Multi-valued t-Norms and t-Conorms

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Abstract

We present a procedure for constructing *multi-valued* t-norms and t-conorms. Our construction uses a pair of single-valued t-norms and the pair of dual t-conorms to construct interval-valued t-norms $\sqcap$ and t-conorms $\sqcup$. In this manner we can combine desirable characteristics of different t-norms and t-conorms; furthermore if we use the t-norm $\wedge$ and t-conorm $\vee$, then $(X, \sqcap, \sqcup)$ is a superlattice, i.e. the multivalued analog of a lattice.

1 Introduction

The fuzzy literature contains many examples of *t-norms*, which are a generalization of (classical) set intersection. All of these t-norms are (as far as we know) *single-valued*. To be precise: given a set (of membership values, truth values etc.) X, a t-norm is a binary function $T : X \times X \to X$ satisfying certain properties. Hence, given two elements of X, call them x, y, then $T(x, y)$ is also an element of X. Note that this is also true in the context of interval-valued fuzzy sets, fuzzy sets of type 2 and other variants. For example, a t-norm which operates on interval-valued fuzzy sets combines two intervals to produce *one* interval. Similar remarks can be made about t-conorms, which are a generalization of (classical) set union. We will refer to both t-norms and t-conorms as *connectives*.

In this paper we introduce *multi-valued* connectives. In other words, we are interested in binary functions which map elements of X to *subsets* of X. Before formally presenting our results let us briefly discuss the reasons for introducing multi-valued connectives.

Fuzzy theorists have often argued that a major motive behind the theory of fuzzy sets has been the treatment of *uncertainty*. Many examples appear in the literature; for instance Nguyen [5] mentions classes with vaguely defined boundaries and numbers which are only known to lie within an interval as two examples where fuzzy sets can be fruitfully applied.

The above examples (and many similar ones appearing throughout the literature) involve uncertainty about the degree to which objects belong to sets; on the other hand the manner in which fuzzy sets are *combined* (e.g. by unions, intersections etc.) *does not involve any uncertainty*. For example, given two fuzzy sets A and B and an element x, the degree to which x belongs to both A and B is given by $A(x) \wedge B(x)$; no uncertainty is involved in the application of the $\wedge$ connective. A natural extension of the principle of fuzziness is to consider *uncertain connectives*; the use of multi-valued t-norms and t-conorms is a simple step in this direction.

Hence the plan of this paper is as follows. We work in the context of a deMorgan lattice $(X, \wedge, \vee, ')$ (where $'$ is *negation*), hence our results will hold equally for fuzzy and *L-fuzzy* sets. We introduce multi-valued operations $\sqcap : X \times X \to \mathbf{P}(X)$ and $\sqcup : X \times X \to \mathbf{P}(X)$ (where $\mathbf{P}(X)$ is the *power set* of X). Then we show that $\sqcap$ has properties which are analogous (in the multi-valued context) of the properties usually required of t-norms; similarly $\sqcup$ has properties analogous to those usually required of t-conorms. Finally, we show that the structure $(X, \sqcap, \sqcup)$ is the analog (in the multi-valued context) of a lattice.

The last remark requires some additional explanation. Let us first remark that there is an extensive literature in the study of multi-valued algebraic operations (called *hyperoperations*) and the corresponding algebraic structures (*hyperalgebras*). The books [1, 2] present an extensive study of hyperalgebras. As will be seen in Section 3, our $(X, \sqcap, \sqcup)$ is a *superlattice* [4].

While multi-valued operations have been studied extensively in the hyperalgebraic literature, we believe (as already mentioned) that they have not been previously discussed in the fuzzy literature. However, our approach is quite similar to the one used by Jenei in [3]. Indeed, the actual construction of the interval-valued t-norms and t-conorms is the same as the one used by us (indeed Jenei's paper has been a major inspiration to us). Jenei argues that his connectives are preferrable to classical ones because they combine a large number of desirable properties; this remark also holds for our $\sqcap$ and $\sqcup$ and can be considered as an additional reason for their introduction. We will discuss the relation of our results to those obtained by Jenei in Section 5.

