p \vee q} b \subseteq a *_p b; \\ p \leq p \vee q &\Rightarrow a *_{p \vee q} b \subseteq a *_q b. \end{aligned}$$

Also, take any  $r \in X$  such that

$$\left. \begin{array}{l} a *_r b \subseteq a *_p b \\ a *_r b \subseteq a *_q b \end{array} \right\} \Rightarrow a *_r b \subseteq (a *_p b) \cap (a *_q b) \Rightarrow a *_r b \subseteq a *_{p \vee q} b.$$

Hence  $\inf(a *_p b, a *_q b) = a *_{p \vee q} b = a *_p b \cap a *_q b$  (the inf being with respect to  $\mathbf{I}_{a,b}(X)$ ). ■

We now introduce a relation  $J_{a,b}$  on  $X \times X$ .

**Definition 5.4** The relation  $J_{a,b} \subseteq X \times X$  is defined by:  $(p, q) \in J_{a,b}$  iff  $(a * b)_p = (a * b)_q$ .

**Proposition 5.5** For all  $a, b \in X$ ,  $J_{a,b}$  is an equivalence relation on  $X$ .

**Proof.** Easy. ■

**Definition 5.6** The classes of  $J_{a,b}$  are defined by  $\bar{p} \doteq \{q : (p, q) \in J_{a,b}\} = \{q : (a * b)_p = (a * b)_q\}$ ,  $p \in X$ .

**Definition 5.7** We denote the quotient of  $X$  with respect to  $J_{a,b}$  by  $X_{a,b}$  and define it by  $X_{a,b} \doteq \{\bar{p}\}_{p \in X}$ .

**Definition 5.8** For all  $a, b \in X$  we define the function  $f_{a,b} : X_{a,b} \rightarrow \mathbf{I}_{a,b}(X)$  by

$$f_{a,b}(\bar{p}) \doteq (a * b)_p = a *_p b.$$

**Proposition 5.9** For all  $a, b \in X$  the function  $f_{a,b}$  is well-defined, 1-1 and onto  $\mathbf{I}_{a,b}(X)$ .

**Proof.** Fix any  $a, b \in X$ . Then  $\bar{p} = \bar{q} \Leftrightarrow a *_p b = a *_q b \Leftrightarrow f_{a,b}(\bar{p}) = f_{a,b}(\bar{q})$ . This shows that  $f_{a,b}$  is well-defined and 1-1. To show that it is onto, take any  $A \in \mathbf{I}_{a,b}(X)$ ; then exists some  $p \in X$  such that  $A = a *_p b = f_{a,b}(\bar{p})$ . ■

**Definition 5.10** For all  $a, b \in X$  and for all  $\bar{p}, \bar{q} \in X_{a,b}$ , we write  $\bar{p} \leq \bar{q}$  iff  $a *_q b \subseteq a *_p b$ .

**Proposition 5.11** For all  $a, b \in X$ ,  $\leq$  is an order on  $X_{a,b}$ .

**Proof.** Since  $\bar{p} \leq \bar{q} \Leftrightarrow a *_q b \subseteq a *_p b$ , it is obvious that: (a)  $\bar{p} \leq \bar{p}$  and (b)  $(\bar{p} \leq \bar{q}, \bar{q} \leq \bar{r}) \Rightarrow \bar{p} \leq \bar{r}$ . It remains to show that  $(\bar{p} \leq \bar{q}, \bar{q} \leq \bar{p}) \Rightarrow \bar{p} = \bar{q}$ . But  $(\bar{p} \leq \bar{q}, \bar{q} \leq \bar{p}) \Rightarrow (a *_q b \subseteq a *_p b, a *_p b \subseteq a *_q b) \Rightarrow a *_p b = a *_q b \Rightarrow \bar{p} = \bar{q}$ . ■

**Proposition 5.12** For all  $a, b \in X$ ,  $f_{a,b}$  is an order anti-isomorphism between  $(X_{a,b}, \leq)$  and  $(\mathbf{I}_{a,b}, \subseteq)$ .

