CONVEXITY IN LATTICES AND AN ISOTONE HYPEROPERATION

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Abstract

Caratheodory has formulated an important theorem regarding the behavior of convex sets in Euclidean spaces [1]. In this paper we discuss a generalization of convexity which is applicable to lattices. This generalization involves a *join* hyperoperation; we show that associativity of this hyperoperation is equivalent to several attractive properties. In particular, we show that, when associativity holds, the join hyperoperation on a finite number of points can be interpreted as the convex hull of these points, and conversely.
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1 Introduction

One of C. Caratheodory's important theorems concerns the properties of *convex* sets in Euclidean spaces [1]. In this paper we present some results connected to a concept of convexity in *lattices*, as seen from the *algebraic hyperstructures* [4] point of view.

The idea of convexity originates in the context of Euclidean spaces R^n: a set $A \subseteq R^n$ is called *convex* iff for all points $a, b \in A$ it is true that the straight line segment $ab \subseteq A$. A derivative concept is that of *betweenness*; we say that c is *between* a and b iff $c \in ab$. The concept of convexity can be generalized by generalization of either "straight line segment" of "betweenness". For example, in *metric* spaces one can define straight line segments in terms of the underlying metric function [3] (*metric betweenness*); in *partially ordered* spaces, one can define betweenness in terms of the underlying order relationship [3] (*order betweenness*). In some cases the two points of view can be combined (for instance, in the case of *metric lattices* [3]).

In this paper we define convexity in terms of an order-based betweenness relation; in particular, given a lattice $(L, \leq)$ we say that x is *between* a, b iff $a \wedge b \leq x \leq a \vee b$. This can also be expressed in the language of algebraic hyperstructures. In this paper we study the hyperoperation $a \cdot b$, which assigns to elements a, b the lattice interval $[a \wedge b, a \vee b]$, i.e. the lattice elements which are between a and b. In [6] we have studied this hyperoperation and have shown that: if $(L, \leq)$ is a *distributive* lattice, then $(L, \leq, \cdot)$ is a *join space* (in the sense of Prenowitz [7, 8]).

2 The Join Hyperoperation

Definition 1 *Given a lattice $(L, \leq)$, the* join *of $a, b \in L$ is denoted by $a \cdot b$ (or, in the interest of brevity, simply by ab) and defined as follows*

$$a \cdot b \doteq [a \wedge b, a \vee b].$$

Theorem 2 *For all $a, b \in L$ we have: (i) $a \cdot a = a$, (ii) $a \cdot b = b \cdot a$, (iii) $\{a, b\} \subseteq a \cdot b$, (iv) $a \cdot L = L$.*

Proof. Omitted in the interest of brevity. ■

Remark. If for all $a, b, c \in L$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (i.e. if the join hyperoperation is *associative*) then $(L, \cdot)$ is a *hypergroup* [4]. Section 4 is devoted to deriving necessary and sufficient conditions for the join to be associative.

3 Convexity

Definition 3 *The set $A \subseteq L$ is called* sl-convex *iff for all $a, b \in A$ we have $a \cdot b \subseteq L$.*

Definition 4 *The set $A \subseteq L$ is called* weakly convex *(or* w-convex*) iff for all $a, b \in A$ such that $a \leq b$, we have $[a, b] \subseteq L$.*

Definition 5 *Given $a_1, a_2, ..., a_N \in L$, we denote the* sl-convex hull *of $a_1, a_2, ..., a_N$ by slCH(a_1, a_2, ..., a_N) and define it by slCH(a_1, a_2, ..., a_N) $\doteq \cap_{A \in \mathbf{Q}} A$, where $\mathbf{Q} \doteq \{A : a_1,\ a_2,\ ...,\ a_N \in A$ and A is sl-convex$\}$.*

Remark. Our definition of "sl-convex sets" coincides with Birkhoff's definition of "convex sublattices" [2]. The next lemma states that every sl-convex set is w-convex; it is easy to see that the converse does not hold. The remaining lemmas describe some useful and easily provable properties of sl-convex sets.