2 Preliminaries

We will present our results in the context of L-fuzzy sets, i.e. all the results presented below hold when membership takes values in a lattice (rather than in the unit interval of real numbers). This generality can be obtained at no additional cost, i.e. the proofs of our results are essentially the same for the cases of real numbers and general lattice[1].

Hence, in what follows we assume the existence of a deMorgan lattice $(X, \wedge, \vee, ')$ (where $'$ denotes *negation*) with a minimum element 0 and a maximum element 1. The order compatible with $\wedge, \vee$ will be denoted by $\leq$. Lattice *intervals* are defined in the standard manner: for every $x, y \in X$ with $x \leq y$ we define $[x, y] = \{z : x \leq z \leq y\}$. The *empty* interval is the empty set $\emptyset$ and can be symbolized as $[x, y]$ for any pair x, y such that $x \not\leq y$. The collection of all intervals of X, including the empty interval, will be symbolized by $\mathbf{I}(X)$. We define, in standard manner, an order on $\mathbf{I}(X)$.

Definition 2.1 *For every $[x, y]$, $[u, v] \in \mathbf{I}(X)$ we write $[x, y] \preceq [u, v]$ iff $x \leq u$ and $y \leq v$.*

Proposition 2.2 *$\preceq$ is an order on $\mathbf{I}(X)$ and $(\mathbf{I}(X), \preceq)$ is a lattice where*

$$\inf([x, y], [u, v]) = [x \wedge u, y \wedge v],$$
$$\sup([x, y], [u, v]) = [x \vee u, y \vee v]$$

for every $[x, y]$, $[u, v] \in \mathbf{I}(X)$.

In the lattice context we can define a t-norm T to be any function $T : X \times X \to X$ which satisfies the following properties.

Definition 2.3 *A function $T : X \times X \to X$ is a* t-norm *iff it satisfies the following for every $x, y, z \in X$.*

1. $T(1, x) = x.$
2. $T(x, y) = T(y, x).$
3. $T(x, T(y, z)) = T(T(x, y), z).$
4. $x \leq y \Rightarrow T(x, z) \leq T(y, z).$

Similarly, a t-conorm S is any function $S : X \times X \to X$ which satisfies the following properties.

Definition 2.4 *A function $S : X \times X \to X$ is a* t-conorm *iff it satisfies the following for every $x, y, z \in X$:*

1. $S(0, x) = x.$
2. $S(x, y) = S(y, x).$
3. $S(x, S(y, z)) = S(S(x, y), z).$
4. $x \leq y \Rightarrow S(x, z) \leq S(y, z).$

[1] In what follows we omit proofs because of space limitations.

Notation 2.5 *We will write $T(x, y, z)$ for $T(T(x, y), z) = T(x, T(y, z))$ and $S(x, y, z)$ for $S(S(x, y), z) = S(x, S(y, z))$ (by associativity).*

Definition 2.6 *Given a t-norm T and a t-conorm S, we say that T and S are* dual *(with respect to the negation $'$) iff $(T(x, y))' = S(x', y')$.*

Definition 2.7 *For every $[x, y] \in \mathbf{I}(X)$, we define $[x, y]' = \{z'\}_{z \in [x,y]}$.*

Remark. In the sequel we will occasionally make use of certain well-known properties of t-norms and t-conorms which follow from Definitions 2.3 and 2.4. For example, $T(0, x) = 0$, $S(1, x) = 1$, $x \leq y \Rightarrow T(z, x) \leq T(z, y)$, $x \leq y \Rightarrow S(z, x) \leq S(z, y)$ etc. Also, using Definition 2.7 it is straightforward that $[x, y]' = [y', x']$. Finally, proofs of the following propositions can be found in [5].

