

# The L-fuzzy Nakano “Hyperlattice”

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## Abstract

In this paper we study the L-fuzzy hyperoperation  $\sqcup$ , which generalizes the crisp Nakano hyperoperation  $\sqcup_1$ . We construct  $\sqcup$  using a family of crisp  $\sqcup_p$  hyperoperations as its  $p$ -cuts. The hyperalgebra  $(X, \sqcup, \wedge)$  can be understood as an *L-fuzzy hyperlattice*.

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## 1 Introduction

In this paper we perform the following construction: on a generalized deMorgan lattice  $(X, \leq, \vee, \wedge, ')$  we construct an *L-fuzzy hyperoperation*  $\sqcup$ . Then  $(X, \sqcup, \wedge)$  has almost all the properties of a fuzzy *hyperlattice* [10].  $(X, \sqcup, \wedge)$  is an example of a *L-fuzzy hyperalgebra* similar to the constructions previously presented by several authors. For example *fuzzy polygroups* have been presented by Zahedi and Hasankhani in [5, 17, 18], the same authors present *fuzzy hyperrings* in [4]; Corsini and Tofan present *fuzzy hypergroups* in [1]; Kehagias presents *L-fuzzy join spaces* in [7].

## 2 Preliminaries

In the remainder of the paper we use some notation and results from the theory of L-fuzzy sets. We present a few basic definitions here; some additional material can be found in [7, 8]. Let us also note that, in the remainder of the paper, some easy proofs are omitted because of space limitations.

In this paper we use a lattice which is defined as follows.

**Definition 2.1** A generalized deMorgan lattice is a structure  $(X, \leq, \vee, \wedge, ')$ , where  $(X, \leq, \vee, \wedge)$  is a complete distributive lattice with minimum element 0 and maximum element 1; the symbol  $'$  denotes a unary operation (“negation”); and the following properties are satisfied.

1. For all  $x \in X, Y \subseteq X$  we have  $x \wedge (\bigvee_{y \in Y} y) = \bigvee_{y \in Y} (x \wedge y)$ ,  $x \vee (\bigwedge_{y \in Y} y) = \bigwedge_{y \in Y} (x \vee y)$ . (Complete distributivity).
2. For all  $x \in X$  we have:  $(x')' = x$ . (Negation is involutory).
3. For all  $x, y \in X$  we have:  $x \leq y \Rightarrow y' \leq x'$ . (Negation is order reversing).
4. For all  $Y \subseteq X$  we have  $(\bigvee_{y \in Y} y)' = \bigwedge_{y \in Y} y'$ ,  $(\bigwedge_{y \in Y} y)' = \bigvee_{y \in Y} y'$  (Complete deMorgan laws).

The following definitions and notation will be used in the sequel.

1. A *fuzzy set* is a function  $\widetilde{M} : X \rightarrow [0, 1]$ , where  $[0, 1]$  is an interval of *real* numbers; a *L-fuzzy set* is a function  $\widetilde{M} : X \rightarrow X$ . The collection of all crisp subsets of  $X$  is denoted by  $\mathbf{P}(X)$  (*power set* of  $X$ ); the collection of all *L-fuzzy sets* (i.e. functions  $\widetilde{M} : X \rightarrow X$ ) by  $\mathbf{F}(X)$ . Hence  $\mathbf{F}(X)$  is a collection of functions which includes, as special case, the (0/1 valued) characteristic functions of crisp sets.
2. Given a set  $A \in \mathbf{P}(X)$ , we denote its inf by  $\wedge A$  and its sup by  $\vee A$ .
3. Given a *L-fuzzy set*  $\widetilde{M} : X \rightarrow X$ , the *p-cut* of  $\widetilde{M}$  is denoted by  $M_p$  and defined by  $M_p \doteq \{x : \widetilde{M}(x) \geq p\}$ . For some basic properties of *p-cuts* see [15]. Two particularly important facts are [15, pp.34-35]: (a) a fuzzy set is uniquely determined by its *p-cuts*; (b) a family of sets  $\{N_p\}_{p \in X}$  which has certain properties (“*p-cut properties*”) can be used to define a fuzzy set  $\widetilde{M}$  in a manner such that for every  $p \in X$  we have  $M_p = N_p$ .

A *crisp hyperoperation* is a mapping  $\circ : X \times X \rightarrow \mathbf{P}(X)$ ; a *L-fuzzy hyperoperation* is a mapping  $\circ : X \times X \rightarrow \mathbf{F}(X)$ .

