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The Construction of Fuzzy-valued t-norms and t-conorms

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Summary. In this paper we present a method to construct *fuzzy-valued* t-norms and t-conorms, i.e. operations which map pairs of lattice elements to fuzzy sets, and are commutative, associative and monotone. The fuzzy-valued t-norm and t-conorm are synthesized from their α-cuts which are obtained from families of *multi-valued* t-norms and t-conorms.

18.1 Introduction

We are interested in generalizations of the concepts of t-norm and t-conorm. In a companion chapter in this volume [14] we have presented a family of *hyper-t-norms* $\wedge_q$ and a family of *hyper-t-conorms* $\vee_p$. The prefix *hyper* is used to indicate *multi-valued* operations, also known as *hyperoperations* (see [5, 14]), i.e. operations which map pairs of elements to *sets* of elements. See [14] for the construction of hyperoperations which generalize t-norms and t-conorms. These hyperoperations are *crisp*, i.e. their output is a crisp set. A natural generalization is to consider *fuzzy* hyperoperations, i.e. operations which map elements to *fuzzy* sets.

Hence in this chapter we will present a procedure to construct fuzzy-valued t-norms and t-conorms.

Relatively little work in this direction has appeared in the literature. A pioneering paper is [4] which introduces fuzzy hypoperations which induce *fuzzy hypergroups*. A fuzzy hypergroup, different from the one used by Corsini, is [13] and a version of fuzzy min and fuzzy max operations appears in [11, 12][1].

As explained also in [14], our motivation to study multi-valued and fuzzy-valued connectives is that, while fuzzy logic is a calculus of uncertain reasoning, not much attention has been paid to the case where uncertainty is

[1] Let us also note that t-norms and t-conorms appropriate for the lattice of fuzzy-valued truth values are studied in [19, 20, 21]. But all of these works concern single- not multi-valued operations. (Also, t-norms and t-conorms for the lattice of real intervals are studied in [1, 2, 3]).

associated with the actual operation of the logical connectives. This is exactly the situation we want to capture with the proposed fuzzy-valued t-norm and t-conorm.

There is some literature on the use of fuzzy *uni*-valued operations, in the context of type-2 fuzzy sets [9, 17] which is related to the present chapter.

The plan of this paper is as follows. In Section 18.2 we present some preliminary concepts. In Section 18.3 we construct the basic objects of our study, namely the fuzzy-valued t-norm $\curlywedge$ and t-conorm $\curlyvee$. In Section 18.4 we study the fuzzy-valued *S-implication* obtained from $\curlyvee$. In Section 18.5 we summarize our results.

18.2 Preliminaries

In this section we review some fundamental concepts which will be used in the main part of the paper.

We will work with a *generalized deMorgan lattice* $(X, \leq, \vee, \wedge, ')$ (for the corresponding definition see [14] in this volume). We will study *lattice-valued fuzzy sets*, also termed *L-fuzzy sets* or, for the sake of brevity, simply *fuzzy sets*. We take these to be identical to their membership functions and we consider the special case where both the domain and range of the membership function is $(X, \leq, \vee, \wedge, ')$. In short, we define fuzzy sets as follows.

Definition 18.1 *An* L-fuzzy set *is a function $M : X \to X$.*

The collection of all L-fuzzy sets is denoted by $\mathbf{F}(X)$.

The *α-cut* of the fuzzy set M is denoted by M_α and defined by $M_\alpha \doteq \{x : M(x) \geq \alpha\}$. We will use the following properties of α-cuts (see [18]).

Proposition 18.1 *Given a fuzzy set $M \in \mathbf{F}(X)$ we have*

1. $M_0 = X$.
2. *For all $p, q \in X$ we have: $p \leq q \Rightarrow M_q \subseteq M_p$.*
3. *For all $p, q \in X$ we have: $M_q \cap M_p = M_{p \vee q}$, for all $P \subseteq X$ we have: $\cap_{p\in P} M_p = M_{\vee P}$.*

Proposition 18.2 *Suppose a family of sets $\left\{\widetilde{M}_p\right\}_{p\in X}$ is given which satisfies*

1. $\widetilde{M}_0 = X$.
2. *For all $p, q \in X$ we have: $p \leq q \Rightarrow \widetilde{M}_q \subseteq \widetilde{M}_p$.*
3. *For all $p, q \in X$ we have: $\widetilde{M}_q \cap \widetilde{M}_p = \widetilde{M}_{p\vee q}$, for all $P \subseteq X$ we have: $\cap_{p\in P} \widetilde{M}_p = \widetilde{M}_{\vee P}$.*

Then, defining for every $x \in X$

$$M(x) = \boldsymbol{sup}\left\{p : x \in \widetilde{M}_p\right\},$$

we obtain the fuzzy set $M \in \mathbf{F}(X)$ which, for every $p \in X$, satisfies $M_p = \widetilde{M}_p$.