Lemma 6 *If $A \subseteq L$ is sl-convex then it is w-convex.*

Proof. Omitted in the interest of brevity. ■

Lemma 7 *For all $a_1, a_2, ..., a_N \in L$ we have: (i) a_1, a_2, ..., a_N $\in$slCH(a_1, a_2, ..., a_N) and (ii) slCH(a_1, a_2, ..., a_N) is sl-convex.*

Proof. Omitted in the interest of brevity. ■

Lemma 8 *For all $a, b \in L$, the set $[a \wedge b, a \vee b]$ is sl-convex.*

Proof. Omitted in the interest of brevity. ■

Lemma 9 *For all $A \subseteq L$ we have: A is sl-convex $\Leftrightarrow A \cdot A = A$.*

Proof. Omitted in the interest of brevity. ■

4 Associativity and Equivalent Properties

We now investigate conditions necessary and sufficient for the join hyperoperation to be associative.

Lemma 10 *If $p \cdot [a, b] = [c, d]$, then $c = p \wedge a$, $d = p \vee b$.*

Proof. Omitted in the interest of brevity. ■

Lemma 11 *The join hyperoperation is associative iff the following property holds:* $(\forall p, q, r \in L$ *such that* $p \leq q$ *there exist* $x, y \in L$ *such that* $r \cdot [p, q] = [x, y])$.

Proof. (i) Suppose that for all $p, q, r \in L$ such that $p \leq q$ there exist $x, y \in L$ such that $r \cdot [p, q] = [x, y]$. Then, for all $a, b, c \in L$ there will exist $u, w \in L$ such that we will have $(a \cdot b) \cdot c = [a \wedge b, a \vee b] \cdot c = [u, w]$ and then, by Lemma 10, $(a \cdot b) \cdot c = [a \wedge b \wedge c, a \vee b \vee c]$. Similarly we get that $a \cdot (b \cdot c) = [a \wedge b \wedge c, a \vee b \vee c]$. Hence $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, i.e. the join hyperoperation is associative.

(ii) Now suppose that for all $a, b, c \in L$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Take any $p, q, r \in L$ such that $p \leq q$. There are several possibilities regarding the placement of r relative to p, q; these are summarized in the following table.

(2)	(6)	(1)	Impos.	(5)	(4)	Impos.	Impos.	(3)
$p \leq r$	$p \leq r$	$p \leq r$	$p\|\|r$	$p\|\|r$	$p\|\|r$	$r \leq p$	$r \leq p$	$r \leq p$
$q \leq r$	$q\|\|r$	$r \leq q$	$q \leq r$	$q\|\|r$	$r \leq q$	$q \leq r$	$q\|\|r$	$r \leq q$

It can be seen that all combinations are exhausted in the above table. It can also be seen that certain combinations are impossible, namely the ones in the fourth, seventh and eighth column (for instance, regarding the fourth column, $q \leq r$ and $p \leq q$ implies that $p \leq r$, which is contrary to $p||r$). The remaining six columns cover all viable placements of p, q and r; the numbering of these cases has been chosen so that they can be examined in an appropriate sequence (as will now become obvious) to establish that in every case $r \cdot [p, q]$ is an interval.

Case 1: $p \leq r \leq q$**.** In this case, it is easy to establish that $r \cdot [p, q] = [p, q]$; hence $r \cdot [p, q]$ is an interval.

Case 2: $p \leq q \leq r$**.** Take any $x \in [p, r]$, then $x \in r \cdot p \subseteq r \cdot (p \cdot q)$. So $[p, r] \subseteq r \cdot [p, q]$. On the other hand, $z \in r \cdot [p, q] \Rightarrow \exists y \in p \cdot q$, s.t. $z \in r \cdot y$. Hence $r \wedge p \leq r \wedge y \leq z \leq r \vee y \leq r \vee q$ which implies $p \leq z \leq r$ and so $z \in [p, r]$. So we finally have $r \cdot [p, q] = [p, r]$; hence $r \cdot [p, q]$ is an interval.

Case 3: $r \leq p \leq q$**.** This case is treated similarly to Case 2 and it is easily proved that $r \cdot [p, q] = [r, q]$; hence $r \cdot [p, q]$ is an interval.