**Definition 2.2** Let  $\circ : X \times X \rightarrow \mathbf{F}(X)$  be a *L-fuzzy hyperoperation*.

1. For all  $a \in X, \widetilde{B} \in \mathbf{F}(X)$  we define the *L-fuzzy set*  $a \circ \widetilde{B}$  by  $(a \circ \widetilde{B})(x) \doteq \bigvee_{b \in X} (\widetilde{B}(b) \wedge (a \circ b)(x))$
2. For all  $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$  we define the *L-fuzzy set*  $\widetilde{A} \circ \widetilde{B}$  by  $(\widetilde{A} \circ \widetilde{B})(x) \doteq \bigvee_{a \in X, b \in X} (\widetilde{A}(a) \wedge \widetilde{B}(b) \wedge [(a \circ b)(x)])$ .

### 3 The Family of $\sqcup_p$ Crisp Hyperoperations

**Definition 3.1** For every  $p \in X$  we define the hyperoperation  $\sqcup_p : X \times X \rightarrow \mathbf{P}(X)$  as follows:

$$\forall a, b \in X : a \sqcup_p b \doteq \{x : a \vee b \vee p' = a \vee x \vee p' = b \vee x \vee p'\}$$

In the the above definition, if we set  $p = 1$  we recover the  $\sqcup_1$  Nakano hyperoperation first presented in [14] and then in [2] and also studied in [3, 6, 11, 12, 13] and several other places. The following proposition summarizes some obvious consequences of the definition of  $\sqcup_p$ .

**Proposition 3.2** For every  $p, a, b, c \in X$  we have:

1.  $c \in a \sqcup_p b \Leftrightarrow c \vee p' \in a \sqcup_p b$ .
2.  $a \sqcup_p b = (a \vee p') \sqcup_p (b \vee p') = (a \vee p') \sqcup_1 (b \vee p')$

**Proposition 3.3** For all  $a, b, p \in X$  there exists some  $f$  such that  $a \sqcup_p b = [f, a \vee b \vee p']$ .

**Proof.** We have:  $\forall c \in a \sqcup_p b : a \vee b \vee p' = c \vee a \vee p' = c \vee b \vee p' \Rightarrow$

$$\begin{aligned} (a \vee b \vee p') &= \bigwedge_{c \in a \sqcup_p b} (c \vee a \vee p') = \bigwedge_{c \in a \sqcup_p b} (c \vee b \vee p') \Rightarrow \\ (a \vee b \vee p') &= (\bigwedge_{c \in a \sqcup_p b} c) \vee a \vee p' = (\bigwedge_{c \in a \sqcup_p b} c) \vee b \vee p' \Rightarrow \\ \bigwedge_{c \in a \sqcup_p b} c &\in a \sqcup_p b \end{aligned}$$

Similarly we can show  $\bigvee_{c \in a \sqcup_p b} c \in a \sqcup_p b$ . Next we show that  $a \sqcup_p b$  is a convex sublattice. Take any  $x, y \in a \sqcup_p b$ . I.e.

$$\begin{aligned} a \vee b \vee p' &= a \vee x \vee p' = b \vee x \vee p' \\ a \vee b \vee p' &= a \vee y \vee p' = b \vee y \vee p'. \end{aligned}$$

Taking the join of the above we obtain  $a \vee b \vee p' = a \vee x \vee y \vee p' = b \vee x \vee y \vee p'$  and so  $x \vee y \in a \sqcup_p b$ . Taking the meet, we obtain

$$\begin{aligned} a \vee b \vee p' &= (a \vee x \vee y \vee p') \wedge (a \vee x \vee y \vee p') = (b \vee x \vee y \vee p') \wedge (b \vee x \vee y \vee p') \\ &\Rightarrow a \vee b \vee p' = a \vee (x \wedge y) \vee p' = b \vee (x \wedge y) \vee p' \end{aligned}$$

and so  $x \wedge y \in a \sqcup_p b$ . Furthermore, take any  $x, y, z$  such that  $x \leq y \leq z$  and  $x, z \in a \sqcup_p b$ . I.e.

$$\begin{aligned} a \vee b \vee p' &= a \vee x \vee p' = b \vee x \vee p' \\ a \vee b \vee p' &= a \vee z \vee p' = b \vee z \vee p'. \end{aligned}$$

