17

A Family of Multi-valued t-norms and t-conorms

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Summary. Given a lattice $(X, \leq, \wedge, \vee)$ we define a *multi-valued* operation $\wedge^Q$ which is analogous to a t-norm (i.e. it is commutative, associative, has one as a neutral element and is monotone). The operation is parametrized by the set Q, hence we actually obtain an entire family of such *multi-valued* t-norms. Similarly we define a family of multi-valued t-conorms $\vee^P$. We show that, when P, Q are chosen appropriately, $\wedge^Q, \vee^P$ (along with a standard negation) form a de Morgan pair. Furthermore $\wedge^Q, \vee^P$ are order generating and $(X, \leq, \wedge^Q, \vee^P)$ is a *superlattice*, i.e. a multi-valued analog of a lattice.

17.1 Introduction

In this paper we generalize the concepts of t-norm and t-conorm. While many variants of t-norm and t-conorm have appeared in the literature, they always are *single-valued* functions. The current work is substantially different because it considers a *multi-valued* generalization.

As is well known, a t-norm is an extension of the classical logical operator AND to the realm of fuzzy logic. While in the classical case AND is a function from $\{0, 1\} \times \{0, 1\}$ to $\{0, 1\}$ (or, equivalently, from $\{\text{True,False}\} \times \{\text{True,False}\}$ to $\{\text{True,False}\}$), a t-norm is a function from $[0, 1] \times [0, 1]$ to $[0, 1]$. The interval $[0, 1]$ can be an interval of reals or, more generally, a lattice with bottom element 0 and top element 1. Recently considerable attention has been paid to t-norms for particular types of lattices, for instance the lattice of real intervals [2], the lattice of type-2 fuzzy sets [23] etc. All of these generalizations are concerned with the domain and range of the t-norm. Similar remarks hold for t-conorms, which extend from $\{0, 1\}$ to $[0, 1]$ the logical OR operator[1].

Many t-norms and t-conorms have been presented in the fuzzy literature but the above remarks apply to practically all of them. This also holds true for t-norms and t-conorms which apply to interval-valued fuzzy sets: in this

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[1] Under another interpretation, a t-norm generalizes set intersection and a t-conorm generalizes set union.

case a t-norm takes as input pairs of intervals and produces as output a single interval. The case with which we deal in the current work is significantly different. We are interested in *multi-valued* t-norms and t-conorms; a simple example would be a t-norm which takes as input two reals and produces as output an *interval* of reals. In this paper we discuss the generalization of this example.

Our motivation for studying multi-valued logical connectives is the following: fuzzy logic is a calculus of uncertain reasoning; but it seems that all the uncertainty is concentrated in the truth values of the logical propositions; the case where uncertainty is associated with the actual operation of the logical connectives has not been studied in the past. The introduction of multi-valued logical connectives is a reasonable way to introduce this type of uncertainty. On the other hand, much of the work presented concerns *intervals* (either of reals or, more generally, of lattice elements) which has been a common theme in the fuzzy and computationbal intelligenece literature; see for instance the use of "hyperboxes" (lattice intervals) in [11, 12, 17] for theoretical aspects and [1, 11, 18] for applications. We present our work in the general framework of lattice theory, which is quite popular for the general analysis of fuzzy systems [6, 8, 19].

There is a considerable literature on algebras equipped with multi-valued operations; see the book [5]. In this area the multi-valued operations are called *hyperoperations* and they yield *hyperalgebras* such as hypergroups [3, 4], hyperrings [15, 22], hyperlattices [14, 16, 20, 21] etc. Using similar terminology we will speak of the *hyper-t-norm*, the *hyper-t-conorm* etc. It is not stressed often enough, but a basic hyperoperation which finds frequent application in Computational Intelligence is *clustering* [17] which takes as input two (or a few) *points* to produce as output a *region*, i.e. a *set of points*.