Case 4: $r||p$; $r, p \leq q$**.** In this case, by associativity, we have $r \cdot [p, q] = r \cdot (p \cdot q) = (r \cdot p) \cdot q = [r \wedge p, r \vee p]q$. But $r, p \leq q$ which implies $r \vee p \leq q$; also $r \wedge p \leq p \leq q$. Seting $a = r \wedge p$, $b = r \vee p$, $c = q$, we have $r \cdot [p, q] =$

$[a,b] \cdot c$. Since $a \leq b \leq c$, from Case 2 we know that $[a,b] \cdot c = c \cdot [a,b]$ is an interval; hence $r \cdot [p,q]$ is an interval.
Case 5: $r||p$; $r||q$. By associativity, we have $r \cdot [p,q] = r \cdot (p \cdot q) = (r \cdot q) \cdot p$ $= [r \wedge q, r \vee q] \cdot p$. We also have $p \leq q \leq r \vee q$. Set $a = p$, $b = r \wedge q$ and $c = r \vee q$; consider the following subcases.

a. $p \leq r \wedge q$. This is impossible, since then $p \leq r \wedge q \leq r$, but we assumed $r||p$.

b. $r \wedge q \leq p$. Then $b \leq a \leq c$ and we conclude, using Case 1, that $[r \wedge q, r \vee q] \cdot p$ is an interval.

c. $p||r \wedge q$. Then $a||b$ and $a \leq c$ and we conclude, using Case 4, that $[r \wedge q, r \vee q] \cdot p$ is an interval.

In every one of the above subcases we conclude that $r \cdot [p,q] = [r \wedge q, r \vee q] \cdot p$ is an interval.
Case 6: $r||q$; $p \leq r, q$. In this case we prove, similarly to Case 4, that $r \cdot [p,q]$ is an interval.

Hence, by examining all possible relative placements of r relative to p, q, we see that associativity of join implies that $r \cdot [p,q]$ is an interval. ■

Lemma 12 *The join hyperoperation is associative iff the following property holds:* ($\forall A \subseteq L, \forall p \in L$ *we have*: A *is sl-convex*$\Rightarrow p \cdot A$ *is sl-convex*).

Proof. (i) Assume that for all $p \in L$ and all sl-convex $A \subseteq L$ the set $p \cdot A$ is also sl-convex. Take any $a, b, c \in L$. Then $a \cdot (b \cdot c) = a \cdot [b \wedge c, b \vee c]$ which is sl-convex by the assumption (since, by Lemma 8, $[b \wedge c, b \vee c]$ is sl-convex). Now, by Lemma 6, $a \cdot [b \wedge c, b \vee c]$ is also w-convex; it is easy to see that $a \wedge b \wedge c$, $a \vee b \vee c \in a \cdot (b \cdot c)$ and so, by w-convexity we have $[a \wedge b \wedge c, a \vee b \vee c] \subseteq a \cdot (b \cdot c)$; furthermore, it is easy to check that $a \cdot (b \cdot c)$ has minimum element $a \wedge b \wedge c$ and maximum element $a \vee b \vee c$, hence $a \cdot (b \cdot c) \subseteq [a \wedge b \wedge c, a \vee b \vee c]$. So $a \cdot (b \cdot c) = [a \wedge b \wedge c, a \vee b \vee c]$. By exactly the same argument we obtain $(a \cdot b) \cdot c = [a \wedge b \wedge c, a \vee b \vee c]$ and so $a \cdot (b \cdot c)$ $= (a \cdot b) \cdot c$.

(ii) On the other hand, assume that the join hyperoperation is associative. Take any $p \in L$ and any sl-convex $A \subseteq L$; also take any $x, y \in p \cdot A$; then there exist $q, r \in A$ such that $x \in p \cdot q$, $y \in p \cdot r$. So $x \cdot y \subseteq (p \cdot q) \cdot (p \cdot r)$ $= p \cdot q \cdot r \subseteq p \cdot A$ (by sl-convexity). Hence $p \cdot A$ is sl-convex. ■

Lemma 13 *If the join hyperoperation is associative, then for all $a, b, c \in L$ the set $(a \cdot b) \cdot c$ is sl-convex.*

Proof. Since the join hyperoperation is associative, by Lemma 11 we have $(a \cdot b) \cdot c = [a \wedge b \wedge c, a \vee b \vee c]$, which is an interval and hence sl-convex by Lemma 8. ■

We are now ready to present our main theorem.

Theorem 14 *In a lattice $(L, \leq)$ the following conditions are equivalent.*

(i) The join hyperoperation is associative.

(ii) For all $p, q, r \in L$ such that $p \leq q$, there exist $x, y \in L$ such that $r \cdot [p, q] = [x, y]$.

(iii) $\forall p \in L$ and $A \subseteq L$, A sl-convex, the set $r \cdot A$ is sl-convex.

(iv) $(L, \cdot)$ is a hypergroup.

(v) For every N we have $(...(a_1 \cdot a_2) \cdot ... \cdot a_{N-1}) \cdot a_N =$slCH$(a_1, a_2, ..., a_N)$.

