L-fuzzy Meet and Join Hyperoperations

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Abstract

In this paper we study two fuzzy hyperoperations, denoted by $\curlyvee$ (which can be seen as a generalization of $\vee$) and $\curlywedge$ (which can be seen as a generalization of $\wedge$). $\curlyvee$ is obtained from a family of crisp $\vee_p$ hyperoperations and $\curlywedge$ is obtained from a family of crisp $\wedge_p$ hyperoperations. The hyperstructure $(X, \curlyvee, \wedge)$ resembles a *hyperlattice* and the hyperstructure $(X, \vee, \curlywedge)$ resembles a *dual hyperlattice*.

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1 Introduction

Starting with a generalized deMorgan lattice $(X, \leq, \vee, \wedge, ')$ we construct two *L-fuzzy hyperoperations*: $\curlyvee$ and $\curlywedge$; $\curlyvee$ is a *fuzzified* version of the join $\vee$ and $\curlywedge$ is a *fuzzified* version of the meet $\wedge$. The *fuzzy hyperalgebra* $(X, \curlyvee, \wedge)$ can be understood as a fuzzy *hyperlattice* and $(X, \curlywedge, \vee)$ can be seen as a fuzzy *dual hyperlattice*. The work presented here is related to previous work in fuzzy hyperalgebras: Zahedi and Hasankhani have studied *fuzzy polygroups* in [3, 9, 10] and *fuzzy hyperrings* in [2]; Corsini and Tofan have studied *fuzzy hypergroups* in [1]; Kehagias has studied *L-fuzzy join spaces* in [4].

2 Preliminaries

Here we present notation and standard results which we will use in the sequel[1].

DeMorgan Lattices. In this paper we use a lattice which is defined as follows.

[1]Let us also note that in the rest of the paper some easy proofs are omitted because of space limitations.

Definition 2.1 *A* generalized deMorgan lattice *is a structure* $(X, \leq, \vee, \wedge, ')$*, where* $(X, \leq, \vee, \wedge)$ *is a complete distributive lattice with minimum element 0 and maximum element 1; the symbol* $'$ *denotes a unary operation ("negation"); and the following properties are satisfied.*

1. *For all* $x \in X$*,* $Y \subseteq X$ *we have* $x \wedge (\vee_{y \in Y} y) = \vee_{y \in Y}(x \wedge y)$*,* $x \vee (\wedge_{y \in Y} y) = \wedge_{y \in Y}(x \vee y)$. (Complete distributivity).

2. *For all* $x \in X$ *we have:* $(x')' = x$. (Negation is involutory).

3. *For all* $x, y \in X$ *we have:* $x \leq y \Rightarrow y' \leq x'$. (Negation is order reversing).

4. *For all* $Y \subseteq X$ *we have* $(\vee_{y \in Y} y)' = \wedge_{y \in Y} y', \quad (\wedge_{y \in Y} y)' = \vee_{y \in Y} y'$ (Complete deMorgan laws).

Intervals. We will be especially interested in (closed) intervals of the lattice $(X, \leq)$. Recall the following facts regarding intervals.

1. The collection of all closed *lattice intervals* of X is denoted by $\mathbf{I}(X)$, i.e. the set of all sets $[a, b] = \{x : a \leq x \leq b\}$. $\mathbf{I}(X)$ includes $X = [0, 1]$ and $\emptyset$, which can be written as $[a, b]$ for any a, b such that $a \not\leq b$.

2. $\cup, \cap$ will denote the usual set-theoretic union and intersection. In addition, we will use $\dot\cup$ to denote the following set operation: $A \dot\cup B \doteq \cap_{C: C \subseteq A, C \subseteq B} C$. Let $[a_1, a_2]$, $[b_1, b_2]$ be two closed intervals of $(X, \leq)$; then we have: $[a_1, a_2] \cap [b_1, b_2] = [a_1 \vee b_1, a_2 \wedge b_2]$, $[a_1, a_2] \dot\cup [b_1, b_2] = [a_1 \wedge b_1, a_2 \vee b_2]$. Also $(I(X), \dot\cup, \cap, \subseteq, ')$ is a generalized deMorgan lattice (here $'$ means set complement).

3. For every $[a_1, a_2]$, $[b_1, b_2]$, $\in \mathbf{I}(X)$, we write $[a_1, a_2] \preceq [b_1, b_2]$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$. Then $\preceq$ is an order on $\mathbf{I}(X)$ and $(\mathbf{I}(X), \preceq)$ is a lattice.