Among all uni-valued t-norms, the min t-norm $\wedge$ has a special place. For example, it is the only idempotent t-norm. Also, in a certain sense, it is the simplest extension of AND. Finally, and perhaps more importantly, it is the only t-norm which *generates an order*, i.e. the double implication

$$aTb = a \Leftrightarrow a \leq b$$

only holds for $T = \wedge$. Similar remarks hold for the max t-conorm $\vee$. In the current work we present a family of multi-valued logical connectives which make essential use of $\wedge$ and $\vee$; our construction can be applied to other t-norms and t-conorms but the results are not as satisfactory. Let us mention however that the current generalization is only one of several ways to define multi-valued t-norms and t-conorms; we have presented other possibilities in [10, 20, 21].

The plan of this paper is as follows. In Section 17.2 we present some preliminary concepts. The rest of the paper assumes a basic de Morgan algebra $(X, \leq, \wedge, \vee, ')$ and constructs on it various algebraic structures. In Section 17.3 we introduce the basic objects of our study, namely the multi-valued

$\wedge^Q$ t-norm and $\vee^P$ t-conorm. In Section 17.4 we study the multi-valued *S-implication* obtained from $\vee^P$. In Section 17.5 we show that $\wedge^Q$ and $\vee^P$ generate a structure analogous to a lattice, the so-called (P,Q)-*superlattice*. In Section 17.6 we obtain additional results for the case when $(X, \leq, \wedge, \vee, ')$ is a *Boolean* algebra. Finally, we summarize our results in Section 17.7.

17.2 Preliminaries

In this section we review some fundamental concepts which will be used in the main part of the paper; we also present some well known propositions (we omit their proofs, which can be found in standard texts [7]).

Given a set X, the *power set* of X will be denoted by $\mathbf{P}(X)$ and will be defined to be the set of all (crisp) subsets of X. We can equip X with an order relationship $\leq$ and thus obtain a *partially ordered set* $(X, \leq)$. If for every pair $x, y \in X$ the $\mathtt{inf}(x, y)$ and $\mathtt{sup}(x, y)$ exist, we say that $(X, \leq)$ is a *lattice*. We denote $\mathtt{inf}(x, y)$ by $x \wedge y$ and $\mathtt{sup}(x, y)$ by $x \vee y$; then $\wedge, \vee$ are binary operations on X and we sometimes say that $(X, \leq, \wedge, \vee)$ or $(X, \wedge, \vee)$ is a lattice. We also use the notation $x_1 \wedge x_2 \wedge ... \wedge x_N$ to denote the infimum of $x_1, x_2, ..., x_N$ (the $\wedge$ operation is associative) and $\wedge P$ to denote $\wedge_{x \in P} x$, the infimum of a set $P \subseteq X$ (if such an infimum exists); similarly we use $x_1 \vee x_2 \vee ... \vee x_N$ and $\vee P$. A lattice is called *distributive* iff $\forall a, b, c \in X$ it is

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

We will use the following proposition repeatedly in what follows.

Proposition 17.1 *In a distributive lattice* $(X, \leq, \vee, \wedge)$ *the following properties hold for all* $a, b, x, y \in X$ *such that* $x \leq y$, $a \leq b$:

$$a \vee [x, y] = [a \vee x, a \vee y]; [a, b] \vee [x, y] = [a \vee x, b \vee y];$$
$$a \wedge [x, y] = [a \wedge x, a \wedge y]; [a, b] \wedge [x, y] = [a \wedge x, b \wedge y].$$

We now introduce several "substructures" within the lattice $(X, \leq, \wedge, \vee)$.

1. A *sublattice* of a lattice is a set $Y \subseteq X$ such that: $x, y \in Y \Rightarrow x \wedge y \in Y, x \vee y \in Y$.
2. A *convex sublattice* is a set $Y \subseteq X$ such that: if $x, y \in Y$ then $[x \wedge y, x \vee y] \subseteq Y$. We denote the set of all convex sublattices of X by $\mathbf{C}(X)$.
3. A *filter* is a set $Y \subseteq X$ which satisfies the following two conditions: (a) if $x \in Y$, $y \in X$ and $x \leq y$, then $y \in Y$ (b) if $x, y \in Y$ then $x \wedge y \in Y$.
4. An *interval* in the lattice $(X, \leq, \wedge, \vee)$ is a set of the form $\{x : a \leq x \leq b\}$, where $a, b \in X$ are the *endpoints* of the interval; we denote this interval by $[a, b]$. We denote the set of all intervals of X by $\mathbf{I}(X)$.