From the above propositions we see that a fuzzy set M is in a 1-to-1 correspondence with its α-cuts $\{M_p\}_{p\in X}$. A special class of fuzzy sets are the fuzzy intervals [10].

Definition 18.2 *A fuzzy set is called a* fuzzy interval *iff all its α-cuts are closed intervals. We denote the set of all fuzzy intervals of X by $\widetilde{\mathbf{I}}(X)$.*

In the rest of the paper we will deal with crisp and fuzzy *hyper*operations. Crisp hyperoperations map pairs of elements to crisp sets; fuzzy hyperoperations map pairs of elements to fuzzy sets. We will use the following.

Notation 18.1 *Let $*$ be a fuzzy operation and $a, b, x \in X$. We will denote the membership grade of x in the fuzzy set $a * b$ by $(a * b)(x)$.*

18.3 Fuzzy Valued t-norm and t-conorm

Our goal in the current section is to define *fuzzy-valued operations* which are analogous to t-norms and t-conorms. Recall the definition of the *hyperoperations* $\wedge_q$, $\vee_p$ (see [14] in this volume).

$$a \wedge_q b = [a \wedge b \wedge q, a \wedge b], \quad a \vee_p b = [a \vee b, a \vee b \vee p'].$$

The fuzzy-valued operations will be defined in terms of the *crisp hyperoperations* $\wedge_q$ and $\vee_p$. To this end, let us first prove the "α-cut properties" for $\wedge_q$ and $\vee_p$.

Proposition 18.3 *Take any $q_1, q_2 \in X$ and $R \subseteq X$. Then, for every $a, b, c \in X$ we have:*

1. $a \wedge_0 b = [0, a \wedge b]\,; a \wedge_1 b = [a \wedge b, a \wedge b]\,;$
2. $q_1 \leq q_2 \Rightarrow a \wedge_{q_2} b \subseteq a \wedge_{q_1} b$;
3. $(a \wedge_{q_1} b) \cap (a \wedge_{q2} b) = a \wedge_{q_1 \vee q_2} b$, $\cap_{q\in R} (a \wedge_q b) = a \wedge_{\vee R} b$.

Proof. For 1 we have: $a \wedge_0 b = [a \wedge b \wedge 0, a \wedge b] = [0, a \wedge b]$, $a \wedge_1 b = [a \wedge b \wedge 1, a \wedge b] = [a \wedge b, a \wedge b]$.

For 2 we have $a \wedge_{q_2} b = [a \wedge b \wedge q_2, a \wedge b] \subseteq [a \wedge b \wedge q_1, a \wedge b] = a \wedge_{q_1} b$ since $a \wedge b \wedge q_1 \leq a \wedge b \wedge q_2$.

For the second (more general) part of 3:

$$\begin{aligned}\cap_{q\in R} (a \wedge_q b) &= \vee_{q\in R} [a \wedge b \wedge q, a \wedge b] = [\vee_{q\in R} (a \wedge b \wedge q), a \wedge b] \\ &= [a \wedge b \wedge (\vee_{q\in R} q), a \wedge b] = [a \wedge b \wedge (\vee R), a \wedge b] = a \wedge_{\vee R} b. \square\end{aligned}$$

Proposition 18.4 *For $p_1, p_2 \in X$ and $\overline{P} \subseteq X$ and $a, b, c \in X$ we have:*

1. $a \vee_1 b = [0, a \vee b]\,; a \vee_1 b = [a \vee b, a \vee b]\,;$
2. $p_1 \leq p_2 \Rightarrow a \vee_{p_2} b \subseteq a \vee_{p_1} b$;
3. $(a \vee_{p_1} b) \cap (a \vee_{p_2} b) = a \vee_{p_1 \vee q_2} b$; $\cap_{p\in R} (a \vee_p b) = a \vee_{\vee R} b$.