4.Since $(X, \leq, \vee, \wedge)$ is a distributive lattice, the following properties hold (for all $a, b, x, y \in X$ such that $x \leq y$, $a \leq b$):

$$\begin{array}{ll} a \vee [x, y] = [a \vee x, a \vee y]; & a \wedge [x, y] = [a \wedge x, a \wedge y]; \\ [a, b] \vee [x, y] = [a \vee x, b \vee y]; & [a, b] \wedge [x, y] = [a \wedge x, b \wedge y]. \end{array} \quad (1)$$

Sets. The following definitions and notation will be used in the sequel.

1. A *fuzzy set* is a function $\widetilde{M} : X \rightarrow [0, 1]$, where [0,1] is an interval of *real* numbers; a L-fuzzy set is a function $\widetilde{M} : X \rightarrow X$.

2. The collection of all crisp subsets of X is denoted by $\mathbf{P}(X)$ (*power set* of X); the collection of all L-fuzzy sets (i.e. functions $\widetilde{M} : X \rightarrow X$) by $\mathbf{F}(X)$. Hence $\mathbf{F}(X)$ is a collection of functions which includes, as special case, the (0/1 valued) characteristic functions of crisp sets.

3. Given a set $A \in \mathbf{P}(X)$, we denote its $\inf$ by $\wedge A$ and its $\sup$ by $\vee A$.

4. Given a L-fuzzy set $\widetilde{M} : X \rightarrow X$, the p*-cut* of $\widetilde{M}$ is denoted by M_p and defined by $M_p \doteq \{x : \widetilde{M}(x) \geq p\}$. For some basic properties of p-cuts see [7]. Two particularly important facts are [7, pp.34-35]: (a) a fuzzy set is uniquely determined by its p-cuts; (b) a family of sets $\{N_p\}_{p\in X}$ which has certain properties ("p-cut properties") can be used to define a fuzzy set $\widetilde{M}$ in a manner such that for every $p \in X$ we have $M_p = N_p$.

A *crisp hyperoperation* is a mapping $\circ : X\times X\rightarrow \mathbf{P}(X)$; a L*-fuzzy hyperoperation* is a mapping $\circ : X\times X\rightarrow \mathbf{F}(X)$.

Definition 2.2 *Let* $\circ : X \times X \rightarrow \mathbf{F}(X)$ *be a* L-fuzzy *hyperoperation .*

1. *For all* $a \in X$, $\widetilde{B} \in \mathbf{F}(X)$ *we define the* L-fuzzy *set* $a \circ \widetilde{B}$ *by* $(a \circ \widetilde{B})(x) \doteq \vee_{b\in X} \left(\widetilde{B}(b) \wedge (a \circ b)(x)\right)$
2. *For all* $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ *we define the* L-fuzzy *set* $\widetilde{A} \circ \widetilde{B}$ *by* $(\widetilde{A} \circ \widetilde{B})(x) \doteq \vee_{a\in X, b\in X} \left(\widetilde{A}(a) \wedge \widetilde{B}(b) \wedge [(a \circ b)(x)]\right)$.

The above definition also covers some special cases. For instance, if $\circ$ is a crisp hyperoperation (i.e. $a \circ b$ is a crisp set for every a, b) and B is a crisp set, then Definition 2.2 reduces to the classical hyperoperation definition $a \circ B = \cup_{b\in B} a \circ b$ (provided that we understand $\widetilde{B}(x)$ to denote the characteristic function of the set B and $(a \circ b)(x)$ to denote the characteristic function of set $a \circ b$). Similarly if $\circ$ is a crisp operation (i.e. $a \circ b$ is an element) and B is a crisp set, then Definition 2.2 reduces to $a \circ B = \cup_{b\in B} \{a \circ b\}$ which is the same as $\{x : \exists b \in B$ such that $x = a \circ b\}$.

3 The $\vee_p$ and $\wedge_p$ Crisp Hyperoperations

Definition 3.1 *For all* $p \in X$ *we define the hyperoperation* $\vee_p : X \times X \rightarrow \mathbf{P}(X)$ *as follows: for all* $a, b \in X$*:* $a \vee_p b \doteq [(a \vee b) \wedge p, (a \vee b) \vee p']$.

Proposition 3.2 *For all* $a, b, c, p \in X$ *we have:* $a\vee_p[b, c] = [(a \vee b)\wedge p, (a\vee c)\vee p']$.