Every filter is a convex sublattice; in a lattice with bottom element 0 and top element 1, every interval of the form $[q, 1]$ is a filter.

Given a poset (or lattice) $(X, \leq)$ we can define two "order-like" relations on $\mathbf{P}(X)$ (i.e. they hold between subsets of X).

1. $A \leqslant_1 B$ means that: $\begin{cases} \exists a \in A \text{ such that } \forall b \in B \text{ we have } a \leq b \\ \exists b \in B \text{ such that } \forall a \in A \text{ we have } a \leq b \end{cases}$.

2. $A \leqslant_2 B$ means that: $\forall a \in A,\, b \in B$ we have $a \vee b \in B$ and $a \wedge b \in A$.

These relations are not independent; also (under appropriate restrictions) they are orders.

Proposition 17.2 *For every $A, B \in \mathbf{C}\,(X)$, we have $A \leqslant_1 B \Rightarrow A \leqslant_2 B$.*

Proposition 17.3 *The relations $\leqslant_i (i = 1, 2)$ are orders on $\mathbf{C}\,(X)$ (and, a fortiori, on $\mathbf{I}\,(X)$).*

In the rest of the paper we will use a *generalized de Morgan lattice* defined as follows.

Definition 17.1 *A* generalized deMorgan lattice *is a structure $(X, \leq, \vee, \wedge, ')$, where $(X, \leq, \vee, \wedge)$ is a complete distributive lattice with minimum element 0 and maximum element 1; the symbol $'$ denotes a unary operation ("negation"); and the following properties are satisfied.*

1. *For all $x \in X$, $Y \subseteq X$ we have $x \wedge (\vee_{y \in Y} y) = \vee_{y \in Y} (x \wedge y)$, $x \vee (\wedge_{y \in Y} y) = \wedge_{y \in Y} (x \vee y)$. (Complete distributivity).*
2. *For all $x \in X$ we have: $(x')' = x$. (Negation is involutory).*
3. *For all $x, y \in X$ we have: $x \leq y \Rightarrow y' \leq x'$. (Negation is order reversing).*
4. *For all $Y \subseteq X$ we have $(\vee_{y \in Y} y)' = \wedge_{y \in Y} y'$, $(\wedge_{y \in Y} y)' = \vee_{y \in Y} y'$ (Complete deMorgan laws).*

In Section 17.6 we will make a stronger assumption, that $(X, \leq, \vee, \wedge, ')$ is a *generalized Boolean lattice*, i.e. it satisfies the following.

Definition 17.2 *A* generalized Boolean lattice *is a generalized deMorgan lattice $(X, \leq, \vee, \wedge, ')$ in which every $x \in X$ satisfies: $x \vee x' = 1$, $x \wedge x' = 0$.*

Notation 17.1 *Given a set $A \subseteq X$ we will denote the set of all negated elements of A by $A' = \{x : x' \in A\}$.*

Proposition 17.4 *Let $A = [a_1, a_2]$; then $A' = [a_2', a_1']$.*

We now turn to t-norms, t-conorms and their multi-valued extensions. Recall that a t-norm on a general lattice $(X, \leq)$ with 0 and 1 is a function $T : X \times X \to X$ which is commutative, associative and satisfies for all $a, b, c \in X$ the following

$$\text{absorption} : aT1 = a, \tag{A1}$$

$$\text{monotonicity} : a \leq b \Rightarrow aTc \leq bTc; \tag{A2}$$

similarly a t-conorm is a function $S : X \times X \to X$ which is commutative, associative and satisfies for all $a, b, c \in X$ the following

$$\text{absorption} : aS0 = a, \tag{B1}$$

$$\text{monotonicity} : a \le b \Rightarrow aSc \le bSc. \tag{B2}$$

Now, t-norms and t-conorms are examples of *operations*, i.e. *uni-valued* mappings from $X \times X$ to X; they map pairs of elements to elements. The following notations are standard in the hyperoperations literature:

1. if $*$ is an operation, for any $a \in X$ and $A, B \subseteq X$ we define $a * B = \cup_{b \in B} \{a * b\}$, $A * B = \cup_{a,b \in B} \{a * b\}$.
2. if $*$ is a hyperoperation, for any $a \in X$ and $A, B \subseteq X$ we define $a * B = \cup_{b \in B} a * b$, $A * B = \cup_{a,b \in B} a * b$ (remember that $a * b$ is a *set*).

