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Abstract

We consider a discrete-time, infinite-horizon, one-good stochastic growth model, and we solve the central planner's optimization problem by applying a stochastic version of Pontryagin's maximum principle for Markov controls. An approximation method is used in order to extend to an infinite horizon a stochastic maximum principle derived by Arkin and Evstigneev (1987) for the finite-horizon case. We obtain efficiency conditions which are expressed in terms of stochastic multipliers, i.e., "shadow"-price functions, and a transversality condition.

We interpret these conditions by treating uncertainty as a source of heterogeneity, i.e., by considering different realizations of the capital stock as different types of capital. A connection can then be established between the stochastic one-good model and a deterministic model with heterogeneous capital goods, where the dimension of the state space increases over time reflecting the history of the stochastic shock. In such a context, the variations of the prices of the realizations, which can be interpreted as capital gains, are shown to play an explicit role, besides rentals, in the conditions for efficiency, both along the optimal path and in the stochastic steady state; in addition, such capital gains reflect not only intertemporal, but also interstate price variations, capturing insurance elements related to the presence of risk.

Introducing generations, we show how the insurance theme appears not only in the conditions for efficiency in the intertemporal allocation of resources, but also in the conditions for Pareto optimality in the distribution of aggregate consumption between generations, which are stated in terms of a stochastic marginal rate of substitution.

Efficiency and Optimality

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1. Introduction

The problem of characterizing the efficiency properties of a stochastic, dynamic economy has recently received increasing attention, especially within the overlapping-generations literature. The renewed interest in the subject is due to the fact that assessing dynamic efficiency is crucial not only for determining long-run economic policy, but also for analyzing a variety of theoretical questions, such as the existence of speculative bubbles, the operativeness of bequest motives and the sustainability of national debt.¹

However, the issue being discussed goes beyond the realm of the model with generations, and concerns directly also the classical growth model, which is the framework within which the question was originally posed by Malinvaud (1953). The overlapping-generations model can be seen as a special application, where the question of dynamic efficiency is faced jointly with the problem of Pareto optimality of resource distribution, which is conventionally examined separately after Samuelson's pure-exchange seminal contribution.

Recent contributions include the following. Abel et al. (1989) address the problem within a stochastic version of the Diamond (1965) model; starting from the consideration that a comparison between the safe rate of interest and the average growth rate is not sufficient in order to detect efficiency under uncertainty, both because of the existence of a risk premium relating the safe rate of interest and the marginal product of capital, and because of the relevance of capital gains and losses; they develop and test an alternative criterion based on flows of investment and return to capital. Zilcha (1988), introducing the notion of stochastic dominance in a similar model, derives a characterization of dynamic efficiency again in terms of interest rates and growth rates, along the lines

¹See Tirole (1985), Weil (1987) and Bertocchi (1988).

of Cass (1972). In a pure exchange framework with overlapping generations, optimality questions have been examined by Peled (1982, 1984), Manuelli (1988), and Aiyagari and Peled (1988). Related work by Balasko and Shell (1981), who deal with a deterministic model with many goods, is also relevant to our approach.

The scope of this paper is to characterize efficient allocations for a dynamic, stochastic model, both in an optimal growth version and in a version with overlapping generations. Following a tradition initiated by Cass (1965), we solve the optimization problem of a central planner by applying a stochastic version of the maximum principle derived by Pontryagin et al. (1962). Hamiltonian dynamical systems, as shown by Cass and Shell (1976a,b) both in a continuous and discrete time version, provide a useful framework for the analysis of economic growth and dynamics. A discrete time version of the maximum principle, which turns out to be crucial for extensions involving a generational structure, is developed in further detail by Weitzman and Schmidt (1971) and Weitzman (1973) for a deterministic, infinite economy. In a stochastic context, Arkin and Evstigneev (1987) derive necessary conditions for a stochastic maximum principle in discrete time; however, their results are limited to a finite horizon.

Using an approximation argument, in Section 2 we extend Arkin and Evstigneev's results to a stochastic, infinite model; our conditions for optimality involve the maximization of the conditional expected value of the Hamiltonian function, and a set of canonical equations for the multipliers, i.e., the "shadow"-price functions.

Next, we suggest the following interpretation of the resulting efficiency conditions: by treating uncertainty as a source of heterogeneity in an otherwise homogeneous state space, different realizations of the single capital good can be viewed as different "types" of capital. A similarity can then be established between the stochastic, one-capital model and a deterministic, heterogeneous-capital model where the dimension of the state space increases with time, reflecting the entire history of the shock. This exercise allows us to interpret the efficiency conditions as equilibrium conditions for the capital market, which equalize gross return, i.e., rental plus capital gain, for each "type" of capital. Such capital gains are shown to be generated by intertemporal, as well as interstate, variations of the "shadow" prices, reflecting an insurance element due to the presence of risk; in addition, they do not fade away in a stochastic steady state, which is defined as a stationary distribution for the capital stock.

In Section 3, we introduce generations and we apply our results to characterize Pareto optimal programs. Since uncertainty creates a source of heterogeneity also with respect to the agents' characteristics, we construct a welfare index which assigns different weights to different individual "types", where each "type" is again defined by the past history of the shock. Our welfare index is general enough to encompass increasing degrees of strength for the implied Pareto optimality criterion, depending on the interpretation of the weights. With respect to the resulting conditions for efficiency, the considerations previously advanced for the growth model also apply to the model with generations. In addition, we find the condition for Pareto optimality, expressed in terms of a stochastic

marginal rate of substitution, also reflects the presence of insurance aspects associated with the potential lack of intertemporal risk sharing.

Finally, in Section 4 we summarize the conclusions of this paper and suggest some questions for further research.

2. A Classical Growth Model

In this section we do the following: In 2.1 we define the infinite-horizon stochastic problem we want to solve. In 2.2 we define and solve an associated finite-horizon stochastic problem. In 2.3 we show how to obtain a suboptimal solution to the infinite horizon problem from a sequence of finite-horizon problems; we also prove that the suboptimal solution can give a level of welfare arbitrarily close to the optimal one. In 2.4 we show that the stochastic one-capital problem can be interpreted as a deterministic heterogeneous capital problem.