Proof. The proof is similar to that of Proposition 18.3 and hence is omitted. □

We now construct the L-fuzzy hyperoperations $\curlyvee$ and $\curlywedge$. Following a standard approach, we will construct $\curlyvee$ and $\curlywedge$ through their α-cuts, which will be the $\vee_p$ and $\wedge_p$ families studied previously. First, for compatibility with the usual interpretation of α-cuts, we redefine for every a, b the symbols

$$a \wedge_0 b = [0,1] \,, \qquad a \vee_0 b = [0,1] \,.$$

Now, for every $a, b \in X$ we can define an L-fuzzy valued hyperoperation.

Definition 18.3 *For all $a, b \in X$*

1. *We define the L-fuzzy set $a \curlyvee b$ by defining for every $x \in X$: $(a \curlyvee b)(x) \doteq \vee\{q : x \in a \vee_q b\}$;*
2. *We define the L-fuzzy set $a \curlywedge b$ by defining for every $x \in X$: $(a \curlywedge b)(x) \doteq \vee\{q : x \in a \wedge_q b\}$;*

Proposition 18.5 *For all $a, b \in X$ and $p \in X$ we have $(a \curlyvee b)_p = a \vee_p b$, $(a \curlywedge b)_p = a \wedge_p b$.*

Proof. It follows from the construction of $a \curlyvee b$, $a \curlywedge b$ in Definition 18.3 (for details see [18]). □

Proposition 18.6 *For all $a, b \in X$, the L-fuzzy sets $a \curlyvee b$ and $a \curlywedge b$ are L-fuzzy intervals.*

Proof. As already mentioned (Proposition 18.5), for any $p \in X$ the p-cut of $a \curlyvee b$ is $(a \curlyvee b)_p = a \vee_p b$ and by construction $a \vee_p b$ is an interval. The same is true for $a \curlywedge b$. □

Before proceeding, we will need an auxiliary definition.

Definition 18.4 *Let $\circ : X \times X \to \mathbf{F}(X)$ be an L-*fuzzy* hyperoperation.*

1. *For all $a \in X$, $B \in \mathbf{F}(X)$ we define the L-*fuzzy* set $a \circ B$ by*

$$(a \circ B)(x) \doteq \vee \left([(a \circ b)(x)] \wedge B(b) \right) .$$

2. *For all $A, B \in \mathbf{F}(X)$ we define the L-*fuzzy* set $A \circ B$ by*

$$(A \circ B)(x) \doteq \vee \left([(a \circ b)(x)] \wedge A(a) \wedge B(b) \right) .$$

Remark 18.1. The above definition also applies to *crisp* operations if we take the view that a crisp operation gives as output not an element but an *indicator function*. An example should make this clear. Take the operation $\wedge$. For any $a, b \in X$ we can write that

$$(a \wedge b)(x) = \begin{cases} 1 \text{ iff } x = \texttt{inf}(a,b) \\ 0 \text{ else.} \end{cases}$$

The same approach can be used for crisp *hyper*operations. In other words, we take crisp sets to be a special case of fuzzy sets and identify every set (crisp or fuzzy) with its membership function.

Remark 18.2. The above construction of the L-fuzzy valued hyperoperations is similar to the construction of fuzzy interval numbers (FINs) used in [7, 8].

Proposition 18.7 *For all $a, p \in X$, for all $A, B \in \mathbf{F}(X)$ we have*

1. $a \vee_p B_p \subseteq (a \curlyvee B)_p$; $A_p \vee_p B_p \subseteq (A \curlyvee B)_p$.
2. $a \wedge_p B_p \subseteq (a \curlywedge B)_p$; $A_p \wedge_p B_p \subseteq (A \curlywedge B)_p$.

Proof. We only prove the first part of 1, since the remaining items are proved similarly. Choose any $x \in a \vee_p B_p$. If $p = 0$, then, by definition, $(a \curlyvee B)_p = X$ and obviously $a \vee_0 B_0 \subseteq X$. If $p > 0$ then there is some $b \in B_p$ and so $B(b) \geq p$. Also $x \in a \vee_p b = (a \curlyvee b)_p$ implies that $(a \curlyvee b)(x) \geq p$. Hence

$$(a \curlyvee B)(x) = \vee_{u \in X}([(a \curlyvee u)(x)] \wedge [B(u)]) \geq [(a \curlyvee b)(x)] \wedge [B(b)] \geq p$$

which implies that $x \in (a \curlyvee B)_p$. We have thus $a \vee_p B_p \subseteq (a \curlyvee B)_p$. □

Let us now prove some simple properties of $\curlyvee, \curlywedge$.