Now we are ready to define hyper-t-norms and hyper-t-conorms by generalizing properties A1, A2 and B1, B2 to their multi-valued analogs.

1. A *hyper-t-norm* is a multi-valued map $\mathbf{T} : X \times X \to \mathbf{P}(X)$ which is commutative, associative and satisfies for all $a, b, c \in X$ the following

$$\text{absorption} : a\mathbf{T}1 \ni a, \tag{C1}$$

$$\text{monotonicity} : a \le b \Rightarrow a\mathbf{T}c \leqslant b\mathbf{T}c. \tag{C2}$$

(*where $\leqslant$ is an order relationship on the range of $\mathbf{T}$*).

2. A *hyper-t-conorm* is a multi-valued map $\mathbf{S} : X \times X \to \mathbf{P}(X)$ which is commutative, associative and satisfies for all $a, b, c \in X$ the following $a \le b \Rightarrow a\mathbf{T}c \leqslant b\mathbf{T}c$.

$$\text{absorption} : a\mathbf{S}0 \ni a, \tag{D1}$$

$$\text{monotonicity} : a \le b \Rightarrow a\mathbf{S}c \leqslant b\mathbf{S}c. \tag{D2}$$

Conditions C1 and D1 are straightforward generalizations of A1 and B1, respectively. Conditions C2 and D2 are more subtle. Note that in both C2 and D2 $\leqslant$ *is an order relationship between sets* (such as the above defined $\leqslant_1, \leqslant_2$ etc.). To complete the definitions of hyper-t-norm and hyper t-conorm, we must specify which order relationship we are using. There are several options for $\leqslant$ and, furthermore, it is only required that $\leqslant$ is an order *on the range of* $\mathbf{T} / \mathbf{S}$. As will be seen in Section 17.3, this will turn out to be significant when we examine specific hyperoperations with a restricted range.

17.3 The $\wedge^Q$ hyper-t-norm and the $\vee^P$ hyper-t-conorm

17.3.1 The $\wedge^Q$ hyper-t-norm

We define a hyperoperation which, under suitable restrictions, has the properties of a multivalued t-norm .

Definition 17.3 *Let Q be some subset of X and define the hyperoperation $\wedge^Q$ as follows:*

$$\forall a, b \in X : a \wedge^Q b \doteq \{a \wedge b \wedge q : q \in Q\} \,.$$

Remark 17.1. Hence we could write $a \wedge^Q b = a \wedge b \wedge Q$.

We will now show that under suitable conditions $\wedge^Q$ is a hyper-t-norm, i.e. it is commutative, associative and satisfies C1, C2. We will proceed in several steps. First we show that $\wedge^Q$ is *always* commutative and associative.

Proposition 17.5 *Take any set $Q \subseteq X$. Then, for all a, b, c we have*

$$a \wedge^Q b = b \wedge^Q a; \quad \left(a \wedge^Q b\right) \wedge^Q c = a \wedge^Q \left(b \wedge^Q c\right).$$

Proof. The first part is obvious. For the second, take any $x \in \left(a \wedge^Q b\right) \wedge^Q c$; then exists some y such that

$$y \in a \wedge^Q b \text{ and } x \in y \wedge^Q c;$$

also, $y \in a \wedge^Q b$ means that there exists $q_1 \in Q$ such that $y = a \wedge b \wedge q_1$; and $x \in y \wedge^Q c$ means that exists $q_2 \in Q$ such that $x = y \wedge c \wedge q_2$. Hence

$$x = (a \wedge b \wedge q_1) \wedge c \wedge q_2 = a \wedge (b \wedge c \wedge q_1) \wedge q_2.$$

But the last part of the above equality shows that $x \in a \wedge^Q \left(b \wedge^Q c\right)$. Hence $\left(a \wedge^Q b\right) \wedge^Q c \subseteq a \wedge^Q \left(b \wedge^Q c\right)$; similarly we can show that $a \wedge^Q \left(b \wedge^Q c\right) \subseteq \left(a \wedge^Q b\right) \wedge^Q c$; and so $a \wedge^Q \left(b \wedge^Q c\right) = \left(a \wedge^Q b\right) \wedge^Q c$. $\square$

Next we give a necessary and sufficient condition on Q for C1 to hold.