2.1 The Optimization Problem: Infinite Horizon

Consider the following **infinite-horizon optimization problem**: given a discrete-time, stochastic growth model, the optimization problem of the central planner consists of the maximization of the following welfare function

$$\sum_{t=0}^{\infty} E_0 \beta^t u(c_t) \quad (2.1)$$

where E_0 is the expected value operator, given the information available at time 0; β is a discount factor, such that $0 < \beta < 1$, c_t is consumption per capita and the utility function u satisfies the following assumptions: $u(0) = 0$, $u'(c_t) > 0$, and $u''(c_t) < 0$.

The resource constraint is given by the following stochastic difference equation in intensive form

$$k_{t+1} = f(k_t, \theta_t) - c_t \quad (2.2)$$

for $t = 0, 1, \dots$, where k_{t+1} is investment at time t and $f(k_t, \theta_t)$ is output at time t , as a function of the stock of capital k_t and of a stochastic shock θ_t . Capital completely depreciates in one period. We impose the following assumptions on the technology: $f(0, \theta) = 0$, for all θ ; $f'(k, \theta) > 0$ and $f''(k, \theta) < 0$ for all $k > 0$; $f'(\infty, \theta) = 0$, $f'(0, \theta) = \infty$ for all θ ; where f' , f'' are the first and second derivatives of f with respect to k ; in addition, f and f' are assumed to be continuous in both arguments; finally, $f_\theta(k, \theta) > 0$, where f_θ is the first derivative of f with respect to θ .

The random variables θ_t are independent and identically distributed; there exists a finite number N of states of nature; the realizations of θ_t are on a set

$S_t = \{\alpha_1, \alpha_2, \dots, \alpha_N\}$, such that $0 < \underline{\alpha} < \bar{\alpha} < \infty$, where $\underline{\alpha}$ and $\bar{\alpha}$ are the minimum and the maximum values of the realizations of θ_t . Let $p = [p_1, \dots, p_N]$ be a probability measure such that $\text{Prob}(\theta_t = \alpha_n) = p_n$, for $n = 1, 2, \dots, N$. Finally, define $\theta^t = (\theta_0, \theta_1, \dots, \theta_t)$ as the history of the shock up to time t , and $S^{t+1} = x_{i=0}^t S_i$ as the cartesian product of S_t , with dimension N^{t+1} .

A central planner maximizes (2.1) subject to (2.2) and the non-negativity constraints

$$c_t, k_t \geq 0 \quad (2.3)$$

for $t = 0, 1, \dots$, given an initial condition $k_0 > 0$.

The problem can be solved by applying a stochastic version of Pontryagin's maximum principle. We shall proceed in two steps. First, we apply a result derived by Arkin and Evstigneev (1987) to a truncation of the model described above, to obtain a set of necessary optimality conditions for a finite-horizon problem. Secondly, we use an approximation argument to show how similar conditions yield a suboptimal policy for an infinite-horizon model; this suboptimal policy can be chosen to guarantee a level of welfare arbitrarily close to the optimal one.

2.2 The Optimization Problem: Finite Horizon

For a finite-horizon T , the welfare function becomes

$$\sum_{t=0}^{T-1} E_0 \beta^t u(c_t) \quad (2.4)$$

to be maximized subject to the resource constraint

$$k_{t+1} = f(k_t, \theta_t) - c_t \quad (2.5)$$

for $t = 0, 1, \dots, T-1$, and to the non-negativity constraints

$$c_t, k_t \geq 0 \quad (2.6)$$

for $t = 0, 1, \dots, T$, given an initial condition $k_0 > 0$ and a final condition $k_T \geq 0$.

The problem described above will be called the **constrained optimization problem**. If we remove the positivity constraint (2.6), we get a similar (but not identical) problem, which will be called the **unconstrained optimization problem**.

Let us consider some properties of the solution to the *constrained* optimization problem. First of all, by strict concavity, *there is always a unique solution*, for any $k_0 > 0$; such solution is given by an optimal consumption policy $\{c_t\}_{t=0}^{T-1}$, where each c_t is based only upon the information available at time t . The optimal consumption c_t can be written in the form

$$c_t = c(k_t, \theta_t) \quad (2.7)$$

and it is associated with an optimal investment k_{t+1} given by

$$k_{t+1} = f(k_t, \theta_t) - c(k_t, \theta_t) \equiv h(k_t, \theta_t) \quad (2.8)$$

The values c_t and k_{t+1} are random variables, and the sequence of capital stocks $\{k_t\}_{t=0}^T$ is a first-order Markov process.²

We proceed to describe necessary optimality conditions that the solution must satisfy. We are going to apply a stochastic maximum principle for a simple Markov control problem. Such a principle is described in Arkin and Evstigneev (1987); however, it applies to the unconstrained problem. To extend it to the constrained problem, we need certain preliminary definitions.

Consider the set of solutions to the constrained problem:

$$C = \{\{c_t\}_{t=0}^{T-1} : c_t \text{ maximizes (2.4), subject to (2.5), (2.6) and (2.7) given } k_0 > 0\}.$$

Analogously, we can define the set of solutions to the unconstrained problem:

$$C^* = \{\{c_t\}_{t=0}^{T-1} : c_t \text{ maximizes (2.4), subject to (2.5) and (2.7), given } k_0 > 0\}.$$

In general, C and C^* have different elements, i.e. the solutions to the constrained problem need not be solutions of the unconstrained problem and vice versa. However, *if* there exists a $\{c_t, k_t\}$ belonging to C^* which is *strictly* positive (i.e. $c_t, k_{t+1} > 0 \forall t$), then it will also belong to C . That is: a strictly positive solution to the unconstrained problem is also a solution to the constrained problem. We call this special class of solutions *interior* solutions. In what follows, we **assume** that for every T there exists an interior solution of the constrained problem. By strict concavity this solution will be the *unique* element of C . Now we proceed, in Theorem 1, to give necessary conditions the interior solution to the constrained problem must satisfy; these are, in fact, exactly the same as the necessary conditions for a solution to the unconstrained problem.