Proof. (i) ⇔(ii) by Lemma 11 and (i)⇔(iii) by Lemma 12; (i) ⇔(iv) by the properties of the join and by the definition of a hypergroup [4]. Hence (i)⇔(ii)⇔(iii)⇔(iv) and it remains to show that (i)⇔(v).

It is easy to show (v)⇒(i). Take any $a, b, c \in L$; then by (v) we have: $(a \cdot b) \cdot c=$ slCH$(a, b, c) =$ slCH$(b, c, a) = (b \cdot c) \cdot a = a \cdot (b \cdot c)$.

So it remains to show (i)⇒(v). First, if (i) holds then $(...(a_1 \cdot a_2) \cdot ... \cdot a_{N-1}) \cdot a_N$ can be written as $a_1 \cdot a_2 \cdot ... \cdot a_N$ and (using induction and the fact (i)⇒(ii)) it can be shown that for any N we have $a_1 \cdot a_2 \cdot ... \cdot a_N = [a_1 \wedge a_2 \wedge ... \wedge a_N \ , a_1 \vee a_2 \vee ... \vee a_N]$; the latter interval is sl-convex and contains $a_1, a_2, ..., a_N$, so slCH$(a_1, a_2, ..., a_N) \subseteq [a_1 \wedge a_2 \wedge ... \wedge a_N, a_1 \vee a_2 \vee ... \vee a_N]$. On the other hand, $a_1 \wedge a_2 \wedge ... \wedge a_N, a_1 \vee a_2 \vee ... \vee a_N \in$ slCH$(a_1, a_2, ..., a_N)$, which is sl-convex, so it follows that $[a_1 \wedge a_2 \wedge ... \wedge a_N, a_1 \vee a_2 \vee ... \vee a_N] \subseteq$slCH$(a_1, a_2, ..., a_N)$. Hence $[a_1 \wedge a_2 \wedge ... \wedge a_N, a_1 \vee a_2 \vee ... \vee a_N] =$ slCH$(a_1, a_2, ..., a_N)$ and we have shown that (i)⇒(v); the proof of the theorem is complete. ■

5 The Join Hyperoperation is Isotone

The order relation $\leq$ (on elements of L), produces a new order $\precsim$ on the set $I(L)$, defined to be the set of *intervals* of elements of L. The new order is compatible with the join operation; in this sense the join hyperoperation is *isotone* with respect to $\leq$.

Definition 15 *We define the $\precsim$ relation on pairs (A, B) where $A, B \in \wp(L)$, as follows*

$$A \precsim B \Leftrightarrow \begin{cases} \forall a_1 \in A \quad \exists b_1 \in B: \quad a_1 \leq b_1 \\ \forall b_2 \in B \quad \exists a_2 \in A: \quad a_2 \leq b_2. \end{cases}$$

Lemma 16 *(i) $\precsim$ is a pre-order on $\wp(L)$; (ii) $\precsim$ is an order on $I(L)$.*

Proof. The proof can be found in [6]. ∎

Lemma 17 *$(I(L), \precsim)$ is a lattice; in particular, for any $A = [a_1,\ a_2]$ and $B = [\ b_1,\ b_2]$, we have*

$$\inf(A, B) = [a_1 \wedge b_1, a_2 \wedge b_2], \qquad \sup(A, B) = [a_1 \vee b_1, a_2 \vee b_2]$$

Proof. The proof can be found in [6]. ∎

We are now ready to state and prove the compatibility of join with the $\precsim$ order.

Definition 18 *$(L, \leq, \cdot)$ is called a* strictly ordered hypergroup *iff:*

(i) $(L, \leq)$ is a lattice,

(ii) $(L, \cdot)$ is a hypergroup,

(iii) for all $x, y \in L$ we have that $x \cdot y$ is an interval,

(iv) for all $a, x, y \in L$ such that $x \leq y$ we have $a \cdot x \precsim a \cdot y$.

The above definition follows [9]. We now have that $(L, \leq, \cdot)$ is a strictly ordered hypergroup.

Theorem 19 *For all $a, b, x, y \in L$ we have: (i) $x \leq y \Rightarrow a \cdot x \precsim a \cdot y$, (ii) $a \leq b$ and $x \leq y \Rightarrow a \cdot x \precsim b \cdot y$.*

Proof. The proof can be found in [6]. ∎

Conclusion 20 *$(L, \leq, \cdot)$ is a strictly ordered hypergroup.*

6 The Case of Distributive Lattice

In case L is *distributive*, then the join hyperoperation has additional properties, as we have shown in [6]. In particular, in [6] we have shown that distributivity implies associativity; hence all the properties (i)-(v) of Theorem 14 hold true in a distributive lattice. On the other hand, distributivity *does not* imply associativity; this can be seen by a counterexample. Consider the lattice $\mathbf{N}_5$, depicted in Figure 1.