Proposition 2.8 *$\wedge$ is a t-norm and $\vee$ is its dual t-conorm.*

Proposition 2.9 *Given a t-norm T and a t-conorm S, for every $x, y \in X$ we have: $T(x, y) \leq x \wedge y$ and $x \vee y \leq S(x, y)$.*

Proposition 2.10 *For all $x, y \in X$ we have: $T(x, y) \leq x \leq S(x, y)$.*

We now present some material relating to *hyperoperations*. For more details see [1].

Definition 2.11 *A* hyperoperation *is a mapping $* : X \times X \to \mathbf{P}(X)$, where $\mathbf{P}(X)$ is the power-set of X.*

Remark. In other words, while an operation maps every pair of elements to an element, a hyperoperation maps every pair of elements to a *set*. The following is a standard notation used in the hyperoperations literature.

Notation 2.12 *If $*$ is a hyperoperation on X, then for every $x, y, z \in X$ we define*

$$x * (y * z) = \cup_{u \in y * z} x * u,$$
$$(x * y) * z = \cup_{u \in x * y} u * z.$$

A particular hyperstructure of interest in this paper is the *superlattice* [4].

Definition 2.13 *Given hyperoperations $\bigtriangledown, \bigtriangleup$ on $(X, \wedge, \vee)$, we say that $(X, \bigtriangledown, \bigtriangleup)$ is a* superlattice *iff the following properties hold for all $x, y, z \in X$.*

A1 $x \in x \bigtriangleup x$, $x \in x \bigtriangledown x$.

A2 $x \bigtriangleup y = y \bigtriangleup x$, $x \bigtriangledown y = y \bigtriangledown x$.

A3 $(x \triangle y) \triangle z = x \triangle (y \triangle z)$, $(x \triangledown y) \triangledown z = x \triangledown (y \triangledown z)$.

A4 $x \in (x \triangle y) \triangledown x$, $x \in (x \triangledown y) \triangle x$.

A5 $x \leq y \Leftrightarrow y \in x \triangledown y \Leftrightarrow x \in x \triangle y$.

Obviously this is a generalization of the concept of lattice to the context of hyperoperations; in particular, every lattice can be seen as a superlattice with "trivial" (single-valued) hyperoperations.

3 Interval-Valued t-Norms and t-Conorms

In the following $T(x,y)$ will denote an arbitrary t-norm and $S(x,y)$ its dual t-conorm (with respect to some arbitrary negation x'). The only condition we impose on $T(x,y)$ and $S(x,y)$ is the following.

Condition 3.1 *For all $x, y, z \in X$ we have:*

1. $T(x \vee y, z) = T(x,z) \vee T(y,z)$.
2. $T(x \wedge y, z) = T(x,z) \wedge T(y,z)$.
3. $S(x \vee y, z) = S(x,z) \vee S(y,z)$.
4. $S(x \wedge y, z) = S(x,z) \wedge S(y,z)$.

Proposition 3.2 *Condition 3.1 is automatically satisfied for every T, S pair if X is the interval $[0,1]$ of real numbers.*

We now define the interval-valued fuzzy connectives $\sqcap, \sqcup$.

Definition 3.3 *For all $x, y \in X$ we define $x \sqcap y = [T(x,y), x \wedge y]$, $x \sqcup y = [x \vee y, S(x,y)]$.*

Proposition 3.4 *For all $x, y, z \in X$ such that $y \leq z$, we have: $x \sqcap [y,z] = [T(x,y), x \wedge z]$ and $x \sqcup [y,z] = [x \vee y, S(x,z)]$.*

The following proposition shows that $\sqcap, \sqcup$ have the analogs of t-norm, t-conorm properties (in the context of hyperoperations).