Theorem 1 *Define a Hamiltonian function H_t as follows:*

$$\begin{aligned} H_{t+1}(c_t, k_t, \mu_{t+1}(\theta_{t+1}, k_{t+1}), \theta_t, \theta_{t+1}) \\ \equiv \beta^t u(c_t) + \mu_{t+1}(\theta_{t+1}, k_{t+1})[f(k_t, \theta_t) - c_t] \end{aligned} \quad (2.9)$$

for $t = 0, 1, \dots, T-1$, where the functions $\mu_{t+1}(\theta_{t+1}, k_{t+1})$ are the multipliers.

Then, for $\{(c_t, k_{t+1})\}_{t=0}^{T-1}$ to be an interior solution of the constrained problem, it is necessary that the conditional expected value of the Hamiltonian, given by the expression

$$E_t\{H_{t+1}(c_t, k_t, \mu_{t+1}(\theta_{t+1}, k_{t+1}), \theta_t, \theta_{t+1}) | \theta_t, k_t\} \quad (2.10)$$

²Theorem 1, p. 83, in Arkin and Evstigneev (1987) establishes the sufficiency of Markov controls.

is maximized, i.e., the following first-order conditions are satisfied

$$\frac{\partial E_t\{H_{t+1}(c_t, k_t, \mu_{t+1}(\theta_{t+1}, k_{t+1}), \theta_t, \theta_{t+1})|\theta_t, k_t\}}{\partial c_t} = 0 \quad (2.11)$$

for $t = 0, 1, \dots, T-1$. In addition, the multipliers must satisfy the conjugate system given below

$$\frac{\partial E_t\{H_{t+1}(c_t, k_t, \mu_{t+1}(\theta_{t+1}, k_{t+1}), \theta_t, \theta_{t+1})|\theta_t, k_t\}}{\partial k_t} = \mu_t(\theta_t, k_t) \quad (2.12)$$

$$\frac{\partial E_t\{H_{t+1}(c_t, k_t, \mu_{t+1}(\theta_{t+1}, k_{t+1}), \theta_t, \theta_{t+1})|\theta_t, k_t\}}{\partial \mu_{t+1}(\theta_{t+1}, k_{t+1})} = k_{t+1} \quad (2.13)$$

for $t = 0, 1, \dots, T-1$, and the transversality condition

$$\mu_T(\theta_T, k_T) = 0. \quad (2.14)$$

Proof:

By definition, an interior solution of the constrained problem is a positive solution of the unconstrained problem. Then, the necessary conditions for an interior solution are the same as the ones for a solution to the unconstrained problem. Now, by Arkin and Evstigneev (1987, Theorem 3, p.94) (2.9)-(2.14) are necessary conditions for such a solution. ■

It will be useful to rewrite (2.10) as

$$\begin{aligned} & E_t\{\beta^t u(c_t) + \mu_{t+1}(\theta_{t+1}, k_{t+1})[f(k_t, \theta_t) - c_t]|\theta_t, k_t\} \\ &= \beta^t u(c_t) + E_t\{\mu_{t+1}(\theta_{t+1}, k_{t+1})|\theta_t, k_t\}[f(k_t, \theta_t) - c_t] \\ &= \beta^t u(c_t) + \sum_{n=1}^N p_n \mu_{t+1}(\alpha_n, k_{t+1})[f(k_t, \theta_t) - c_t] \end{aligned} \quad (2.15)$$

for $t = 0, 1, \dots, T-1$.³ Conditions (2.11) - (2.13) then become simply

$$\beta^t u'(c_t) = \sum_{n=1}^N p_n \mu_{t+1}(\alpha_n, k_{t+1}) \quad (2.16)$$

$$\sum_{n=1}^N p_n \mu_{t+1}(\alpha_n, k_{t+1}) f'(k_t, \theta_t) = \mu_t(\theta_t, k_t) \quad (2.17)$$

$$k_{t+1} = f(k_t, \theta_t) - c_t \quad (2.18)$$

³Note that the second equality follows from the fact that $k_{t+1} = f(k_t, \theta_t) - c(k_t, \theta_t)$.

2.3 The Approximation Argument: From Finite to Infinite Horizon

In this section, we use an approximation method in order to extend the conditions derived above to a problem with an infinite horizon. In such a context, we show that an appropriately defined suboptimal policy can be chosen in such a way that guarantees a level of welfare arbitrarily close to the optimal one.

Assume that for every positive integer T there is an interior solution to the constrained optimization problem with finite horizon T , initial condition $k_0 > 0$ and final condition $k_T = 0$. Denote this solution by $\hat{c}_t^T = \{\hat{c}_t^T\}_{t=0}^{T-1}$.

Assume, also, that there exists a solution for the infinite-horizon optimization problem, described in 2.1. Denote this solution by $c^\infty = \{c_t^\infty\}_{t=0}^\infty$.