Then  $a \vee b \vee p' = a \vee x \vee p' \leq a \vee y \vee p' \leq a \vee z \vee p' = a \vee x \vee p'$  and so  $a \vee b \vee p' = a \vee y \vee p'$ . Similarly we show  $a \vee b \vee p' = b \vee y \vee p'$  and hence  $y \in a \sqcup_p b$ . In short we have shown that

$$a \sqcup_p b = [\wedge_{c \in a \sqcup_p b} c, \vee_{c \in a \sqcup_p b} c].$$

Let  $f = \wedge_{c \in a \sqcup_p b} c$ ,  $g = \vee_{c \in a \sqcup_p b} c$ . Since  $a \vee b \vee p' \in a \sqcup_p b$ , we have  $a \vee b \vee p' \leq g$ . On the other hand  $g \in a \sqcup_p b$  and so  $a \vee b \vee p' = a \vee g \vee p' = b \vee g \vee p' \geq g$ . Hence  $g = a \vee b \vee p'$ . ■

The following properties are related to distributivity.

**Proposition 3.4** *For all  $a, b, c, p \in X$  the following properties hold.*

1.  $(a \sqcup_p b) \vee (a \sqcup_p c) \subseteq a \sqcup_p (b \vee c)$ .
2.  $a \wedge (b \sqcup_p c) \subseteq (a \wedge b) \sqcup_p (a \wedge c)$ .
3.  $a \vee (b \sqcup_p c) \subseteq (a \vee b) \sqcup_p (a \vee c)$ .

**Proof.** In this proof we make use of some distributivity properties of  $\sqcup_1$ , established in [13]. For part 1 we have:

$$\begin{aligned} (a \sqcup_p b) \vee (a \sqcup_p c) &= ((a \vee p') \sqcup_1 (b \vee p')) \vee ((a \vee p') \sqcup_1 (c \vee p')) \\ &\subseteq (a \vee p') \sqcup_1 ((b \vee p') \vee (c \vee p')) \\ &= (a \vee p') \sqcup_1 (b \vee c \vee p') \\ &= a \sqcup_p (b \vee c). \end{aligned}$$

where the set inclusion in the second line has been obtained using the previously mentioned results of [13]. For part 2: from  $b \sqcup_p c = (b \vee p') \sqcup_1 (c \vee p')$  we get

$$\begin{aligned} a \wedge (b \sqcup_p c) &= a \wedge ((b \vee p') \sqcup_1 (c \vee p')) \\ &\subseteq (a \wedge (b \vee p')) \sqcup_1 (a \wedge (c \vee p')) \\ &= ((a \wedge b) \vee (a \wedge p')) \sqcup_1 ((a \wedge c) \vee (a \wedge p')) \\ &= ((a \wedge b) \vee (a' \vee p')) \sqcup_1 ((a \wedge c) \vee (a' \vee p')) \\ &= (a \wedge b) \sqcup_{a' \vee p} (a \wedge c) \\ &\subseteq (a \wedge b) \sqcup_p (a \wedge c); \end{aligned}$$

(in the last step we have used Proposition 3.10.2). For part 3:

$$\begin{aligned} a \vee (b \sqcup_p c) &= a \vee ((b \vee p') \sqcup_1 (c \vee p')) \\ &\subseteq (a \vee b \vee p') \sqcup_1 (a \vee c \vee p') \\ &= (a \vee b) \sqcup_p (a \vee c). \end{aligned}$$

■

**Definition 3.5** For all  $a, b, p \in X$  we write  $a \leq_p b$  (and  $b \geq_p a$ ) iff  $a \vee p' \leq b \vee p'$ .

**Proposition 3.6** The relation  $\leq_p$  is a preorder on  $X$ . The associated relation  $=_p$  (defined by:  $a =_p b$  iff  $a \leq_p b$  and  $b \leq_p a$ ) is an equivalence relation and we have  $a =_p b \Leftrightarrow a \vee p' = b \vee p'$ .

**Proposition 3.7** For all  $a, b, c, p \in X$  we have:

$$(a \sqcup_p c = b \sqcup_p c \text{ and } a \wedge c = b \wedge c) \Rightarrow a =_p b.$$

**Proof.** Since  $a \sqcup_p c = [x, a \vee c \vee p']$  and  $b \sqcup_p c = [y, b \vee c \vee p']$  we have  $a \vee c \vee p' = b \vee c \vee p'$ . Hence  $(a \vee p') \vee (c \vee p') = (b \vee p') \vee (c \vee p')$ . From  $a \wedge c = b \wedge c$  we get  $(a \wedge c) \vee p' = (b \wedge c) \vee p'$  which gives  $(a \vee p') \wedge (c \vee p') = (b \vee p') \wedge (c \vee p')$ . Hence, by distributivity,  $a \vee p' = b \vee p'$ . ■