**Proof.**  $\bar{p} \leq \bar{q} \Leftrightarrow a *_q b \subseteq a *_p b \Leftrightarrow f_{a,b}(\bar{q}) \subseteq f_{a,b}(\bar{p})$ . Also,  $f_{a,b}$  is 1-1 and onto. ■

**Proposition 5.13** For all  $a, b \in X$ ,  $J_{a,b}$  is a congruence, i.e. for all  $p, q, r \in X$  we have:

$$\bar{p} = \bar{q} \Rightarrow \begin{cases} \overline{p \vee r} = \overline{q \vee r}, \\ \overline{p \wedge r} = \overline{q \wedge r}. \end{cases}$$

**Proof.** Assume  $\bar{p} = \bar{q}$ . Then  $\bar{p} = \bar{q} \Rightarrow a *_p b = a *_q b \Rightarrow a *_p b \cap a *_r b = a *_q b \cap a *_r b \Rightarrow a *_p \vee_r b = a *_q \vee_r b \Rightarrow \overline{p \vee r} = \overline{q \vee r}$ . Similarly we can show  $\overline{p \wedge r} = \overline{q \wedge r}$ . ■

**Definition 5.14** For given  $a, b \in X$  and for all  $p, q \in X$  define  $\bar{p} \vee \bar{q} = \overline{p \vee q}$ ,  $p \bar{\wedge} q = \overline{p \wedge q}$ .

**Remark.** The  $\bar{p} \vee \bar{q}$ ,  $p \bar{\wedge} q$  are well defined in view of Proposition 5.13.

**Proposition 5.15** For all  $a, b \in X$ ,  $f_{a,b}$  is a lattice anti-isomorphism between  $(X_{a,b}, \leq, \vee, \bar{\wedge})$  and  $(\mathbf{I}_{a,b}, \subseteq, \dot{\cup}, \cap)$ .

**Proof.** This is obvious in light of the fact that  $(\mathbf{I}_{a,b}, \subseteq, \dot{\cup}, \cap)$  is a lattice (Proposition 5.3) and Propositions 5.12 and 5.13. ■

## 6 Conclusion

Working on a complete generalized de Morgan lattice  $(X, \leq, \vee, \wedge, ')$ , we introduced a family of join hyperoperations  $*_p$  and a family of extension hyperoperations  $/_p$ . Hence we obtained a family of join spaces indexed by  $p \in X$ . We used this family to construct the cuts of the respective  $L$ -fuzzy join “ $*$ ” and  $L$ -fuzzy extension “ $/$ ” hyperoperations and thus obtained a  $L$ -fuzzy join space. In addition, we proved that for any  $a, b \in X$  the  $L$ -fuzzy sets  $a * b$  and  $a/b$  are  $L$ -fuzzy intervals. The present work can be extended in several directions; let us indicate some possibilities.

1. In [11] we have shown that the crisp join space  $(X, *_1)$  is a lattice-ordered hypergroup with respect to the  $\leq$  order and also obtained a number of distributivity properties of the crisp hyperoperations  $*_1, /_1$ . One may extend this research to: (a) the study of  $(X, *_p)$  for a fixed  $p$ , (b) the study of the family  $\{(X, *_p)\}_{p \in X}$ , (c) the study of analogous order and distributivity properties of the  $L$ -fuzzy join space  $(X, *)$ .
2. In [12] we have examined convexity properties of crisp join hyperoperations. One may, analogously, explore *fuzzy convexity* properties of the  $L$ -fuzzy  $*$  hyperoperation.
3. Choosing a particular pair  $a, b \in X$  several issues can be explored. For example: what are the properties of the cuts of  $a * b$ ,  $a/b$ ? what are the properties of the congruence  $J_{a,b}$  introduced in Section 5?

4. One may also investigate the families  $\{a * b\}_{a,b \in X}$ ,  $\{a/b\}_{a,b \in X}$  and obtain  $L$ -fuzzy interval properties.