Proposition 18.8 *For all $a, b, c \in X$ the following hold.*

1. $(1 \curlyvee a)(1) = 1$, $(1 \curlywedge a)(a) = 1$, $(0 \curlyvee a)(a) = 1$, $(0 \curlywedge a)(0) = 1$.
2. $a \curlyvee b = b \curlyvee a$, $a \curlywedge b = b \curlywedge a$.
3. $a \vee_p b \vee_p c \subseteq (a \curlyvee (b \curlyvee c))_p \cap ((a \curlyvee b) \curlyvee c)_p$, $a \wedge_p b \wedge_p c \subseteq ((a \curlywedge b) \curlywedge c) \cap (a \curlywedge (b \curlywedge c))$.
4. $(a \curlyvee a)(a) = 1$, $(a \curlywedge a)(a) = 1$.
5. $(a \curlywedge b)(a \wedge b) = 1$, $(a \curlyvee b)(a \vee b) = 1$.
6. $[(a \curlywedge b) \vee a](a) = 1$,$[(a \curlyvee b) \wedge a](a) = 1$.
7. $((a \curlywedge b) \curlyvee a)(a) = 1$, $((a \curlyvee b) \curlywedge a)(a) = 1$.

Proof. 1. We have: $(1 \curlyvee a)(1) \doteq \vee\{q : 1 \in 1 \vee_q a\}$. Since $1 \in 1 \vee_1 a = [(1 \vee a), (1 \vee a) \vee 1']$, it follows that $1 \in \{q : 1 \in 1 \vee_q a\}$ and so $(1 \curlyvee a)(1) = 1$. The remaining parts of 1 are proved similarly.
2. This is obvious.
3. We apply Proposition 18.7.1 using $B = b \curlyvee c$; in this manner we show that $a \vee_p b \vee_p c = a \vee_p (b \vee_p c) = a \vee_p (b \curlyvee c)_p \subseteq (a \curlyvee (b \curlyvee c))_p$. Similarly $a \vee_p b \vee_p c \subseteq ((a \curlyvee b) \curlyvee c)_p$ and we are done.
4. Note that $a \in [a, a] = a \vee_1 a = (a \curlyvee a)_1$ and so $(a \curlyvee a)(a) \geq 1$. Similarly we show $(a \curlywedge a)(a) = 1$.
5. Note that $(a \curlywedge b)(a \wedge b) = \vee\{q : a \wedge b \in a \wedge_q b\} = 1$ (since $a \wedge b \in a \wedge_1 b$); the case $(a \curlyvee b)(a \vee b) = 1$ is proved similarly.

6. From 5 we have $(a \curlywedge b)(a \wedge b) = 1 > 0$. Also $[a \vee (a \wedge b)](a) = 1$. Now

$$[(a \curlywedge b) \vee a](a) = \vee_{z \in X} [(a \curlywedge b)(z)] \wedge [(z \vee a)(a)] \geq \tag{18.1}$$
$$[(a \curlywedge b)(a \wedge b)] \wedge [((a \wedge b) \vee a)(a)] = 1 \wedge 1 = 1 \tag{18.2}$$

   Similarly we can prove $[(a \curlyvee b) \wedge a](a) = 1$.
7. We prove the first part as follows. We already have $(a \curlywedge b)(a \wedge b) = 1$ and

$$[((a \wedge b) \curlyvee a)(a)] = \vee_{z \in X} [(z \curlyvee a)(a)] \wedge [(a \wedge b)(z)] \geq \tag{18.3}$$
$$[((a \wedge b) \curlyvee a)(a)] \wedge [(a \wedge b)(a \wedge b)] = 1 \wedge 1 = 1; \tag{18.4}$$

   hence

$$[(a \curlywedge b) \curlyvee a](a) = \vee [(a \curlywedge b)(z)] \wedge [(z \curlyvee a)(a)] \geq \tag{18.5}$$
$$[(a \curlywedge b)(a \wedge b)] \wedge [((a \wedge b) \curlyvee a)(a)] = 1 \wedge 1 = 1. \tag{18.6}$$

   The second part is proved similarly.□

Proposition 18.9 *For all $a, b, c, p \in X$ we have*

1. $a \vee_p (b \wedge_p c) \subseteq (a \curlyvee (b \curlywedge c))_p \cap ((a \curlyvee b) \curlywedge (a \curlyvee c))_p$.
2. $a \wedge_p (b \vee_p c) \subseteq (a \curlywedge (b \curlyvee c))_p \cap ((a \curlywedge b) \curlyvee (a \curlywedge c))_p$.