**Proposition 3.8** For all  $a, b, c, p \in X$  we have:

$$a \leq b \Rightarrow (\forall w \in a \sqcup_p c \quad \exists u : b \sqcup_p c : w \leq u).$$

**Proof.**  $a \leq b \Rightarrow a \vee c \vee p' \leq b \vee c \vee p'$ . Since  $a \sqcup_p c = [x, a \vee c \vee p']$  and  $b \sqcup_p c = [y, b \vee c \vee p']$  the required result follows immediately. ■

The hyperstructure  $(X, \sqcup_p, \wedge, \leq_p)$  has some interesting properties.

**Proposition 3.9** For all  $a, b, c, p \in X$  the following hold.

1.  $a \in a \sqcup_p a, a = a \wedge a.$
2.  $a \sqcup_p b = b \sqcup_p a, a \wedge b = b \wedge a.$
3.  $(a \sqcup_p b) \sqcup_p c = a \sqcup_p (b \sqcup_p c), (a \wedge b) \wedge c = a \wedge (b \wedge c),$
4.  $a \in (a \sqcup_p b) \wedge a, a \in (a \wedge b) \sqcup_p a,$
5.  $b \leq_p a \Leftrightarrow a \in a \sqcup_p b.$

**Proof.** 1 and 2 are obvious. For 3 take any  $y \in (a \sqcup_p b) \sqcup_p c$  then there exists  $x \in a \sqcup_p b$  such that  $y \in x \sqcup_p c$ . Hence

$$\begin{aligned} x \vee p' &\in (a \vee p') \sqcup_1 (b \vee p') \\ y \vee p' &\in (x \vee p') \sqcup_1 (c \vee p') \subseteq ((a \vee p') \sqcup_1 (b \vee p')) \sqcup_1 (c \vee p') \\ &= (a \vee p') \sqcup_1 ((b \vee p') \sqcup_1 (c \vee p')) = \cup_{z \in b \sqcup_p c} (a \vee p') \sqcup_1 z \\ &= \cup_{z \in b \sqcup_p c} (a \vee p') \sqcup_1 (z \vee p') = \cup_{z \in b \sqcup_p c} a \sqcup_p z = a \sqcup_p (b \sqcup_p c) \end{aligned}$$

(where we have used the associativity of the  $\sqcup_1$  hyperoperation<sup>1</sup>). Hence we have shown  $(a \sqcup_p b) \sqcup_p c \subseteq a \sqcup_p (b \sqcup_p c)$ . Siimilarly we show  $a \sqcup_p (b \sqcup_p c) \subseteq (a \sqcup_p b) \sqcup_p c$  and we have proved the first part of 3; the second part is obvious. For 4 we have  $a = ((a \vee p') \vee (b \vee p')) \wedge a \in ((a \vee p') \sqcup_1 (b \vee p')) \wedge a = (a \sqcup_p b) \wedge a$ . Also  $(a \wedge b) \vee a \vee p' = (a \wedge b) \vee a \vee p' = a \vee a \vee p' \Rightarrow a \in (a \wedge b) \sqcup_p a$ . For 5, we have  $a \in a \sqcup_p b \Leftrightarrow a \vee b \vee p' = a \vee a \vee p' = b \vee a \vee p' \Leftrightarrow b \vee p' \leq a \vee p'$ . ■

Hence  $(X, \sqcup_p, \wedge, \leq_p)$  is “nearly” a hyperlattice [10]. The only difference is that  $\leq_p$  is a preorder, not an order. Next we show that, for any  $a, b \in X$ ,  $a \sqcup_p b$  has the  $p$ -cut properties.

**Proposition 3.10** *The following properties hold for all  $a, b, p, q \in X$ ,  $P \subseteq X$ .*

1.  $a \sqcup_0 b = [0, 1]$ .
2.  $p \leq q \Rightarrow a \sqcup_q b \subseteq a \sqcup_p b$ .
3.  $a \sqcup_{p \vee q} b = (a \sqcup_p b) \cap (a \sqcup_q b)$ ; more generally  $a \sqcup_{\vee P} b = \cap_{p \in P} (a \sqcup_p b)$ .