Proof. By definition, $a\vee_p[b, c] = \cup_{b\leq z\leq c} a\vee_p z = \cup_{b\leq z\leq c}[(a \vee z)\wedge p, (a\vee z)\vee p']$. Take any $u \in a \vee_p [b, c]$. Then there exists some z such that: $b \leq z \leq c$ and $(a \vee z)\wedge p \leq u \leq (a \vee z)\vee p'$. Hence $(a \vee b)\wedge p \leq (a \vee z)\wedge p \leq u \leq (a \vee z)\vee p' \leq (a \vee c)\vee p'$, i.e. $u \in [(a \vee b)\wedge p, (a\vee c)\vee p']$. So $a\vee_p[b, c] \subseteq [(a \vee b)\wedge p, (a\vee c)\vee p']$. On the other hand, take any $u \in [(a \vee b)\wedge p, (a\vee c)\vee p']$ and define $z = (u\vee b)\wedge c = (u\wedge c)\vee b$ (by distributivity). Clearly $b \leq z \leq c$. Also $z\vee a\vee p' = (u\wedge c)\vee b\vee a\vee p' =$

$(u\vee b\vee a\vee p')\wedge(c\vee b\vee a\vee p')$. Since $u\leq u\vee b\vee a\vee p'$ and $u\leq c\vee a\vee p'=c\vee b\vee a\vee p'$, it follows that $u\leq z\vee a\vee p'$. Also $(z\vee a)\wedge p=((u\wedge c)\vee b\vee a)\wedge p=(u\wedge c\wedge p)\vee((b\vee a)\wedge p)$. Since $u\wedge c\wedge p\leq u$ and $(b\vee a)\wedge p\leq u$, it follows that $(z\vee a)\wedge p\leq u$. Hence we have shown $(z\vee a)\wedge p\leq u\leq z\vee a\vee p'$ and so $u\in a\vee_p z\subseteq a\vee_p[b,c]$. I.e. $[(a\vee b)\wedge p,(a\vee c)\vee p']\subseteq a\vee_p[b,c]$. ■

Definition 3.3 *For all* $a,b,p\in X$ *we write* $a\leq^p b$ *(and* $b\geq^p a$*) iff* $a\wedge p\leq b\wedge p$*.*

Proposition 3.4 *For all* $a,b,c,p\in X$ *the following hold.*

A1 $a\in a\vee_p a$.

A2 $a\vee_p b=b\vee_p a$.

A3 $(a\vee_p b)\vee_p c=a\vee_p(b\vee_p c)$.

A4 $a\in(a\vee_p b)\wedge a$, $a\in(a\wedge b)\vee_p a$.

A5 $b\leq^p a\Leftrightarrow a\in a\vee_p b$.

Proof. A1 and **A2** are obvious. For **A3** we have:

$$\begin{aligned}(a\vee_p b)\vee_p c&=\cup_{x\in[(a\vee b)\wedge p,(a\vee b)\vee p']}[(x\vee c)\wedge p,(x\vee c)\vee p']\\&=[(((a\vee b)\wedge p)\vee c)\wedge p,(((a\vee b)\vee p')\vee c)\vee p']\\&=[(a\vee b\vee c)\wedge p,(a\vee b\vee c)\vee p']\end{aligned}$$

where we have used Proposition 3.2. Similarly we can show $a\vee_p(b\vee_p c)$ = $[(a\vee b\vee c)\wedge p,(a\vee b\vee c)\vee p']$. For **A4** we have $(a\vee_p b)\wedge a$= $[(a\vee b)\wedge p,(a\vee b)\vee p']\wedge a$= $[(a\vee b)\wedge a\wedge p,((a\vee b)\vee p')\wedge a]$ = $[a\wedge p,a]\ni a$; we have used (1). Also: $(a\wedge b)\vee_p a$ = $[((a\wedge b)\vee a)\wedge p,(a\wedge b)\vee a\vee p']$ = $[a\wedge p,a\vee p']\ni a$. For **A5,** $b\leq^p a\Rightarrow b\wedge p\leq a\wedge p\leq a\Rightarrow(b\wedge p)\vee(a\wedge p)\leq a\Rightarrow(b\vee a)\wedge p\leq a\Rightarrow a\in[(b\vee a)\wedge p,(b\vee a)\vee p']=a\vee_p b$.On the other hand, assume $a\in a\vee_p b$. Then $(b\vee a)\wedge p\leq a\leq(b\vee a)\vee p'\Rightarrow(b\vee a)\wedge p\leq a\wedge p\Rightarrow(b\wedge p)\vee(a\wedge p)\leq a\wedge p\Rightarrow b\wedge p\leq a\wedge p$. ■

The next proposition shows that $\{a\vee_p b\}_{p\in X}$ has the "p-cut properties".

Proposition 3.5 *The following properties hold for all* $a,b\in X$*.*

B1 $a\vee_1 b=\{a\vee b\}, a\vee_0 b=[0,1]$.

B2 *For all* $p,q\in X$*:* $p\leq q\Rightarrow a\vee_q b\subseteq a\vee_p b$.