Proof. From Proposition 18.7.1 we have

$$a \vee_p (b \wedge_p c) \subseteq (a \curlyvee (b \curlywedge c))_p. \tag{18.7}$$

From Proposition 28 of [14], with $p = q$, we have

$$a \vee_p (b \wedge_p c) \subseteq (a \vee_p b) \wedge_p (a \vee_p c) = (a \curlyvee b)_p \wedge_p (a \curlyvee c)_p \subseteq ((a \curlyvee b) \curlywedge (a \curlyvee c))_p. \tag{18.8}$$

From (18.7) and (18.8) follows the first part of the proposition; the second part can be proved similarly. □

Remark 18.3. Part 3 of Proposition 18.8 shows that the associativity of $\curlyvee, \curlywedge$ holds in a *limited* sense. Proposition 18.9 shows a *limited* form of distributivity. Both of these limitations can be traced to the fact that, in Proposition 18.7, we do not have equality of sets but set inclusion.

Proposition 18.10 *For all $a, b, c \in X$ we have*

$$\left.\begin{array}{l} a \curlyvee c = b \curlyvee c \\ a \curlywedge c = b \curlywedge c \end{array}\right\} \Rightarrow a = b.$$

Proof. We have

$$a \curlyvee c = b \curlyvee c \Rightarrow \left(\forall p \in X : (a \curlyvee c)_p = (b \curlyvee c)_p\right) \Rightarrow \tag{18.9}$$
$$(\forall p \in X : a \vee_p c = b \vee_p c) \Rightarrow a \vee_1 c = b \vee_1 c \Rightarrow a \vee c = b \vee c; \tag{18.10}$$

similarly $a \curlywedge c = b \curlywedge c \Rightarrow a \wedge c = b \wedge c$; finally, as is well known, in a distributive lattice we have

$$\left.\begin{array}{l} a \vee c = b \vee c \\ a \wedge c = b \wedge c \end{array}\right\} \Rightarrow a = b. \qquad \square$$

In [14] we have introduced an order $\leqslant_2$ on crisp intervals. We now extend this order to $\widetilde{\mathbf{I}}(X)$, the collection of all L-fuzzy intervals of X.

Definition 18.5 *For every A, $B \in \widetilde{\mathbf{I}}(X)$, we write $A \preceq B$ iff for all $p \in X$ we have $A_p \leqslant_2 B_p$.*

Proposition 18.11 *$\preceq$ is an order on $\widetilde{\mathbf{I}}(X)$ and $(\widetilde{\mathbf{I}}(X), \preceq)$ is a lattice.*

Proof. This follows from the fact that a fuzzy set is uniquely specified by its α-cuts. $\square$

The $\curlyvee$, $\curlywedge$ hyperoperations are monotone in the following sense.

Proposition 18.12 *For all $a, b \in X$ we have: $a \leq b \Rightarrow \begin{cases} a \curlyvee c \preceq b \curlyvee c, \\ a \curlywedge c \preceq b \curlywedge c. \end{cases}$.*

Proof. Take any $p \in X$. Then

$$a \leq b \Rightarrow \left\{\begin{array}{c} a \vee c \leq b \vee c \\ (a \vee c) \vee p' \leq (b \vee c) \vee p' \end{array}\right\} \Rightarrow a \vee_p c \preceq b \vee_p c \Rightarrow (a \curlyvee c)_p \preceq (b \curlyvee c)_p .$$

Since the above is true for every $p \in X$, it follows that $a \curlyvee c \preceq b \curlyvee c$. Similarly we show that $a \curlywedge c \preceq b \curlywedge c$. $\square$

$\curlyvee$, $\curlywedge$ and $'$ are interrelated as seen by Proposition 18.14.