**Proof.** 1 is obvious. For 2:  $p \leq q \Rightarrow q' \leq p'$ . Now

$$\begin{aligned} x \in a \sqcup_q b &\Rightarrow \\ a \vee b \vee q' &= a \vee x \vee q' = b \vee x \vee q' \Rightarrow \\ a \vee b \vee q' \vee p' &= a \vee x \vee q' \vee p' = b \vee x \vee q' \vee p' \Rightarrow \\ a \vee b \vee p' &= a \vee x \vee p' = b \vee x \vee p' \Rightarrow \\ x &\in a \sqcup_p b. \end{aligned}$$

Regarding 3 we will prove the (more general)  $a \sqcup_{\vee P} b = \cap_{p \in P} (a \sqcup_p b)$ . Take any  $P \subseteq X$ . Since for every  $p \in P$  we have  $p \leq \vee P$ , it follows from 2 that

$$\forall p \in P : a \sqcup_{\vee P} b \subseteq a \sqcup_p b \Rightarrow a \sqcup_{\vee P} b \subseteq \cap_{p \in P} (a \sqcup_p b).$$

On the other hand

$$\begin{aligned} x \in \cap_{p \in P} (a \sqcup_p b) &\Rightarrow \forall p \in P : x \in a \sqcup_p b \Rightarrow \\ \forall p \in P : a \vee b \vee p' &= a \vee x \vee p' = b \vee x \vee p' \Rightarrow \\ \bigwedge_{p \in P} (a \vee b \vee p') &= \bigwedge_{p \in P} (a \vee x \vee p') = \bigwedge_{p \in P} (b \vee x \vee p') \Rightarrow \\ a \vee b \vee (\bigwedge_{p \in P} p') &= a \vee x \vee (\bigwedge_{p \in P} p') = b \vee x \vee (\bigwedge_{p \in P} p') \Rightarrow \\ a \vee b \vee (\bigvee_{p \in P} p)' &= a \vee x \vee (\bigvee_{p \in P} p)' = b \vee x \vee (\bigvee_{p \in P} p)' \Rightarrow x \in a \sqcup_{\vee P} b \end{aligned}$$

where we have used complete distributivity and the fact that  $\bigwedge_{p \in P} p' = (\bigvee_{p \in P} p)' = (\vee P)'$ . ■

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<sup>1</sup>This has been established independently by Nakano [14] and Comer [2].

**Definition 3.11** We define the operation  $\dot{\cup}$  between intervals as follows: for all intervals  $A, B$  we set

$$A \dot{\cup} B = \cap_{C: A \subseteq C, B \subseteq C} C.$$

**Proposition 3.12** For all  $a, b \in X$ ,  $(\{a \sqcup_p b\}_{p \in X}, \dot{\cup}, \cap, \subseteq)$  is a lattice.

**Proof.** Because of Proposition 3.10,  $\{a \sqcup_p b\}_{p \in X}$  is a closure system. ■

**Remark.** Let us note that for every  $p \in X$  we can also define a dual hyperoperation  $\sqcap_p$  as follows:

$$\forall a, b \in X : a \sqcap_p b = \{x : a \wedge b \wedge p = a \wedge x \wedge p = b \wedge x \wedge p\}$$

Each  $\sqcap_p$  has properties analogous to the ones presented above for  $\sqcup_p$ . Furthermore, there are some interesting properties of the hyperstructure  $(X, \sqcup_p, \sqcap_p)$ , especially with regard to the combination of the  $\sqcup_p$  and  $\sqcap_p$  hyperoperations. We postpone the study of  $(X, \sqcup_p, \sqcap_p)$  to a future publication.

## 4 The L-Fuzzy Hyperoperation $\sqcup$

We now proceed to synthesize the L-Fuzzy hyperoperation  $\sqcup$  using the crisp hyperoperations  $\sqcup_p$ . We will use a form of the classical construction presented in [15].

**Definition 4.1** For all  $a, b \in X$  we define the L-fuzzy set  $a \sqcup b$  by defining for every  $x \in X$ :  $(a \sqcup b)(x) \doteq \vee \{q : x \in a \sqcup_q b\}$ .

**Proposition 4.2** For all  $a, b, p \in X$  we have:  $(a \sqcup b)_p = a \sqcup_p b$ .

**Proof.** See [15]. ■

**Proposition 4.3** For all  $a, p \in X$ , for all  $\tilde{A}, \tilde{B} \in \mathbf{F}(X)$  we have: (i)  $a \sqcup_p B_p \subseteq (a \sqcup \tilde{B})_p$ , (ii)  $A_p \sqcup_p B_p \subseteq (\tilde{A} \sqcup \tilde{B})_p$ .