Proposition 17.6 *We have:* $\left(\forall a : a \in a \wedge^Q 1\right) \Leftrightarrow (1 \in Q)$.

Proof. If $1 \in Q$, then $a = a \wedge 1 \wedge 1 \in a \wedge^Q 1$. On the other hand, if $\forall a : a \in a \wedge^Q 1$, then $1 \in 1 \wedge^Q 1$, i.e. exists $q_1 \in Q$ such that $1 = 1 \wedge 1 \wedge q_1 = q_1$. $\square$

Finally we examine the question of monotonicity (i.e. condition C2). To this end we need an auxiliary proposition, regarding the nature of $a \wedge^Q b$.

Proposition 17.7 *Take any $Q \subseteq X$ such that $1 \in Q$. Then we have the following*

1. $(Q$ is a convex sublattice$) \Leftrightarrow \left(\forall a, b \in X : a \wedge^Q b$ is a convex sublattice$\right)$;
2. $(Q$ is a filter$) \Rightarrow \left(\forall a, b \in X : a \wedge^Q b$ is a convex sublattice$\right)$;
3. $(Q$ is an interval$) \Leftrightarrow \left(\forall a, b \in X : a \wedge^Q b$ is an interval$\right)$.

Proof. For 1, suppose Q is a convex sublattice. Take any $a, b \in X$, then take any $x, y \in a \wedge^Q b$; i.e. $x = a \wedge b \wedge q_1$, $q_1 \in Q$; and $y = a \wedge b \wedge q_2$, $q_2 \in Q$. Now consider $[x \wedge y, x \vee y]$; we have

$$[x \wedge y, x \vee y] = [a \wedge b \wedge (q_1 \wedge q_2), a \wedge b \wedge (q_1 \vee q_2)]$$
$$= a \wedge b \wedge [q_1 \wedge q_2, q_1 \vee q_2] \subseteq a \wedge b \wedge Q = a \wedge^Q b.$$

(since Q is a convex sublattice, $[q_1 \wedge q_2, q_1 \vee q_2] \subseteq Q$). Hence $a \wedge^Q b$ is a convex sublattice. Conversely, if $\forall a, b \in X : a \wedge^Q b$ is a convex sublattice, then $1 \wedge^Q 1 = 1 \wedge 1 \wedge Q = Q$ is a convex sublattice. For 2, simply note that every filter is a convex sublattice. For 3, suppose $Q = [q_1, q_2]$. Then

$$a \wedge^Q b = a \wedge b \wedge [q_1, q_2] = [a \wedge b \wedge q_1, a \wedge b \wedge q_2]$$

is an interval; and conversely, if $\forall a, b \in X : a \wedge^Q b$ is an interval, then $1 \wedge^Q 1 = 1 \wedge 1 \wedge Q = Q$ is an interval. $\square$

Remark 17.2. In fact, the third part of Proposition 17.7 can be strengthened. As will be seen in the sequel, we are mainly interested in the Q's which contain 1; if such a Q is also an interval, then $Q = [q, 1]$. Then we have the following.

Proposition 17.8 *If $Q = [q, 1]$ then $a \wedge^Q b = [a \wedge b \wedge q, a \wedge b]$ ($\forall a, b \in X$).*

Proof. $a \wedge^Q b = a \wedge b \wedge [q, 1] = [a \wedge b \wedge q, a \wedge b]$. $\square$

Under certain conditions $\leqslant_1, \leqslant_2$ are orders *on the collection* $\{a \wedge^Q b\}_{a,b \in X}$.