At a more general level, let us remark that in this paper we have studied a particular example of a  $L$ -fuzzy join space. This may serve as motivation for a general study of fuzzy and  $L$ -fuzzy join spaces, which, as far as we know, has not been undertaken so far.

Last but not least, let us remark that fuzzy lattices and join hyperoperations have been used in several engineering and computer science applications [10, 16, 17]. Hence we expect that the results of this work may in the future prove useful in similar applications.

## A Appendix: L-Fuzzy Lattices and Intervals

We start with some informal remarks. Consider two complete lattices  $(X, \leq, \vee, \wedge)$  and  $(L, \sqsubseteq, \sqcup, \sqcap)$ . *Fuzzy sublattices*, *fuzzy convex sublattices* and *fuzzy intervals* are fuzzy sets  $M : X \rightarrow L$  which satisfy certain special properties. We reserve the term “fuzzy sublattice” (fuzzy convex sublattice, fuzzy interval) for the case where  $L = [0, 1] \subseteq R$  (i.e. real number valued fuzzy sets). For the more general case of an arbitrary complete lattice  $(L, \sqsubseteq, \sqcup, \sqcap)$  we use the term “ $L$ -fuzzy sublattice” ( $L$ -fuzzy convex sublattice,  $L$ -fuzzy interval). This conforms with standard usage in the fuzzy sets literature.

In [13] we have introduced fuzzy intervals as a special case of fuzzy sublattices [1]. An easy generalization yields  $L$ -fuzzy intervals, as a generalization of  $L$ -fuzzy sublattices [24]. As remarked in Section 4, for every  $a, b \in X$ , the fuzzy sets  $a * b$  and  $a/b$  are  $L$ -fuzzy intervals. Some of the results obtained in [13] hold specifically for the case  $L = [0, 1] \subseteq R$ . However, a number of properties of fuzzy intervals can be extended to the case of  $L$ -fuzzy intervals.

Hence, in this Appendix we present some properties of  $L$ -fuzzy intervals, omitting proofs, since these are identical to the proofs presented in [13] for the case  $L = [0, 1] \subseteq R$ . The presentation is in terms of fuzzy sets  $M : X \rightarrow L$  and, obviously, they also hold for the special case  $(X, \leq, \vee, \wedge) = (L, \sqsubseteq, \sqcup, \sqcap)$  which conforms with the setting of this paper. Hence, all the following propositions hold for any fuzzy set  $M = a * b$ ,  $M = a/b$  (for any  $a, b \in X$ ).

We repeat that  $(X, \leq, \vee, \wedge)$  and  $(L, \sqsubseteq, \sqcup, \sqcap)$  are assumed to be complete lattices. We define  $L$ -fuzzy sublattices and  $L$ -fuzzy convex sublattices in terms of their  $p$ -cuts; this is different from, but equivalent to Ajmal’s approach [1].

**Definition A.1** We say  $M : X \rightarrow L$  is a  $L$ -fuzzy sublattice of  $(X, \leq)$  iff  $\forall p \in L$  the set  $M_p$  is a sublattice of  $(X, \leq)$ .

**Definition A.2** We say  $M : X \rightarrow L$  is a  $L$ -fuzzy convex sublattice of  $(X, \leq)$  iff  $\forall p \in L$  the set  $M_p$  is a convex sublattice of  $(X, \leq)$ ; (i.e.  $\forall p \in L, \forall x, y \in M_p$  we have  $[x \wedge y, x \vee y] \subseteq M_p$ ).