**Proof.** We only prove (i). Choose any  $x \in a \sqcup_p B_p$ . Then there exists some  $b \in B_p$  such that  $x \in a \sqcup_p b = (a \sqcup b)_p$ . Hence  $\tilde{B}(b) \geq p$  and  $(a \sqcup b)(x) \geq p$  and so

$$p \leq \tilde{B}(b) \wedge ((a \sqcup b)(x)) \leq \vee_{u \in X} [\tilde{B}(u) \wedge ((a \sqcup u)(x))] = (a \sqcup b)(x).$$

■

**Proposition 4.4** For all  $a, b, c, p \in X$  we have:

$$(a \sqcup b)(c) \geq p \Leftrightarrow ((a \vee p') \sqcup (b \vee p'))(c) \geq p \Leftrightarrow (a \sqcup b)(c \vee p') \geq p. \quad (1)$$

**Proof.** (1) can be restated as

$$c \in a \sqcup_p b \Leftrightarrow c \in (a \vee p') \sqcup_p (b \vee p') \Leftrightarrow c \vee p' \in a \sqcup_p b$$

which is simply a restatement of Proposition 3.2. ■

The following proposition presents some distributivity properties of  $\sqcup$ .

**Proposition 4.5** For all  $a, b, c \in X$  we have

1.  $(a \sqcup b) \vee (a \sqcup c) \subseteq a \sqcup (b \vee c).$
2.  $a \wedge (b \sqcup c) \subseteq (a \wedge b) \sqcup (a \wedge c).$
3.  $a \vee (b \sqcup c) \subseteq (a \vee b) \sqcup (a \vee c).$

**Proof.** For 1 it suffices to note that for all  $p \in X$  we have (from Proposition 3.4)  $(a \sqcup_p b) \vee (a \sqcup_p c) \subseteq a \sqcup_p (b \vee c)$ . Regarding 2, we will use the (easy to prove) property  $(a \wedge \tilde{B})_p = a \wedge B_p$ . Now, for all  $p \in X$  we have

$$\begin{aligned} (a \wedge (b \sqcup c))_p &= a \wedge (b \sqcup c)_p = a \wedge (b \sqcup_p c) \\ &\subseteq (a \wedge b) \sqcup_p (a \wedge c) \\ &= ((a \wedge b) \sqcup (a \wedge c))_p; \end{aligned}$$

now the required result follows from the equality of all  $p$ -cuts. 3 is proved similarly to 2. ■

**Proposition 4.6** For all  $a, b, c \in X$  we have:  $(a \sqcup c = b \sqcup c \text{ and } a \wedge c = b \wedge c) \Rightarrow a = b$ .

**Proof.** Suppose that  $a \sqcup c = b \sqcup c$  and  $a \wedge c = b \wedge c$ . Then, for every  $p \in X$  we have  $a \sqcup_p c = b \sqcup_p c$  and  $a \wedge c = b \wedge c$ . In particular we have  $a \sqcup_1 c = b \sqcup_1 c$  and  $a \wedge c = b \wedge c$  and so (by Proposition 3.7)  $a = b$ . ■

**Proposition 4.7** For all  $a, b, c, p \in X$  we have:

$$a \leq b \Rightarrow (\forall w : (a \sqcup c)(w) \geq p \quad \exists u : (b \sqcup c)(u) \geq p : w \leq u).$$

**Proof.** This is simply a restatement of Proposition 3.8. ■

**Proposition 4.8** *For all  $a, b, c, p \in X$  the following hold.*

1.  $(1 \sqcup a)(1) = 1; (0 \sqcup a)(a) = 1; (a \sqcup a)(a) = 1.$
2.  $(a \sqcup b)(a \vee b) = 1.$

**Proof.** For 1 we have:  $1 = 1 \vee a \in 1 \sqcup_1 a \Rightarrow (1 \sqcup a)(1) = \vee \{p : 1 \in 1 \sqcup_p a\} \geq 1$ . The remaining parts of 1 are proved similarly. For 2, we have:  $a \vee b \in a \sqcup_1 b \Rightarrow (a \sqcup b)(a \vee b) = \vee \{p : a \vee b \in a \sqcup_p b\} \geq 1$ . ■

The next proposition states the basic properties of  $\sqcup$ .