Now consider for the infinite-horizon problem a sequence of suboptimal solutions $c^T = \{c_t^T\}_{t=0}^\infty$, such that

$$c_t^T = \hat{c}_t^T \quad \text{for } t = 0, 1, \dots, T \quad (2.19)$$

and

$$c_t^T = 0 \quad \text{for } t = T + 1, T + 2, \dots \quad (2.20)$$

It is obvious that every one of the suboptimal solutions satisfies both the resource constraints (2.2) and the non-negativity constraints (2.3). On the other hand these solutions are *suboptimal*: let $J(c)$ be the welfare function for an infinite-horizon problem associated with a policy c . Obviously, since $\{c_t^\infty\}$ is the optimal solution, we have

$$J(c^\infty) \geq J(c^T). \quad (2.21)$$

However, we will show that we can get arbitrarily close to the global maximum $J(c^\infty)$ using suboptimal solutions of the form (2.19)-(2.20):

Theorem 2 Define $\hat{f}(k_t) \equiv f(k_t, \bar{\alpha})$, where $\bar{\alpha}$ is the maximum of $\{a_1, \dots, a_N\}$. Assume there exist constants K' , K'' and δ , with $1 < \delta < \frac{1}{\beta}$, such that

$$\underbrace{\hat{f}(\hat{f}(\dots \hat{f}(k_0)))}_{t \text{ times}} \leq K' + K''\delta^t \quad (2.29)$$

Then for any $\epsilon > 0$, there exists a time T such that

$$J(c^T) + \epsilon \geq J(c^\infty) \geq J(c^T) \quad (2.22)$$

Proof:

Step 1. Assume there exist constants A, B and δ , with $1 < \delta < \beta^{-1}$ such that

$$u(c_t) \leq A + B\delta^t. \quad (2.23)$$

Then we have the following inequality

$$\sum_{t=T+1}^{\infty} \beta^t u(c_t) \leq A \sum_{t=T+1}^{\infty} \beta^t + B \sum_{t=T+1}^{\infty} (\beta\delta)^t \quad (2.24)$$

where the right-hand side tends to 0 as T tends to infinity. So, we obtain

$$\begin{aligned} J(c^\infty) &= \sum_{t=0}^T \beta^t u(c_t^\infty) + \sum_{t=T+1}^{\infty} \beta^t u(c_t^\infty) \leq \sum_{t=0}^T \beta^t u(c_t^T) \\ &+ A \sum_{t=T+1}^{\infty} \beta^t + B \sum_{t=T+1}^{\infty} (\beta\delta)^t \end{aligned} \quad (2.25)$$

where the first part of the inequality obtains because $\{c_t^T\}$ is the optimal solution to the finite horizon problem, while the second part follows from (2.23).

On the other hand, we have

$$\sum_{t=0}^T \beta^t u(c_t^T) = \sum_{t=0}^{\infty} \beta^t u(c_t^T) = J(c^T) \quad (2.26)$$

because $c_t^T = 0$ and $u(c_t^T) = 0$ for $t > T$. Therefore, we obtain

$$J(c^T) \leq J(c^\infty) \leq J(c^T) + A \sum_{t=T+1}^{\infty} \beta^t + B \sum_{t=T+1}^{\infty} (\beta\delta)^t \quad (2.27)$$

which, since β and $\beta\delta$ are less than one, implies that for any $\epsilon > 0$ there exists a T such that

$$J(c^T) \leq J(c^\infty) \leq J(c^T) + \epsilon \quad (2.28)$$

which confirms (2.22).

Step 2. Therefore, in order to complete the proof we need to show the existence of A, B and δ such that (2.23) holds. By the conditions of the theorem, there exist constants K', K'' and δ , with $1 < \delta < \frac{1}{\beta}$, such that

$$\underbrace{\hat{f}(\hat{f}(\dots \hat{f}(k_0)))}_{t \text{ times}} \leq K' + K''\delta^t \quad (2.29)$$

Also, since $u''(c_t) < 0$,

$$\max_{c_t \in [0, \infty)} u'(c_t) = u'(0) = M. \quad (2.30)$$

Moreover, by concavity, we have

$$u(c_t) - u(0) \leq u'(0)(c_t - 0). \quad (2.31)$$

So, (2.30) and (2.31) imply that

$$u(c_t) \leq M c_t \quad (2.32)$$

Now, clearly, the policy that maximizes $c_{\bar{t}}$, for a fixed time \bar{t} is to invest all capital until $t = \bar{t} - 1$ and set $c_{\bar{t}} = k_{\bar{t}+1}$. In that case, we have

$$c_{\bar{t}} \leq \underbrace{\hat{f}(\hat{f}(\dots \hat{f}(k_0)))}_{t \text{ times}} \leq K' + K'' \delta^{\bar{t}} \quad (2.33)$$

where the inequality follows from (2.29), and in turn implies

$$u(c_{\bar{t}}) \leq M(K' + K'' \delta^{\bar{t}}) = M K' + M K'' \delta^{\bar{t}}. \quad (2.34)$$

Now, (2.23) is satisfied with $A \equiv M K'$ and $B \equiv M K''$. ■

Remark 1. We have shown that it is possible to construct an infinite-horizon, suboptimal consumption policy which guarantees a level of welfare within an $\epsilon > 0$ of the level which would be associated with the infinite-horizon, optimal consumption policy.

Remark 2. The conditions which characterize this suboptimal policy are given by equations (2.11) - (2.14), as established by Arkin and Evstigneev (1987) for a finite-horizon, optimal policy.

The above analysis completes the description of a suboptimal solution to the discrete time, infinite horizon, stochastic classical growth model.

2.4 Uncertainty as a Source of Heterogeneity

Intuitively, the presence of a stochastic shock in a one-capital growth model can be interpreted as a source of heterogeneity within an otherwise homogeneous state space. Namely, following Debreu (1959), ch. 7, a state variable can be defined not only by its physical properties and its date, but also by the event on the occurrence of which it is conditional: different realizations of the single capital stock can then be viewed as stocks of different types of capital, as illustrated in Figure 1.

The figure shows the evolution of the space of the state variable k_t , for a case where the shock takes on only two possible values, α and β ; the dimension of the space of the subsequent realizations will depend both on the number of the states of nature, in this case $N = 2$, and by the timing: given a unique initial condition k_0 , at $t = 0$ the shock θ_0 will occur, determining two possible realizations for the stock of capital

at $t = 1$, $k_1(\alpha)$ and $k_1(\beta)$; at $t = 2$, it is the entire history of the stochastic process, $\theta^1 = (\theta_0, \theta_1)$ that determines the four possible realizations of k_2 .