Definition 18.6 *For every $A \in \mathbf{F}(X)$ define A' by its α-cuts, i.e. A' is the (unique) fuzzy set which for every $p \in X$ satisfies*

$$(A')_p = (A_p)' = \{x'\}_{x \in A_p}.$$

Proposition 18.13 *If A is a fuzzy interval, then A' is also a fuzzy interval.*

Proof. Take any $p \in X$. Suppose that $A_p = [a_1, a_2]$. Then

$$(A')_p = (A_p)' = \{x' : a_1 \leq x \leq a_2\} = [a_2', a_1'] . \qquad \square$$

Proposition 18.14 *For every $a, b \in X$ we have:*

$$(a \curlyvee b)' = a' \curlywedge b', \qquad (a \curlywedge b)' = a' \curlyvee b'.$$

Proof. Choose any $p \in X$. Then

$$\left((a \curlyvee b)'\right)_p = \left((a \curlyvee b)_p\right)' = (a \vee_p b)' = a' \wedge_p b' = (a' \curlywedge b')_p .$$

Since for all $p \in X$ the fuzzy sets $(a \curlyvee b)'$ and $a' \curlywedge b'$ have the same cuts, we have $(a \curlyvee b)' = a' \curlywedge b'$. $\square$

In conclusion, we have constructed a *fuzzy hyperoperation* $\curlywedge$. Let us now see in what sense it can be called a fuzzy t-norm. To see the similarity consider the following table, which compares the properties of any crisp t-norm T with those of $\curlywedge$; the properties of $\curlywedge$ are obtained from Proposition 18.8.

Table 1

A crisp t-norm T	**The fuzzy t-norm** $\curlywedge$
$aTb = bTa$	$a \curlywedge b = b \curlywedge a$
$(aTb)Tc = aT(bTc)$	$(a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$
$a = aT1$	$(a \curlywedge 1)(a) = 1$
$a \leq b \Rightarrow aTc \leq bTc$	$a \leq b \Rightarrow a \curlywedge c \preceq b \curlywedge c$

A similar table can be used to compare $\curlyvee$ with some t-conorm S. The analogies between the "classical" operations T, S and the fuzzy-valued hyperoperations $\curlywedge, \curlyvee$ are obvious. Hence we can justifiably say that $\curlywedge$ is a fuzzy t-norm and $\curlyvee$ is a fuzzy t-conorm.

18.4 Fuzzy-valued S-implication

We can also construct a fuzzy hyperoperation which behaves like an S-implication. This is done as follows.

Definition 18.7 *The fuzzy implication is denoted by* $\rightsquigarrow$ *and defined for every* $a, b \in X$ *by*

$$(a \rightsquigarrow b) = (a' \curlyvee b).$$

It is easy to prove the following.

Proposition 18.15 *For all* $a, b, p \in X$ *we have*

$$(a \rightsquigarrow b)_p = a \rightarrow_p b.$$

Proof. Indeed $(a \rightsquigarrow b)_p = (a' \curlyvee b)_p = a' \vee_p b = a \rightarrow_p b$. □

From Proposition 18.15 and the properties of $\vee_p$, described in Section 3 of [14] we can immediately prove the following proposition, which summarizes the properties of $\rightsquigarrow$; it can be easily seen that these are analogous to the classical implication.

Proposition 18.16 *We have for every* $a, b, c \in X$ *the following.*

1. $a \leq b \Rightarrow ((a \rightsquigarrow c) \succeq (b \rightsquigarrow c)$ *and* $(c \rightsquigarrow a) \preceq (c \rightsquigarrow b))$,
2. $(0 \rightsquigarrow a)(1) = 1$, $\left(1 \rightarrow^P a\right)(a) = 1$,
3. $(a \rightsquigarrow b) = (b' \rightsquigarrow a')$.

Proof. Straightforward. □

Remark 18.4. The above are similar to the properties proved in [6] and other places about (crisp, uni-valued) implications.

18.5 Conclusion

We have introduced two fuzzy-valued hyperoperations, $\curlywedge$ and $\curlyvee$, which are natural generalizations of the t-norm $\wedge$ and the t-conorm $\vee$. Clearly, the new (crisp and fuzzy) hyperoperations have a great potential for applications to computational intelligence, where they can extend the concepts and procedures of fuzzy reasoning.

In particular, the definitions and results of Section 18.4 constitute a first step in the study of fuzzy-valued implications; in the future we plan to work further in this direction and apply the resulting implications to the study of *fuzzy cognitive maps* [15, 16].

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