**Proposition 4.9** *For all  $a, b, c, p \in X$  the following hold.*

1.  $(a \sqcup a)(a) = 1.$
2.  $a \sqcup b = b \sqcup a$
3.  $a \sqcup_p b \sqcup_p c \subseteq (a \sqcup (b \sqcup c))_p \cap ((a \sqcup b) \sqcup c)_p.$
4.  $((a \sqcup b) \wedge a)(a) = 1; ((a \wedge b) \sqcup a)(a) = 1.$
5.  $b \leq_p a \Leftrightarrow (a \sqcup b)(a) \geq p.$

**Proof.** For 1 note that  $a \in a \sqcup_1 a = (a \sqcup a)_1$  and so  $(a \sqcup a)(a) \geq 1$ . 2 is immediate. To prove 3, we apply Proposition 4.3.(i) using  $\tilde{B} = b \sqcup c$ ; in this manner we show that  $a \sqcup_p b \sqcup_p c = a \sqcup_p (b \sqcup_p c) = a \sqcup_p (b \sqcup c)_p \subseteq (a \sqcup (b \sqcup c))_p$ ; similarly  $a \sqcup_p b \sqcup_p c \subseteq ((a \sqcup b) \sqcup c)_p$ . and we are done. From Proposition 4.8 we have  $(a \sqcup b)(a \vee b) = 1$ ; also  $(a \vee b) \wedge a = a$ . Hence

$$((a \sqcup b) \wedge a)(a) = \vee_{x: x \wedge a = a} ((a \sqcup b)(x)) \geq (a \sqcup b)(a \vee b) = 1$$

and we have proved the first part of 4. For the second part, note that  $a = (a \wedge b) \vee a \in (a \wedge b) \sqcup_1 a$ , hence  $((a \wedge b) \sqcup a)(a) \geq 1$ . Finally, 5 is simply a restatement of the last part of Proposition 3.9. ■

## 5 The Crisp Hyperalgebra $(X, \sqcup_p, \wedge)$ and the L-fuzzy Hyperalgebra $(X, \sqcup, \wedge)$

In conclusion, let us note that the crisp hyperalgebra  $(X, \sqcup_p, \wedge)$ , as well as the L-fuzzy hyperalgebra  $(X, \sqcup, \wedge)$  are very similar to a *hyperlattice*. According to the definition given in [10], a hyperlattice is a crisp hyperalgebra  $(X, \nabla, \wedge)$ , where  $\nabla$  is a crisp hyperoperation which satisfies (for every  $a, b, c \in X$ ) the properties of Table 1.

$a \in a \nabla a, a = a \wedge a$
$a \nabla b = b \nabla a, a \wedge b = b \wedge a$
$(a \nabla b) \nabla c = a \nabla (b \nabla c)$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$a \in (a \nabla b) \wedge a$
$a \in (a \wedge b) \nabla a$
$b \leq a \Leftrightarrow a \in a \vee b$

**Table 1**

In the first column of Table 2 we list the basic properties (satisfied for every  $a, b, c, p \in X$ ) of the crisp hyperalgebra  $(X, \sqcup_p, \wedge)$ . In the second column of Table 2 we list the corresponding properties of the L-fuzzy hyperalgebra  $(X, \sqcup, \wedge)$ .

$(X, \sqcup_p, \wedge)$	$(X, \sqcup, \wedge)$
$a \in a \sqcup_p a, a = a \wedge a$	$(a \sqcup a)(a) = 1, a = a \wedge a$
$a \sqcup_p b = b \sqcup_p a, a \wedge b = b \wedge a$	$a \sqcup b = b \sqcup a, a \wedge b = b \wedge a$
$(a \sqcup_p b) \sqcup_p c = a \sqcup_p (b \sqcup_p c)$	$(a \sqcup_p b) \sqcup_p c \subseteq (a \sqcup (b \sqcup c))_p \cap ((a \sqcup b) \sqcup c)_p$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$a \in (a \sqcup_p b) \wedge a$	$((a \sqcup b) \wedge a)(a) = 1$
$a \in (a \wedge b) \sqcup_p a$	$((a \wedge b) \sqcup a)(a) = 1$
$a \in a \sqcup_p b \Leftrightarrow b \leq_p a$	$(a \sqcup b)(a) \geq p \Leftrightarrow b \leq_p a$

**Table 2**

The reader will observe the similarity between the properties of  $(X, \nabla, \wedge)$ ,  $(X, \sqcup_p, \wedge)$  and  $(X, \sqcup, \wedge)$ .  $(X, \sqcup_p, \wedge)$  is “almost” a hyperlattice; indeed the only difference between the properties of  $(X, \nabla, \wedge)$  and  $(X, \sqcup_p, \wedge)$  is the use of the pre-order  $\leq_p$  in Table 2.