Proposition 3.5 *For all $x, y, z \in X$ we have:*

1. $x \in 1 \sqcap x$, $0 \in 0 \sqcap x$, $x \in 0 \sqcup x$, $1 \in 1 \sqcup x$.
2. $x \sqcap y = y \sqcap x$, $x \sqcup y = y \sqcup x$.
3. *If $x \leq y$, then $x \sqcap z \preceq y \sqcap z$ and $x \sqcup z \preceq y \sqcup z$.*
4. $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z) = [T(x,y,z), x \wedge y \wedge z]$
5. $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z) = [x \vee y \vee z, S(x,y,z)]$.

Proposition 3.6 *For all $x, y \in X$ we have:*

1. $x \in x \sqcap x$, $x \in x \sqcup x$.
2. $x \in x \sqcap (x \sqcup y)$, $x \in x \sqcup (x \sqcap y)$.
3. $x \leq y \Leftrightarrow y \in x \sqcup y \Leftrightarrow x \in x \sqcap y$.

Proposition 3.7 *For all $x, y \in X$ we have: $(x \sqcup y)' = x' \sqcap y'$ and $(x \sqcap y)' = x' \sqcup y'$.*

Proposition 3.8 *For all $x, y, z \in X$ we have:*

$$
\begin{aligned}
&[T(x, y \vee z), x \wedge (y \vee z)] \\
&\subseteq (x \sqcap (y \sqcup z)) \cap ((x \sqcap y) \sqcup (x \sqcap z))
\end{aligned}
$$

and

$$
\begin{aligned}
&[x \vee (y \wedge z), S(x, y \wedge z)] \\
&\subseteq (x \sqcup (y \sqcap z)) \cap ((x \sqcup y) \sqcap (x \sqcup z)).
\end{aligned}
$$

4 Generalizations

We can generalize the construction of the multi-valued connectives (Definition 3.3) in the following manner. Suppose that T_1, T_2 are t-norms and S_1, S_2 their dual t-conorms. Furthermore, suppose that for all $x, y \in X$ we have $T_1(x,y) \leq T_2(x,y)$ and $S_2(x,y) \leq S_1(x,y)$. For all $x, y \in X$ define

$$
x \sqcap y = [T_1(x,y), T_2(x,y)], x \sqcup y = [S_2(x,y), S_1(x,y)]. \tag{1}
$$

Then it is still possible that $\sqcap, \sqcup$ have the t-norm, t-conorm properties of Proposition 3.5. As an example take

$$
\begin{aligned}
T_1(x,y) &= \max(0, x+y-1), \\
T_2(x,y) &= xy, \\
S_1(x,y) &= \min(1, x+y), \\
S_2(x,y) &= x+y-xy.
\end{aligned}
$$

It is easy to check that all the properties of Proposition 3.5 still hold.

However, an additional attractive point of our construction is that $(X, \sqcap, \sqcup)$ behaves similarly to a lattice (i.e. it is a superlattice). Can we obtain this behavior for T_2 different from $\wedge$ and S_2 different from $\vee$? A first answer turns out to be negative.

Proposition 4.1 *Suppose that T_1, T_2 are t-norms and S_1, S_2 their dual t-conorms. Furthermore, suppose that for all $x, y \in X$ we have $T_1(x,y) \leq T_2(x,y)$ and $S_2(x,y) \leq S_1(x,y)$. For all $x, y \in X$ define $x \sqcap y$ and $x \sqcup y$ as in (1). Then $(\forall x, y \in X : T_2(x,y) = x \wedge y)$* if and only if *$(\forall x, y \in X : x \leq y \Leftrightarrow x \in x \sqcap y)$; also $(\forall x, y \in X : S_2(x,y) = x \vee y)$* if and only if *$(\forall x, y \in X : x \leq y \Leftrightarrow y \in x \sqcup y)$.*

From the above proposition we see that $(X, \sqcap, \sqcup)$ is a superlattice *compatible with the original order* $\leq$ iff $x \sqcap y$ and $x \sqcup y$ are defined according to Definition 3.3.

However it may still be possible to define $x \sqcap y$ and $x \sqcup y$ in such a manner that $(X, \sqcap, \sqcup)$ is a superlattice in a more general sense. Namely, suppose that **A1-A4** are satisfied and **A5** is replaced by the following conditions.