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Figure 1

Associativity of the join hyperoperation holds in $\mathbf{N}_5$, as can be checked by exhaustive computation. However, clearly $\mathbf{N}_5$ is not distributive (in fact it is not even modular).

For completeness, we list two theorems proved in [6], which concern the distribution: (a) of $\vee, \wedge$ on join; (b) of join on $\vee, \wedge$.

Theorem 21 *If L is distributive, then for all $a, b, c \in L$ we have: $(a \cdot b) \vee c = (a \vee c) \cdot (b \vee c)$ and $(a \cdot b) \wedge c = (a \wedge c) \cdot (b \wedge c)$.*

Proof. The proof can be found in [6]. ■

Theorem 22 *If L is distributive, then for all $a, b, c \in L$ we have: $a \cdot (b \vee c) = a \cdot b \vee a \cdot c$ and $a \cdot (b \wedge c) = a \cdot b \wedge a \cdot c$.*

Proof. The proof can be found in [6]. ■

Remark. Following [7, 8] we can use the join hyperoperation to define an associated *extension* hyperoperation as follows: the extension of a through b is denoted by a/b and defined by: $a/b \doteq \{x : a \in b \cdot x\}$. In [6] we have shown that in a distributive lattice, the join and extension hyperoperations satisfy the *extension property: (*for all $a, b, c, d \in L$ we have: $a/b \cap c/d \neq \emptyset \Rightarrow a \cdot d \cap b \cdot c \neq \emptyset$) Hence, in a distributive lattice L, $(L, \cdot)$ is a *join space*, in the sense of [7, 8].

7 Discussion

Several authors have studied the ternary "betweennes" relation $B(a, x, b)$ (x is between a and b) in various contexts. For example, one can define betweenness in

(A) vector spaces: $B(a, x, b)$ iff $x = \lambda a + (1 - \lambda)b$, (where $0 \leq \lambda \leq 1$);

(B) metric spaces: $B(a, x, b)$ iff $d(a, x) + d(x, b) = d(a, b)$ (where $d(.,.)$ is the distance function);

(C) lattices: $B(a, x, b)$ iff $a \wedge b \leq x \leq a \vee b$;

(D) lattices (an alternative defintion): $B(a, x, b)$ iff $(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b)$.

In all of the above cases a join hyperoperation can be defined by $a \circ b \doteq \{x : B(a, x, b) \text{ is true }\}$ (such operations appear, for instance, in [4, 5, 6, 7, 8]) and convexity can be defined as follows: A is convex iff for all $a, b \in A$ we have $a \circ b \subseteq A$. It is a natural question whether results analogous to our Theorem 14 hold true. In certain cases analogs of Theorem 14 hold true "automatically", in the sense that some underlying property of the space ensures the validity of conditions (i)-(v). For example, in Euclidean spaces the vector join of (A) is associative and it is also true that the join of a point with a convex set is a convex set; in distributive lattices, the lattice join of (C) is associative and the join of a lattice element with a lattice interval is a lattice interval.

An interesting research direction, then, is to obtain analogs of Theorem 14 for various types of join hyperoperations (and the respective types of convexity). If a theorem analogous to Theorem 14 holds for a particular join hyperoperation, then we conclude that associativity is equivalent to an interpretation of the join $a_1 \circ a_2 \circ ... \circ a_N$ as the convex hull of $a_1, a_2, ..., a_N$ (using the appropriate definition of convex hull).

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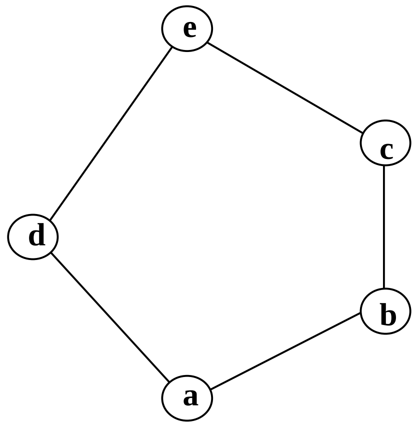

Figure 1