Similarly, the properties of  $(X, \sqcup, \wedge)$  are the “L-fuzzy versions” of the  $(X, \nabla, \wedge)$  properties. The main differences are that  $\sqcup$  is weakly associative (this is similar to  $H_v$  associativity [16]) and the ordering property induced by  $\sqcup$  concerns the pre-order  $\leq_p$  rather than the order  $\leq$ . Hence  $(X, \sqcup, \wedge)$  can be considered as an L-fuzzy version of  $(X, \sqcup_p, \wedge)$ .

We have already mentioned the possibility of constructing a family of  $\sqcap_p$  hyperoperations; these can also be used to construct an L-fuzzy hyperoperation  $\sqcap$ . Then one could compare the properties of the crisp hyperalgebra  $(X, \sqcap_p, \vee)$  and the L-fuzzy hyperalgebra  $(X, \sqcap, \wedge)$  and conclude that  $(X, \sqcap_p, \vee)$  and  $(X, \sqcap, \vee)$  have properties similar to those of a crisp *dual* hyperlattice  $(X, \triangle, \vee)$ .

## References

- [1] P. Corsini and I. Tofan. “On fuzzy hypergroups”. *P.U.M.A.* vol.8, pp.29-37, 1997.
- [2] S.D. Comer, “Multi-valued algebras and their graphical representations”, preprint, Dep. of Mathematics and Computer Science, The Citadel, 1986.
- [3] G. Cialaguareanu and V. Leoreanu. “Hypergroups associated with lattices”. *Ital. J. Pure Appl. Math.*, vol. 9, pp.165–173, 2001.
- [4] A. Hasankhani and M.M. Zahedi. “ $F$ -Hyperrings”. *Ital. Journal of Pure and Applied Math.*, vol. 4, pp.103-118, 1998.
- [5] A. Hasankhani and M.M. Zahedi. “On  $F$ -polygroups and fuzzy sub- $F$ -polygroups”. *J. Fuzzy Math.*, vol. 6, pp. 97–110, 1998.
- [6] Ath. Kehagias, K. Serafimidis and M. Konstantinidou. “A note on the congruences of the Nakano superlattice and Some Properties of the Associated Quotients”. *Rend. Circ. Mat. Palermo*, vol.51, pp.333-354, 2002.
- [7] Ath. Kehagias. “An example of L-fuzzy join space”. *Rend. Circ. Mat. Palermo*, vol. 51, pp.503-526, 2002.
- [8] Ath. Kehagias. “L-fuzzy meet and join hyperoperations”. (In this volume).
- [9] Ath. Kehagias. “The lattice of fuzzy intervals and sufficient conditions for its distributivity”. *arXiv:cs.OH/0206025*, at <http://xxx.lanl.gov/find/cs>.
- [10] M. Konstantinidou and J. Mittas. “An introduction to the theory of hyperlattices”. *Math. Balkanica*, vol.7, pp.187-193, 1977.

- [11] M. Konstantinidou and K. Serafimidis. “Hyperstructures dérivées d’un treillis particulier”. *Rend. Mat. Appl.* vol. 7, pp. 257–265, 1993.
- [12] V. Leoreanu. “Direct limits and inverse limits of hypergroups associated with lattices”. To appear in *Italian J. of Pure and Appl. Math.*
- [13] J. Mittas and M. Konstantinidou. “Contributions à la théorie des treillis avec des structures hypercompositionnelles y attachées”. *Riv. Mat. Pura Appl.* vol. 14, pp.83–114, 1994.
- [14] T. Nakano, “Rings and partly ordered systems”. *Math. Zeitschrift*, vol.99, pp.355–376, 1967.
- [15] H.T. Nguyen and E.A. Walker. *A First Course on Fuzzy Logic*, CRC Press, Boca Raton, 1997.
- [16] S. Spartalis, A. Dramalides and T. Vougiouklis. “On  $H_V$ -group rings”. *Algebras Groups Geom.*, vol.15, pp.47–54, 1998.
- [17] M.M. Zahedi and A. Hasankhani. “ $F$ -Polygroups”. *J. Fuzzy Math.*, vol. 4, pp.533–548. 1996.
- [18] M.M. Zahedi and A. Hasankhani. “ $F$ -Polygroups (II)”. *Inf. Sciences*, vol.89, pp.225–243, 1996.