A6 $y \in x \bigtriangledown y \Leftrightarrow x \in x \bigtriangleup y$.

A7 $(x \in x \bigtriangledown y$ and $y \in x \bigtriangledown y) \Rightarrow x = y$.

A8 $(x \in x \bigtriangledown y$ and $y \in y \bigtriangledown z) \Rightarrow x \in x \bigtriangledown z$.

If **A1-A4** and **A6-A8** hold, then we can define a relation $\leqslant$ on X as follows: "$x \leqslant y$ iff $y \in x \bigtriangledown y$". It turns out that using **A6-A8** it can be shown that $\leqslant$ is an order on X, which will, in general, be different from $\leq$; in fact **A1-A4** and **A6-A8** do not use $\leq$ at all, hence the hyperoperations $\bigtriangledown, \bigtriangleup$ can be defined in a general set X (not necessarily a lattice).

In this light, it may be possible for some pairs T_1, T_2 and S_1, S_2 to define $\sqcup$ and $\sqcap$ as in (1) and then show that **A1-A4** and **A6-A8** hold; in such a case $\sqcup$ and $\sqcap$ will define an order $x \leqslant y$ on X as follows: "$x \leqslant y$ iff $y \in x \sqcup y$" and $\sqcup$, $\sqcap$ are reasonable candidates for multi-valued t-norm and t-conorm on X. However, we emphasize again that $\sqcup$, $\sqcap$ will not fully respect the "intrinsic" order $\leq$.

5 Conclusion

We have presented a procedure for constructing *multi-valued* t-norms and t-conorms. Our construction uses a pair of single-valued t-norms and the pair of dual t-conorms and constructs interval-valued t-norms $\sqcap$ and t-conorms $\sqcup$. In this manner we can combine desirable characteristics of different t-norms and t-conorms; furthermore if we use the t-norm $\wedge$ and t-conorm $\vee$, then $(X, \sqcap, \sqcup)$ is a superlattice, i.e. the multivalued analog of a lattice.

The results of Section 3 have been presented earlier by Jenei [3], but from a different point of view. As already mentioned, Jenei introduces fuzzy connectives which operate on *intervals*. While $(\mathbf{I}(X), \preceq)$ is a lattice, the associated inf and sup operators do *not* belong to the family of Jenei connectives, except if we take the somewhat trivial case of using the *same* t-norm (namely $\wedge$) as lower and upper bound of the interval (and similarly for the t-conorm). In fact, if it is required that a single-valued t-norm and a a single-valued t-conorm generate a lattice structure, then the only choice for the t-norm (resp. t-conorm) is the "natural" inf operator (resp. "natural" sup operator). We believe the main contribution of the current paper is to point out that the use of *multi-valued* t-norms and t-conorms allows the introduction of a more *general* ordered structure, namely the *superlattice.*

Let us close with some issues which require further research. First, it will be interesting to obtain further "deMorgan-like" properties of $(X, \sqcap, \sqcup, ')$ and develop a logic based on multi-valued connectives. Of particular interest is the study of the resulting implication operator, the law of excluded middle and the law of contradiction. Second, note that the fuzzy implication operator is closely connected to the *fuzzy inclusion measure*, so it would be interesting to consider interval-valued inclusion measures. Third, we are interested in analyzing $(X, \sqcap, \sqcup)$ from a geometric point of view, paying special attention to issues such as metric properties, continuity, convexity and betweenness. Finally, it will be interesting to develop a procedure for developing a *family* of interval-valued t-norms $\{\sqcap_a\}_{a\in[0,1]}$ which have the *a-cut properties*, because the $\sqcap_a$'s can then be used to construct a *fuzzy-valued t-norm* $\overline{\sqcap}$. Similarly, one could use a family $\{\sqcup_a\}_{a\in[0,1]}$ to construct a *fuzzy-valued t-conorm* $\underline{\sqcup}$.

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