Exploiting this simple intuition, which was also suggested by Peled (1982) for a pure exchange model, we can “rewrite” a stochastic, one-capital model as a deterministic model with heterogeneous capital goods, where the dimension of the state space increases with time reflecting the entire history of the shock.

In this section, the intuitive connection that can be established between a deterministic model with many capital goods and a stochastic one-capital model will be made more rigorous and explicitly exploited in order to suggest an interpretation for the optimality conditions previously derived.

The following additional notation will be useful. Let σ_t be a realization of the random variable θ_t ; since θ_t takes values on the set $S = \{\alpha_1, \alpha_2, \dots, \alpha_N\}$, σ_t can also take on N possible values on S ; let $\sigma^t = (\sigma_0, \sigma_1, \dots, \sigma_t)$ denote a history of realizations up to time t .

Consider now again the solution we obtained for the stochastic one-capital model with finite horizon: at the final date T , we have N^T sequences of past realizations of the shock, given by $\sigma^{T-1} = (\sigma_0, \sigma_1, \dots, \sigma_{T-1})$, where σ^{T-1} belongs to an N^T -dimensional set S^T . Each sequence σ^{T-1} is associated to an optimal consumption and investment policy $\{(c_t(\sigma^{T-1}), k_{t+1}(\sigma^{T-1}))\}_{t=0}^{T-1}$ and to a sequence of multipliers $\{\mu_{t+1}(\sigma^{T-1})\}_{t=0}^{T-1}$, such to satisfy the following relationships, for $t = 0, 1, \dots, T-1$.

$$\beta^t u'(c_t(\sigma^t)) = \sum_{n=1}^N p_n \mu_{t+1}(\alpha_n, k_{t+1}(\sigma^t)). \quad (2.35)$$

$$\sum_{n=1}^N p_n \mu_{t+1}(\alpha_n, k_{t+1}(\sigma^t)) f'(k_t(\sigma^{t-1}), \sigma_t) = \mu_t(\sigma_t, k_t(\sigma^{t-1})). \quad (2.36)$$

$$k_{t+1}(\sigma^t) = f(k_t(\sigma^{t-1}), \sigma_t) - c_t(\sigma^t). \quad (2.37)$$

It should be noted that $c_t(\sigma^{T-1})$ really depends only on $\sigma^t = (\sigma_0, \sigma_1, \dots, \sigma_t)$; this justifies the notation $c_t(\sigma^t)$ introduced above.

It can now be noted that the expressions obtained are in fact the necessary conditions for optimization of a deterministic problem based on the maximization of the following welfare function

$$\sum_{t=0}^{T-1} \sum_{\sigma^t \in S^{t+1}} p(\sigma_0) \dots p(\sigma_t) \beta^t u(c_t(\sigma^t)) \quad (2.38)$$

where $p(\sigma_i)$ is a function $p : \{\alpha_1, \alpha_2, \dots, \alpha_N\} \rightarrow [0, 1]$, defined by $p(\sigma_i) = p_n$ for $\sigma_i = \alpha_n$, $i = 0, 1, \dots, T-1$, subject to the constraints

$$k_{t+1}(\sigma^t) = f(k_t(\sigma^{t-1}), \sigma_t) - c_t(\sigma^t) \quad (2.39)$$

for $t = 0, 1, \dots, T-1$, which are dynamical equations in k_t on an expanding state space (the state vector k_t is of size N^t).

The solution to the problem is given by a set of $N^1 + N^2 + \dots + N^{T-1}$ control variables $c_t(\sigma^t)$, $t = 0, 1, \dots, T-1$, such that (2.38) is maximized subject to (2.39) and the non-negativity constraints. In particular, each realization of k_t , which can be interpreted as a different type of capital, is appropriately weighted within the welfare function according to the probability of occurrence of the history σ^t (by independence, $p(\sigma^t) = p(\sigma_0) \dots p(\sigma_t)$). The problem, which is now free of randomness and therefore formally identical with a purely deterministic problem, can be solved by application of the deterministic maximum principle (Pontryagin et al., (1962)): define the Hamiltonian H_t

$$H_t \equiv \sum_{\sigma^t \in S^{t+1}} p(\sigma_0) \dots p(\sigma_t) \beta^t u(c_t(\sigma^t)) \\ + \sum_{\sigma^{t+1} \in S^{t+2}} \hat{\mu}_{t+1}(\sigma^{t+1}) [f(k_t(\sigma^{t-1}), \sigma_t) - c_t(\sigma^t)] \quad (2.40)$$

where the $\hat{\mu}_{t+1}(\sigma^{t+1})$ are the multipliers.

Then, the first-order conditions for optimality are given in terms of maximizing H_t . Taking derivatives and proceeding in the standard manner, we get:

$$p(\sigma_0) \dots p(\sigma_t) \beta^t u'(c_t(\sigma^t)) = \sum_{\sigma_{t+1} \in S} \hat{\mu}_{t+1}(\sigma^{t+1}) \quad (2.41)$$

$$\hat{\mu}_t(\sigma^t) = \sum_{\sigma_{t+1} \in S} \hat{\mu}_{t+1}(\sigma^{t+1}) f'(k_t(\sigma^{t-1}), \sigma_t) \quad (2.42)$$

for $t = 0, 1, \dots, T-1$.

Define now

$$\mu_{t+1}(\sigma^{t+1}) \equiv \frac{\hat{\mu}_{t+1}(\sigma^{t+1})}{p(\sigma_0) \dots p(\sigma_{t+1})} \quad (2.43)$$

Substituting (2.43) into (2.41) and (2.42), we obtain

$$\beta^t u'(c_t(\sigma^t)) = \sum_{\sigma_{t+1} \in S} p(\sigma_{t+1}) \mu_{t+1}(\sigma^{t+1}) \quad (2.44)$$

$$\mu_t(\sigma^t) = \sum_{\sigma_{t+1} \in S} p(\sigma_{t+1}) \mu_{t+1}(\sigma^{t+1}) f'(k_t(\sigma^{t-1}), \sigma_t) \quad (2.45)$$

which correspond to expressions (2.16) and (2.17).