B3 *For all* $p, q \in X$*:* $a \vee_{p \vee q} b = (a \vee_p b) \cap (a \vee_q b)$*; for all* $P \subseteq X$*:* $a \vee_{\vee P} b = \bigcap_{p \in P} (a \vee_p b)$.

B4 *For all* $p, q \in X$*:* $a \vee_{p \wedge q} b = (a \vee_p b) \dot{\cup} (a \vee_q b)$.

Proof. **B1** is obvious. For **B2** assume $p \leq q$. Then also $q' \leq p'$ hence $(a \vee b) \wedge p \leq (a \vee b) \wedge q$ and $(a \vee b) \vee q' \leq (a \vee b) \wedge p'$. So $[(a \vee b) \wedge q, (a \vee b) \vee q'] \subseteq [(a \vee b) \wedge p, (a \vee b) \wedge p']$ and we are done. Next, we will prove the (more general) second part of **B3**. We have

$$\begin{aligned} \cap_{p \in P} (a \vee_p b) &= \cap_{p \in P} [(a \vee b) \wedge p, (a \vee b) \wedge p'] \\ &= [\vee_{p \in P} ((a \vee b) \wedge p), \wedge_{p \in P} ((a \vee b) \wedge p')] \\ &= [(a \vee b) \wedge (\vee_{p \in P} p), (a \vee b) \wedge (\wedge_{p \in P} p')] \\ &= [(a \vee b) \wedge (\vee P), (a \vee b) \wedge (\vee P)'] = a \vee_{\vee P} b. \end{aligned}$$

Finally, with respect to **B4** we have

$$\begin{aligned} a \vee_{p \wedge q} b &= [(a \vee b) \wedge (p \wedge q), (a \vee b) \vee (p \wedge q)'] \\ &= [((a \vee b) \wedge p) \wedge ((a \vee b) \wedge q), ((a \vee b) \vee p') \vee ((a \vee b) \vee q')] \\ &= [(a \vee b) \wedge p, (a \vee b) \vee p'] \dot{\cup} [a \vee b) \wedge q, (a \vee b) \vee q'] \\ &= (a \vee_p b) \dot{\cup} (a \vee_q b). \end{aligned}$$

■

Proposition 3.6 *For all* $a, b, c, p \in X$ *the following properties hold.*

1. $a \vee_p (b \wedge c) = (a \vee_p b) \wedge (a \vee_p c)$.
2. $a \wedge (b \vee_p c) = (a \wedge b) \vee_p (a \wedge c)$.

Proof. Omitted for the sake of brevity. ■

Proposition 3.7 *For all* $a, b, c, p \in X$ *we have:* $a \leq b \Rightarrow a \vee_p c \preceq b \vee_p c$.

Proof. Indeed, if $a \leq b$ then $(a \vee c) \wedge p \leq (b \vee c) \wedge p$ and $(a \vee c) \vee p' \leq (b \vee c) \vee p'$ and so $[(a \vee c) \wedge p, (a \vee c) \vee p'] \preceq [(b \vee c) \wedge p, (b \vee c) \vee p']$. ■

The family of crisp hyperoperations $\wedge_p$has properties analogous to the ones of $\vee_p$; hence proofs of the following propositions are omitted.

Definition 3.8 *For all* $p \in X$ *we define the hyperoperation* $\wedge_p : X \times X \to \mathbf{P}(X)$ *as follows. For all* $a, b \in X$*:* $a \wedge_p b \doteq [(a \wedge b) \wedge p, (a \wedge b) \vee p']$

Proposition 3.9 *For all* $a, b, c, p \in X$ *we have:* $a \wedge_p [b, c] = [(a \wedge b) \wedge p, (a \wedge c) \vee p']$.

Definition 3.10 *For all* $a, b, p \in X$ *we write* $a \leq_p b$ *(and* $b \geq_p a$*) iff* $a \vee p' \leq b \vee p'$.

Proposition 3.11 *For all* $a, b, c, p \in X$ *the following hold.*

C1 $a \in a \wedge_p a$.

C2 $a \wedge_p b = b \wedge_p a$.

C3 $(a \wedge_p b) \wedge_p c = a \wedge_p (b \wedge_p c)$.

C4 $a \in (a \wedge_p b) \vee a, a \in (a \vee b) \wedge_p a$.

C5 $b \leq_p a \Leftrightarrow a \in a \wedge_p b$.

Proposition 3.12 *The following properties hold for all* $a, b \in X$.

D1 $a \wedge_1 b = \{a \wedge b\}, a \wedge_0 b = [0, 1]$.

D2 *For all* $p, q \in X$*:* $p \leq q \Rightarrow a \wedge_q b \subseteq a \wedge_p b$.