**Proposition A.3**  $M : X \rightarrow L$  is a  $L$ -fuzzy sublattice of  $(X, \leq)$  iff

$$\forall x, y \in X : M(x \wedge y) \sqcap M(x \vee y) \sqsupseteq M(x) \sqcap M(y).$$

**Proposition A.4** Let  $M : X \rightarrow L$  be a  $L$ -fuzzy sublattice of  $(X, \leq)$ . It is a  $L$ -fuzzy convex sublattice of  $(X, \leq)$  iff

$$\forall x, y \in X, \forall z \in [x \wedge y, x \vee y] : M(z) \sqsupseteq M(x \wedge y) \sqcap M(x \vee y) = M(x) \sqcap M(y).$$

**Definition A.5** We say  $M : X \rightarrow L$  is a  $L$ -fuzzy interval of  $(X, \leq)$  iff

$$\forall p \in L : M_p \text{ is a closed interval of } (X, \leq).$$

The collection all  $L$ -fuzzy intervals will be denoted by  $\tilde{\mathbf{I}}(X, L)$  or simply by  $\tilde{\mathbf{I}}$ .

Arbitrary intersections of  $L$ -fuzzy intervals yield a  $L$ -fuzzy interval.

**Proposition A.6** For all  $\tilde{\mathbf{J}} \subseteq \tilde{\mathbf{I}}$  we have:  $\bigwedge_{M \in \tilde{\mathbf{J}}} M \in \tilde{\mathbf{I}}$

Since  $\tilde{\mathbf{I}} \subseteq \mathbf{F}$ , it follows that  $(\tilde{\mathbf{I}}, \sqsubseteq)$  is an ordered set. In fact,  $(\tilde{\mathbf{I}}, \sqsubseteq)$  is a lattice as seen in the following.

**Definition A.7** For all  $M, N \in \tilde{\mathbf{I}}$  we define  $M \dot{\vee} N$  as follows. We define  $\tilde{\mathbf{S}}(M, N) \doteq \{A : A \in \tilde{\mathbf{I}}, M \sqsubseteq A, N \sqsubseteq A\}$  and then define

$$M \dot{\vee} N \doteq \bigcap_{A \in \tilde{\mathbf{S}}(M, N)} A.$$

**Proposition A.8**  $(\tilde{\mathbf{I}}, \sqsubseteq, \dot{\vee}, \dot{\wedge})$  is a complete lattice.

The following propositions establish some properties of  $L$ -fuzzy intervals.

**Definition A.9** For every fuzzy set  $M$  we define  $L_M \doteq \{p : M_p \neq \emptyset\}$ .

**Proposition A.10** (i) Let  $M$  be a  $L$ -fuzzy convex sublattice. If we have

$$\forall p \in L_M : M(\bigwedge M_p) \supseteq \bigcap_{x \in M_p} M(x), \quad M(\bigvee M_p) \supseteq \bigcap_{x \in M_p} M(x),$$

then  $M$  is a  $L$ -fuzzy interval.

(ii) If  $M$  is a  $L$ -fuzzy interval, then it is a  $L$ -fuzzy convex sublattice and we have

$$\forall p \in L_M : M(\bigwedge M_p) \supseteq \bigcap_{x \in M_p} M(x), \quad M(\bigvee M_p) \supseteq \bigcap_{x \in M_p} M(x).$$

**Corollary A.11** If  $M$  is a  $L$ -fuzzy interval, then  $\forall p \in L_M$  we have  $M(\bigwedge M_p) \cap M(\bigvee M_p) = \bigcap_{x \in M_p} M(x)$ .

**Corollary A.12** If  $X$  is finite, every  $L$ -fuzzy convex sublattice is a  $L$ -fuzzy interval and conversely.

**Proposition A.13** Let  $M$  be a  $L$ -fuzzy convex sublattice. If  $M$  is a  $L$ -fuzzy interval, then  $\forall p \in L_M$  we have  $M_p = M_{p_1 \cap p_2}$ , where  $p_1 = M(\bigwedge M_p)$ ,  $p_2 = M(\bigvee M_p)$ .

As remarked previously, when replacing  $L, \sqsubseteq, \dot{\vee}, \dot{\wedge}$  with  $X, \leq, \vee, \wedge$ , the above propositions hold true for any fuzzy set  $M = a * b$  or  $M = a/b$  (with any  $a, b \in X$ ).

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