In particular, (2.45) can be conveniently rearranged to yield

$$f'(k_t(\sigma^{t-1}), \sigma_t) = 1 + \frac{\mu_t(\sigma^t) - \sum_{\sigma_{t+1} \in S} p(\sigma_{t+1}) \mu_{t+1}(\sigma^{t+1})}{\sum_{\sigma_{t+1} \in S} p(\sigma_{t+1}) \mu_{t+1}(\sigma^{t+1})} \quad (2.46)$$

which can be interpreted as an equilibrium condition for the capital market.

Remark 3. Consistently with standard results in multi-capital models (see for example Shell and Stiglitz, 1967), two elements appear in the expression above: the marginal product of each “type” of capital, i.e., the rental, and the capital gain generated by intertemporal variations of prices, which are represented here by the multipliers. However (as already noticed by Jeanjean, 1974), there exists a qualitative difference between deterministic and stochastic multipliers: the latter, in fact, reflect not only intertemporal, but also interstate variations of relative prices, and therefore capture an insurance element due the presence of risk.

Remark 4. Another important departure from the purely deterministic setup, where the degree of heterogeneity of the state space is invariant, becomes apparent when stationarity is imposed: in the standard heterogeneous-capital model, in fact, with stationary prices capital gains disappear, and the optimality conditions are simply characterized by the equalization of rentals; on the other hand, in a stochastic model, where a stationary state is defined in terms of a stationary distribution which involves lagged values of the capital stock and of its price, the optimality conditions never reduce to a simple equalization of rentals. The “stochastic golden rule” for this model is in fact described by (2.45), if restricted on the support for the stationary distribution of the capital stock; the corresponding system of stationary prices is implicitly defined, through (2.44), by the marginal utilities of consumption over the possible levels of capital which constitute such support.

Example. A simple example will be useful at this stage. Consider the system of equations given by (2.46), and set $t = 0$ and $N = 2$, i.e., $\sigma_t = \{\alpha, \beta\}$; this implies $N^t = 1$ and $N^{t+1} = 2$. The history at time 0 is simply $\sigma^0 = (\sigma_0)$, and can take on two values, α or β ; the history at time 1 is given by $\sigma^1 = (\sigma_0, \sigma_1)$, which can take on values (α, α) , (α, β) , (β, α) , (β, β) . The system is, therefore, given by the following expression

$$f'(k_0, \alpha) + \frac{\sum_{\sigma_1} p(\sigma_1) \mu_1(\sigma^1) - \mu_0(\alpha)}{\sum_{\sigma_1} p(\sigma_1) \mu_1(\sigma^1)} = f'(k_0, \beta) + \frac{\sum_{\sigma_1} p(\sigma^1) \mu_1(\sigma^1) - \mu_0(\beta)}{\sum_{\sigma_1} p(\sigma_1) \mu_1(\sigma^1)} \quad (2.47)$$

where $\sum_{\sigma_1} p(\sigma_1) \mu_1(\sigma^1) = p(\alpha) \mu_1(\alpha, \alpha) + p(\beta) \mu_1(\alpha, \beta) + p(\alpha) \mu_1(\beta, \alpha) + p(\beta) \mu_1(\beta, \beta)$.

Remark 5. At time 1, there exist two types of capital, one for each realization of the current stock, given k_0 : the equation derived above dictates that gross returns, i.e., the sum of the own rates of return and the capital gains have to be the same for both types of capital.

3. An Overlapping-Generations Model

In this section we apply our results to characterize Pareto optimal programs in an overlapping-generations model, and to explore the connection between Pareto optimality in the distribution among consumers and dynamic efficiency of resource allocation. In 3.1 we define a stochastic optimization problem for the central planner and we obtain a suboptimal solution by solving a finite-horizon approximation. The method is exactly analogous to that used in the previous section. In 3.2 we discuss the resulting conditions. In 3.3 we consider Pareto optimality.

3.1 The Optimization Problem

We introduce now an overlapping-generations structure into the infinite-time model previously developed. In period t there are L_t identical individuals who live for two periods, t and $t+1$. Without loss of generality, we shall refer to a single representative agent for each generation. The population growth rate is assumed to be equal to zero, i.e., $L_t = L_{t+1}$. Each young individual is endowed with one unit of labor, which he supplies inelastically; utility is derived from consumption in each period, according to a time-additive utility function with standard concavity properties, given by

$$u(c_t^1) + E_t v(c_t^2) \quad (3.1)$$

where c_t^1 and c_t^2 are the levels of consumption in the first and in the second period, respectively, and $u(0) = 0$, $u' > 0$, $u'' < 0$, $v(0) = 0$, $v' > 0$, $v'' < 0$.

The technology is unchanged, and so is the stochastic process assumed for the production shock θ_t . The model, therefore, represents a stochastic version of the Diamond (1965) model.

Once again, it is possible to treat uncertainty of a source of heterogeneity: each generation, in fact, is characterized by a specific history of realizations of the random variable. In particular, at time t we can distinguish between S^{t+1} different “types” of individuals, each characterized by an history $\sigma^t = \{\sigma_0, \sigma_1, \dots, \sigma_t\}$ of realizations of $\theta^t = \{\theta_0, \theta_1, \dots, \theta_t\}$.