D3 *For all* $p, q \in X$*:* $a \wedge_{p \vee q} b = (a \wedge_p b) \cap (a \wedge_q b)$*; for all* $P \subseteq X$*:* $a \wedge_{\vee P} b = \bigcap_{p \in P} (a \wedge_p b)$.

D4 *For all* $p, q \in X$*:* $a \wedge_{p \wedge q} b = (a \wedge_p b) \dot{\cup} (a \wedge_q b)$

Proposition 3.13 *For all* $a, b, c, p \in X$ *the following properties hold.*

1. $a \vee (b \wedge_p c) = (a \vee b) \wedge_p (a \vee c)$.
2. $a \wedge_p (b \vee c) = (a \wedge_p b) \vee (a \wedge_p c)$.

Proposition 3.14 *For all* $a, b, c, p \in X$ *we have:* $a \leq b \Rightarrow a \wedge_p c \preceq b \wedge_p c$.

Proposition 3.15 *For all* $a, b, c, p, q \in X$ *the following properties hold.*

1. $a \vee_p (b \wedge_q c) \subseteq (a \vee_{p \wedge q} b) \wedge_{p \vee q} (a \vee_{p \wedge q} c)$ *(when* $p \leq q$*, the* $\subseteq$ *becomes* $=$*).*
2. $a \wedge_p (b \vee_q c) \subseteq (a \wedge_{p \wedge q} b) \vee_{p \vee q} (a \wedge_{p \wedge q} c)$ *(when* $p \leq q$*, the* $\subseteq$ *becomes* $=$*).*
3. $(a \vee_{p \vee q} b) \wedge_{p \wedge q} (a \vee_{p \vee q} c) = (a \wedge_{p \wedge q} b) \vee_{p \vee q} (a \wedge_{p \wedge q} c)$.
4. $(a \wedge_{p \wedge q} b) \vee_{p \vee q} (a \wedge_{p \wedge q} c) = (a \wedge_{p \vee q} b) \vee_{p \wedge q} (a \wedge_{p \vee q} c)$.

5. $a \vee_{p\vee q} b \vee_{p\vee q} c \subseteq \left\{ \begin{array}{c} (a \vee_q b) \vee_p c \\ a \vee_q (b \vee_p c) \end{array} \right\} \subseteq a \vee_{p\wedge q} b \vee_{p\wedge q} c.$

6. $a \wedge_{p\vee q} b \wedge_{p\vee q} c \subseteq \left\{ \begin{array}{c} (a \wedge_q b) \wedge_p c \\ a \wedge_q (b \wedge_p c) \end{array} \right\} \subseteq a \wedge_{p\wedge q} b \wedge_{p\wedge q} c.$

Proof. Omitted for the sake of brevity. ■

The next proposition shows that $\vee_p$ and $\wedge_p$ have a "deMorgan" property.

Definition 3.16 *For every $A \in \mathbf{P}(X)$, we define $A' \doteq \{x'\}_{x\in A}$.*

Proposition 3.17 *For every $p, a, b \in X$ we have: (i) $(a \vee_p b)' = a' \wedge_p b'$, (ii) $(a \wedge_p b)' = a' \vee_p b'$.*

Proof. We only prove (i) ((ii) is proved similarly). We have

$$\begin{aligned} (a \vee_p b)' &= \{x' : (a \vee b) \wedge p \leq x \leq a \vee b \vee p'\} \\ &= \{x' : ((a \vee b) \wedge p)' \geq x' \geq (a \vee b \vee p')'\} \\ &= \{z : a' \wedge b' \wedge p \leq z \leq (a' \wedge b') \vee p'\} = a' \wedge_p b'. \end{aligned}$$

■

4 The $\curlyvee$ and $\curlywedge$ L-fuzzy Hyperoperations

We now construct the L-fuzzy hyperoperations $\curlyvee$ and $\curlywedge$ using the $\vee_p$ and $\wedge_p$ families as their p-cuts. This is possible because of Propositions 3.5 and 3.12.

Definition 4.1 *For all $a, b \in X$ we define the L-fuzzy sets $a \curlyvee b$ and $a \curlywedge b$ as follows: for every $x \in X$ set $(a \curlyvee b)(x) \doteq \vee\{q : x \in a \vee_q b\}$ and $(a \curlywedge b)(x) \doteq \vee\{q : x \in a \wedge_q b\}$.*

Proposition 4.2 *For all $a, b, p \in X$ we have $(a \curlyvee b)_p = a \vee_p b$, $(a \curlywedge b)_p = a \wedge_p b$.*

Proof. Follows from the construction of $a \curlyvee b$, $a \curlywedge b$ [7, pp.34-35]. ■

Proposition 4.3 *For all $a, p \in X$, for all $\widetilde{A}, \widetilde{B} \in \mathbf{F}(X)$ we have*

1. $a \vee_p B_p \subseteq \left(a \curlyvee \widetilde{B}\right)_p$, $A_p \vee_p B_p \subseteq \left(\widetilde{A} \curlyvee \widetilde{B}\right)_p$.

2. $a \wedge_p B_p \subseteq \left(a \curlywedge \widetilde{B}\right)_p$, $A_p \wedge_p B_p \subseteq \left(\widetilde{A} \curlywedge \widetilde{B}\right)_p$.

Proof. We only prove the first part of 1 (the remaining items are proved similarly). Choose any $x \in a \vee_p B_p$; then there exists $b \in B_p$ such that $x \in a \vee_p b$. Now $x \in a \vee_p b = (a \curlyvee b)_p$ implies that $(a \curlyvee b)(x) \geq p$. Also, $B(b) \geq p$. Then $(a \curlyvee B)(x)$= $\vee_u (B(u) \wedge [(a \curlyvee u)(x)]) \geq B(b) \wedge [(a \curlyvee b)(x)] \geq p$, hence $x \in (a \curlyvee B)_p$. ■

Proposition 4.4 *For all $a \in X$ the following hold.*

1. $(1 \curlyvee a)(1) = 1$, $(1 \curlywedge a)(a) = 1$.

2. $(0 \curlyvee a)(a) = 1$, $(0 \curlywedge a)(0) = 1$.

3. $(a \curlywedge b)(a \wedge b) = 1$, $(a \curlyvee b)(a \vee b) = 1$.

Proof. For 1 we have: $(1 \curlyvee a)(1) \doteq \vee\{q : 1 \in 1 \vee_q a\}$. $1 \in 1 \vee_1 a \Rightarrow 1 \in \{q : 1 \in 1 \vee_q a\} \Rightarrow (1 \curlyvee a)(1) \geq 1$. The remaining part of 1, as well as 2 are proved similarly. Regarding 3, we note that $(a \curlywedge b)(a \wedge b)$= $\vee \{q : a \wedge b \in a \wedge_q b\} \geq 1$ (since $a \wedge b \in a \wedge_1 b$). $(a \curlyvee b)(a \vee b) = 1$ is proved similarly. ■

We are now ready to establish some basic properties of $\curlyvee$ and $\curlywedge$.

Proposition 4.5 *For all $a, b, c, p \in X$ the following hold.*

E1 $(a \curlyvee a)(a) = 1$, $(a \curlywedge a)(a) = 1$.

E2 $a \curlyvee b = b \curlyvee a$, $a \curlywedge b = b \curlywedge a$.

E3.1 $a \vee_p b \vee_p c \subseteq (a \curlyvee (b \curlyvee c))_p \cap ((a \curlyvee b) \curlyvee c)_p$.

E3.2 $a \wedge_p b \wedge_p c \subseteq ((a \curlywedge b) \curlywedge c)_p \cap (a \curlywedge (b \curlywedge c))_p$.

E4.1 $((a \curlywedge b) \curlyvee a)(a) = 1$, $((a \curlyvee b) \curlywedge a)(a) = 1$.

E4.2 $((a \curlywedge b) \vee a)(a) = 1$, $((a \vee b) \curlywedge a)(a) = 1$.

E4.3 $((a \curlyvee b) \wedge a)(a) = 1$, $((a \wedge b) \curlyvee a)(a) = 1$.

E5 $b \leq^p a \Leftrightarrow (a \curlyvee b)(a) \geq p$; $b \leq_p a \Leftrightarrow (a \curlywedge b)(b) \geq p$.

Proof. For **E1** note that $a \in [a, a] = a \vee_1 a = (a \curlyvee a)_1$ and so $(a \curlyvee a)(a) \geq 1$. Similarly we can show $(a \curlywedge a)(a) = 1$. **E2** is obvious. To prove **E3.1**, we apply Proposition 4.3.1 using $\widetilde{B} = a \curlyvee b$; in this manner we show that $a \vee_p b \vee_p c = a \vee_p (b \vee_p c) = a \vee_p (b \curlyvee c)_p \subseteq (a \curlyvee (b \curlyvee c))_p$; similarly $a \vee_p b \vee_p c \subseteq ((a \curlyvee b) \curlyvee c)_p$ and we are done. For **E3.2** we apply Proposition 4.3.2 using $\widetilde{B} =$