An appropriate welfare index for the central planner would then involve the expected utility of each type, which is indexed by σ^t and can be written as

$$W(\sigma^t) = u(c_t^1(\sigma^t)) + E_t v(c_t^2(\sigma^t, \theta_{t+1})) \quad (3.2)$$

for $t = 0, 1, \dots$. Given the intrinsic heterogeneity of the individuals, at each t the central planner will assign to each type a positive weight $w(\sigma^t)$; introducing a social discount factor R such that $0 < R < 1$, we obtain the following social welfare function

$$v(c_0^2) + \sum_{t=0}^{\infty} R^t \sum_{\sigma^t \in S^{t+1}} W(\sigma^t) w(\sigma^t). \quad (3.3a)$$

where $v(c_0^2)$ is the utility of the agent who was born old at time 0. Denote the value of the welfare function under the policy $c = \{c_t\}_{t=0}^\infty$ by $J(c)$. Consider also the finite-horizon social welfare function:

$$v(c_0^2) + \sum_{t=0}^T R^t \sum_{\sigma^t \in S^{t+1}} W(\sigma^t) w(\sigma^t). \quad (3.3b)$$

We will denote by \hat{c}^T the finite-horizon policy $\{\hat{c}_t^T\}_{t=0}^{T-1}$ and by c^T the infinite-horizon policy $\{c_t^T\}_{t=0}^\infty$, with $c_t = \hat{c}_t$ for $t = 0, 1, \dots, T-1$, $c_t = 0$ for $t = T, T+1, \dots$. Denote the value of the welfare function under the finite-horizon policy by $J(\hat{c}^T)$ and note that $J(\hat{c}^T) = J(c^T)$.

The set of resource constraints is given by

$$k_{t+1}(\sigma^t) = f(k_t(\sigma^{t-1}), \sigma_t) - c_t^1(\sigma^t) - c_{t-1}^2(\sigma^t) \quad (3.4)$$

for $t = 0, 1, \dots, \sigma^{t-1} \in S^t$ and $\sigma^t \in S^{t+1}$. The policy must also satisfy the non-negativity constraints

$$c_t^1, c_t^2, k_t \geq 0 \quad (3.5)$$

The **infinite-horizon problem** consists of the maximization of (3.3a) under (3.4) and (3.5), given an initial condition $k_0 > 0$. The **finite-horizon problem** consists of the maximization of (3.3b) under (3.4)-(3.5), initial condition $k_0 > 0$ and final condition $k_T = 0$. We can solve the infinite-horizon problem in a way that is directly analogous to the one employed in Section 2; namely by a sequence of finite-horizon solutions we can approximate the optimal infinite-horizon welfare level. The solution is summarized in the following theorem, which corresponds to Theorems 1 and 2:

Theorem 3 *Assume that for every positive integer T the finite-horizon problem has an interior solution \hat{c}^T . Assume that the infinite horizon problem has a solution c . Finally, define \hat{f} as in Theorem 2 and assume the existence of constants K' , K'' and δ that satisfy (2.29). Then for any $\epsilon > 0$ there is a time T such that*

$$J(c^T) + \epsilon \geq J(c^\infty) \geq J(c^T) \quad (2.22)$$

Here c^T is an infinite-horizon policy with $c_t^T = \hat{c}_t^T$ for $t = 0, 1, \dots, T-1$ and $c_t^T = 0$ for $t = T, 1, \dots$. Also \hat{c}_t^T is chosen to maximize the Hamiltonian function:

$$\begin{aligned} H_{t+1}(c_t^1, c_t^2, k_t, \lambda_{t+1}(\sigma^{t+1}), \sigma^t) &\equiv R^t \sum_{\sigma^t \in S^{t+1}} w(\sigma^t) [u(c_t^1(\sigma^t)) \\ &+ E_t v(c_t^2(\sigma^t, \theta_{t+1}))] + \sum_{\sigma^{t+1} \in S^{t+2}} \lambda_{t+1}(\sigma^{t+1}) [f(k_t(\sigma^{t-1}), \sigma_t) - c_t^1(\sigma^t) \\ &- c_{t-1}^2(\sigma^t)] \end{aligned} \quad (3.6)$$

for $t = 0, 1, \dots, T-1$, $\sigma^{t-1} \in S^t$, $\sigma^t \in S^{t+1}$, $\sigma^{t+1} \in S^{t+2}$.

The necessary conditions for optimization of (3.6) are:

$$R^t w(\sigma^t) u'(c_t^1(\sigma^t)) = \sum_{\sigma^{t+1} \in S^{t+2}} \lambda_{t+1}(\sigma^{t+1}) \quad (3.7)$$

$$R^{t-1} w(\sigma^{t-1}) p(\sigma_t) v'(c_{t-1}^2(\sigma^t)) = \sum_{\sigma^{t+1} \in S^{t+2}} \lambda_{t+1}(\sigma^{t+1}) \quad (3.8)$$

$$\lambda_t(\sigma^t) = \sum_{\sigma^{t+1} \in S^{t+2}} \lambda_{t+1}(\sigma^{t+1}) f'(k_t(\sigma^{t-1}), \sigma_t) \quad (3.9)$$

$$k_{t+1}(\sigma^t) = f(k_t(\sigma^{t-1}), \sigma_t) - c_t^1(\sigma^t) - c_{t-1}^2(\sigma^t) \quad (3.10)$$

$$\lambda_T(\sigma^T) = 0 \quad (3.11)$$

Proof:

The proof is similar to the proof of optimality for the classical growth problem, so only a sketch will be given. The approximation argument of Theorem 2 allows us to construct a suboptimal solution for the infinite-horizon problem from optimal interior solutions to the finite horizon problem. This suboptimal solution, as explained in Theorem 2, will give a welfare level that is within ϵ of the optimal level. The interior solution $\{\hat{c}_t^T\}$ to the problem with finite horizon T must satisfy the necessary optimality conditions (3.6)-(3.11). ■

The analogy between a stochastic onecapital problem and a deterministic heterogeneous capital problem holds in this case, too. Define

$$\hat{\lambda}_{t+1}(\sigma^{t+1}) \equiv \frac{\lambda_{t+1}(\sigma^{t+1})}{w(\sigma^{t+1})} \quad (3.12)$$

which can be substituted into (3.7) - (3.9) to yield

$$R^t w(\sigma^t) u'(c_t^1(\sigma^t)) = \sum_{\sigma_{t+1} \in S} w(\sigma^{t+1}) \hat{\lambda}_{t+1}(\sigma^{t+1}) \quad (3.13)$$

$$R^{t-1} w(\sigma^{t-1}) p(\sigma_t) v'(c_{t-1}^2(\sigma^t)) = \sum_{\sigma_{t+1} \in S} w(\sigma^{t+1}) \hat{\lambda}_{t+1}(\sigma^{t+1}) \quad (3.14)$$

$$w(\sigma^t) \hat{\lambda}_t(\sigma^t) = \sum_{\sigma_{t+1} \in S} w(\sigma^{t+1}) \hat{\lambda}_{t+1}(\sigma^{t+1}) f'(k_t(\sigma^{t-1}), \sigma_t) \quad (3.15)$$

which represent the analogue of equations (2.44) and (2.45), previously derived for the growth model. In particular, since in the overlapping-generations model we have two variables representing consumption for different generations, (3.13) and (3.14) correspond to (2.44), while (3.15) corresponds to (2.45).