$a \curlywedge b$. For **E4.1** we have $((a \curlywedge b) \curlyvee a)(a)$= $\vee_{x\in X}([(a \curlywedge b)(x)] \wedge [(x \curlyvee a)(a)])$. Now $(a \curlywedge b)(a \wedge b) = 1$ and $((a \wedge b) \curlyvee a)(a) = 1$. Hence $((a \curlywedge b) \curlyvee a)(a)$=1. Similarly $((a \wedge b) \curlyvee a)(a) = 1$. For **E4.2** note that $a \wedge b \in a \wedge_1 b \Rightarrow (a \curlywedge b)(a \wedge b) = 1$. Also $a = a \vee (a \wedge b)$. Hence $((a \curlywedge b) \vee a)(a)$= $\vee_{u:a\vee u=a}(a \curlywedge b)(u)$= 1. Similarly we can prove $((a \vee b) \curlywedge a)(a) = 1$. **E4.3** is proved in exactly analogous manner. Finally, we prove the first part of **E5** (the second is proved similarly) as follows. First: $b \leq^p a \Rightarrow b \wedge p \leq a \wedge p \Rightarrow a \in a \vee_p b \Rightarrow p \in \{q : a \in a \vee_q b\}$. Hence $(a \curlyvee b)(a) = \vee\{q : a \in a \vee_q b\} \geq p$. Conversely, $(a \curlyvee b)(a) \geq p \Rightarrow a \in (a \curlyvee b)_p = a \vee_p b$. Hence $(a \vee b) \wedge p \leq a \Rightarrow (a \vee b) \wedge p \leq a \wedge p \Rightarrow (a \wedge p) \vee (b \wedge p) \leq a \wedge p \Rightarrow b \wedge p \leq a \wedge p \Rightarrow b \leq^p a$. ■

Proposition 4.6 *For all* $a, b, c \in X$*:* $(a \curlyvee c = b \curlyvee c$ *and* $a \curlywedge c = b \curlywedge c) \Rightarrow a = b$.

Proof. $a \curlyvee c = b \curlyvee c \Rightarrow (\forall p \in X : (a \curlyvee c)_p = (b \curlyvee c)_p) \Rightarrow (\forall p \in X : a \vee_p c$= $b \vee_p c) \Rightarrow a \vee_1 c$= $b \vee_1 c \Rightarrow a \vee c$= $b \vee c$; also $a \curlywedge c = b \curlywedge c \Rightarrow a \wedge c$= $b \wedge c$; and $(a \vee c = b \vee c, a \wedge c = b \wedge c) \Rightarrow a = b$ by distributivity. ■

Definition 4.7 *We say* $\widetilde{M} : X \rightarrow X$ *is a* L-fuzzy interval *of* $(X, \leq)$ *iff* $\forall p \in X$: M_p *is a closed interval of* $(X, \leq)$.

Definition 4.8 *We denote the collection of L-fuzzy intervals of* X *by* $\widetilde{\mathbf{I}}(X)$.

Proposition 4.9 *For all* $a, b \in X$ *,* $a \curlyvee b$ *and* $a \curlywedge b$ *are* L*-fuzzy intervals.*

In Section 2 we have introduced the $\preceq$ order on crisp intervals. We now extend this order to $\widetilde{\mathbf{I}}(X)$, the collection of all L-fuzzy intervals of X.

Definition 4.10 *For all* $\widetilde{A}$*,* $\widetilde{B} \in \widetilde{\mathbf{I}}(X)$*, we write* $\widetilde{A} \precsim \widetilde{B}$ *iff* $\forall p \in X$ *we have* $A_p \preceq B_p$.

Proposition 4.11 $\preceq$ *is an order on* $\widetilde{\mathbf{I}}(X)$ *and* $(\widetilde{\mathbf{I}}(X), \precsim)$ *is a lattice.*

Proof. This follows from the fact that a fuzzy set is specified by its p-cuts. ■

The $\curlyvee$, $\curlywedge$ hyperoperations are isotone in the sense of the following proposition.

Proposition 4.12 *For all* $a, b \in X$ *such that* $a \leq b$ *we have* $a \curlyvee c \precsim b \curlyvee c$ *and* $a \curlywedge c \precsim b \curlywedge c$.

Proof. $a \leq b \Rightarrow a \vee c \leq b \vee c$. Hence for any p we have $(a \vee c) \wedge p \leq (b \vee c) \wedge p$ and $(a \vee c) \vee p' \leq (b \vee c) \vee p'$ which imply $a \vee_p c \preceq b \vee_p c \Rightarrow (a \curlyvee c)_p \preceq (b \curlyvee c)_p$ .Since the above is true for every p, it follows that $a \curlyvee c \precsim b \curlyvee c$. Similarly we show that $a \curlywedge c \precsim b \curlywedge c$. ■

$\curlyvee$, $\curlywedge$ and $'$ are related as seen by the next "deMorgan-like" proposition.