A closer examination of the above conditions leads to the following considerations.

3.2 Efficiency and Pareto Optimality

Consider first the condition for dynamic efficiency given by (3.15), which has to be satisfied simultaneously with the condition for Pareto optimality given by (3.13) and (3.14). The interpretation is analogous to the one previously suggested within a classical growth model. In summary, (3.15) can be interpreted as a capital market-clearing condition, which involves rentals and capital gains; the latter are due to intertemporal as well as interstate variations of the “shadow” prices, and do not fade away even in the steady state.

Next, we observe that (3.13) and (3.14) taken together imply the following condition for a Pareto-optimal distribution of consumption between generations

$$\frac{u'(c_t^1(\sigma^t))}{v'(c_{t-1}^2(\sigma^t))} = \frac{w(\sigma^{t-1})p(\sigma_t)}{w(\sigma^t)R} \quad (3.16)$$

where the left-hand side is the intertemporal marginal rate of substitution. As noted by Manuelli (1988), such expression is not the conventional one, since at the denominator we find the marginal utility in old age of the individual born at $t - 1$, rather than at t : in other words, rather than a ratio of marginal utilities for the same individual in two different periods, we have a ratio of marginal utilities for individuals of different ages who coexist at time t .

A precise relationship between the two alternative definitions of the marginal rate of substitution is given by

$$\frac{u'(c_t^1(\sigma^t))}{v'(c_{t-1}^2(\sigma^t))} = \frac{u'(c_t^1(\sigma^t))}{v'(c_t^2(\sigma^{t+1}))} \frac{v'(c_t^2(\sigma^{t+1}))}{v'(c_{t-1}^2(\sigma^t))} \quad (3.17)$$

where the first element on the right-hand side corresponds to the conventional definition of the marginal rate of substitution.

It is worth noting that in a deterministic model, stationarity would imply that the second element on the right-hand side is equal to 1, since $c_t^2 = c_{t-1}^2$. Introducing uncertainty, however, the equivalence between intertemporal and interpersonal transfers which characterizes the deterministic case breaks down, since individuals are affected by different realizations of the random variable. The resulting expression for the intertemporal marginal rate of substitution therefore includes an additional element which reflects the presence of risk.⁴

Consider now the right-hand side of equation (3.16). Again, a comparison with the corresponding expression for a deterministic stationary state can be useful: the marginal rate of substitution would in fact be equalized to the rate of population growth. In the

⁴As shown in Manuelli (1988), the insurance aspect which is captured by the intertemporal marginal rate of substitution in a stochastic model has important implications for the conditions that allow money to be valued.

stochastic case, instead, we find a complex expression which involves the ratio of the weights that the central planner assigns to subsequent generations. (Obviously, such ratio would be equal to one in the deterministic, stationary case).

However, it is interesting to note that, if the weight $w(\sigma^t)$ are normalized to sum up to one, they can be interpreted as probabilities. In other words, rather than arbitrarily selecting the weights at each t , the planner could set them according to the following criterion

$$w(\sigma^t) = p(\sigma^t) \quad (3.18)$$

where $p(\sigma^t) = p(\sigma_0)p(\sigma_1)...p(\sigma_t)$ by independency. Substituting (3.18) into (3.16), we obtain a more familiar expression given by

$$\frac{u'(c_t^1(\sigma^t))}{v'(c_{t-1}^2(\sigma^t))} = \frac{1}{R} \quad (3.19)$$

There is in fact a precise economic interpretation of (3.18): it corresponds to the selection of a stronger, non-conditional Pareto optimality criterion (see Muerch, 1977, Peled, 1982, and Peck, 1988), according to which the planner maximizes, at each time t

$$E_0[u(c_t^1(\sigma^t)) + v(c_t^2(\sigma^t, \theta_{t+1}))] = \sum_{\sigma^t \in S^{t+1}} p(\sigma^t)[u(c_t^1(\sigma^t)) + v(c_t^2(\sigma^t, \theta_{t+1}))] \quad (3.20)$$

Summing (3.20) over time yields (3.3). To conclude, condition (3.16) reflects the selection of the welfare weights, and the welfare function given by (3.3) turns out to be general enough to encompass different degrees of strength for the implied Pareto criterion, depending on the interpretation given to such weights.

4. Summary and Conclusions

We have studied the question of dynamic efficiency in a discrete-time, infinite-horizon, one-capital stochastic model.

In an optimal-growth version of the model, we have derived efficiency conditions expressed in terms of “shadow”-price functions. By treating each possible realization of the capital good, for any given history of the shock, as a different “type” of capital, our efficiency conditions can be interpreted as market-clearing conditions for the capital market, which dictate that gross returns, i.e., rentals plus capital gains, should be equalized for each “type” of capital. However, there exists a qualitative difference between deterministic “shadow” prices and stochastic ones: the latter, in fact, reflect an additional insurance element. Introducing generations, the same insurance theme appears also in the conditions for Pareto optimality, which are stated in terms of a stochastic marginal rate of substitution.

Our results can be applied in order to investigate phenomena such as the existence of money, the feasibility of speculative bubbles and the sustainability of national debt in a stochastic, production economy.

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