Definition 4.13 *For every $\widetilde{A} \in \mathbf{F}(X)$ define $\widetilde{A}'$ by its p-cuts, i.e. $\widetilde{A}'$ is the (unique) fuzzy set which for every $p \in X$ satisfies $\left(\widetilde{A}'\right)_p = (A_p)' = \{x'\}_{x \in A_p}$.*

Proposition 4.14 *For every $a, b \in X$ we have: (i) $(a \curlyvee b)' = a' \curlywedge b'$, (ii) $(a \curlywedge b)' = a' \curlyvee b'$.*

Proof. Choose any $p \in X$. Then $\left((a \curlyvee b)'\right)_p = \left((a \curlyvee b)_p\right)' = (a \vee_p b)' = a' \wedge_p b' = (a' \curlywedge b')_p$. Since for all $p \in X$ the fuzzy sets $(a \curlyvee b)'$ and $a' \curlywedge b'$ have the same cuts, we have $(a \curlyvee b)' = a' \curlywedge b'$. ■

5 The Crisp Hyperalgebra $(X, \vee_p, \wedge)$ and the L-fuzzy Hyperalgebra $(X, \curlyvee, \wedge)$

Let us now point out that the crisp hyperalgebra $(X, \vee_p, \wedge)$ and the L-fuzzy hyperalgebra $(X, \curlyvee, \wedge)$ are very similar to a *hyperlattice*. Recall that, given a hyperoperation $\bigtriangledown$, the hyperalgebra $(X, \bigtriangledown, \wedge)$ is called a hyperlattice [6] if it satisfies (for every $a, b, c \in X$) the properties listed in Table 1.

$a \in a \bigtriangledown a$, $a = a \wedge a$
$a \bigtriangledown b = b \bigtriangledown a$, $a \wedge b = b \wedge a$
$(a \bigtriangledown b) \bigtriangledown c = a \bigtriangledown (b \bigtriangledown c)$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$a \in (a \bigtriangledown b) \wedge a$
$a \in (a \wedge b) \bigtriangledown a$
$b \leq a \Leftrightarrow a \in a \vee b$

Table 1

Now consider Table 2. The first column lists some properties (satisfied for every $a, b, c, p \in X$) of the crisp hyperalgebra $(X, \vee_p, \wedge)$ (the $\vee_p$ properties are the ones described in Proposition 3.4 and the $\wedge$ properties are standard). The second column lists the corresponding properties of the L-fuzzy hyperalgebra $(X, \curlyvee, \wedge)$ (the $\curlyvee$ properties are the ones described in Proposition 4.5 and the $\wedge$ properties are standard).

$a \in a \vee_p a$, $a = a \wedge a$	$(a \curlyvee a)(a) = 1$, $a = a \wedge a$
$a \vee_p b = b \vee_p a$, $a \wedge b = b \wedge a$	$a \curlyvee b = b \curlyvee a$, $a \wedge b = b \wedge a$
$(a \vee_p b) \vee_p c = a \vee_p (b \vee_p c)$	$a \vee_p b \vee_p c \subseteq (a \curlyvee (b \curlyvee c))_p \cap ((a \curlyvee b) \curlyvee c)_p$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$a \in (a \vee_p b) \wedge a$	$((a \curlyvee b) \wedge a)(a) = 1$
$a \in (a \wedge b) \vee_p a$	$((a \wedge b) \curlyvee a)(a) = 1$
$b \leq^p a \Leftrightarrow a \in a \vee_p b$	$b \leq^p a \Leftrightarrow (a \curlyvee b)(a) \geq p$

Table 2

The correspondence between the properties of $(X, \bigtriangledown, \wedge)$ and $(X, \vee_p, \wedge)$ is obvious. $(X, \vee_p, \wedge)$ is "almost" a hyperlattice, except in that $\leq^p$ in the last row of Table 2 is a *preorder* rather than an order. Similarly, $(X, \curlyvee, \wedge)$ has the L-fuzzy versions of the $(X, \bigtriangledown, \wedge)$ properties and can be considered as an L-fuzzy relative of $(X, \vee_p, \wedge)$. Note however that: $\curlyvee$ has a weak form of associativity (similar to H_v associativity, see [8]) and the ordering property induced by $\curlyvee$ concerns the preorder $\leq^p$ rather than the order $\leq$.

A table similar to Table 2 can be constructed for the properties of the crisp hyperalgebra $(X, \wedge_p, \vee)$ and the L-fuzzy hyperalgebra $(X, \vee, \curlywedge)$. Similar remarks can be made regarding the similarities and differences of $(X, \wedge_p, \vee)$ and $(X, \curlywedge, \vee)$ to a crisp *dual* hyperlattice $(X, \triangle, \vee)$.

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