Proposition 17.9 *Take any $Q \subseteq X$ such that $1 \in Q$.*

1. *If Q is a convex sublattice, then $\leqslant_2$ is an order on $\{a \wedge^Q b\}_{a,b \in X}$.*
2. *If Q is an interval, then $\leqslant_1, \leqslant_2$ are orders on $\{a \wedge^Q b\}_{a,b \in X}$.*

Proof. First we prove 1. Suppose Q is a convex sublattice.

i. Pick any $a, b \in X$ and any $x, y \in a \wedge^Q b$. Then $a \wedge^Q b$ is a convex sublattice (Prop. 17.7) and so $x \wedge y \in a \wedge^Q b$ and $x \vee y \in a \wedge^Q b$ and so $a \wedge^Q b \leqslant_2 a \wedge^Q b$.

ii. Pick any $a, b, c, d \in X$ such that $a \wedge^Q b \leqslant_2 c \wedge^Q d$ and $c \wedge^Q d \leqslant_2 a \wedge^Q b$. Now choose any $x \in a \wedge^Q b$ and any $y \in c \wedge^Q d$. Then $x \wedge y \in a \wedge^Q b$ and $x \vee y \in a \wedge^Q b$. So $y \in [x \wedge y, x \vee y] \subseteq a \wedge^Q b$ which implies that $c \wedge^Q d \subseteq a \wedge^Q b$. Similarly we can show that $a \wedge^Q b \subseteq c \wedge^Q d$ and so $c \wedge^Q d = a \wedge^Q b$.

iii. Pick any $a, b, c, d, e, f \in X$ such that $a \wedge^Q b \leqslant_2 c \wedge^Q d$ and $c \wedge^Q d \leqslant_2 e \wedge^Q f$. Now choose any $x \in a \wedge^Q b$, any $y \in c \wedge^Q d$ and any $z \in e \wedge^Q f$. Then $x \vee y \in c \wedge^Q d$ and so $x \vee y \vee z \in e \wedge^Q f$. So $x \vee z \in [z, x \vee y \vee z] \subseteq e \wedge^Q f$. Similarly we can show $x \wedge z \in a \wedge^Q b$. Hence $a \wedge^Q b \leqslant_2 e \wedge^Q f$.

2 follows from 1: if Q is an interval, it is also a convex sublattice. $\square$

Now we can prove a monotonicity property of $a \wedge^Q b$.

Proposition 17.10 *Take any $Q \subseteq X$ with $1 \in Q$.*

1. *If Q is a filter, then we have: $a \leq b \Leftrightarrow \left(\forall c \in X : a \wedge^Q c \leqslant_2 b \wedge^Q c\right)$.*
2. *If Q is an interval, then (for $i = 1, 2$) we have:*

$$a \leq b \Leftrightarrow \left(\forall c \in X : a \wedge^Q c \leqslant_i b \wedge^Q c\right).$$

Proof. 1. Let Q be a filter. Suppose that $a \leq b$ and take $x \in a \wedge^Q c$, $y \in b \wedge^Q c$. That is, $x = a \wedge c \wedge q_1$ and $y = b \wedge c \wedge q_2$. Then

$$x \wedge y = a \wedge b \wedge c \wedge (q_1 \wedge q_2) = a \wedge c \wedge (q_1 \wedge q_2);$$

since $q_1, q_2 \in Q$ and Q is a filter, then $q_1 \wedge q_2 \in Q$ and $x \wedge y \in a \wedge^Q c$. Also

$$
\begin{aligned}
x \vee y &= (a \wedge c \wedge q_1) \vee (b \wedge c \wedge q_2) = ((a \wedge q_1) \vee (b \wedge q_1)) \wedge c \\
&= ((a \vee b) \wedge (q_1 \vee b) \wedge (a \vee q_2) \wedge (q_1 \vee q_2)) \wedge c \\
&= (b \wedge (q_1 \vee b) \wedge (a \vee q_2) \wedge (q_1 \vee q_2)) \wedge c \\
&= (b \wedge (a \vee q_2) \wedge (q_1 \vee q_2)) \wedge c.
\end{aligned}
$$

Now, $a \vee q_2 \geq q_2 \in Q$ and so $a \vee q_2 \in Q$ (Q is a filter); also $q_1 \vee q_2 \in Q$ (Q is a sublattice) hence $q = (a \vee q_2) \wedge (q_1 \vee q_2) \in Q$. This means

$$x \vee y = b \wedge c \wedge q \in b \wedge^Q c.$$

Since $x \wedge y \in a \wedge^Q c$ and $x \vee y \in b \wedge^Q c$, it follows that $a \wedge^Q c \leqslant_2 b \wedge^Q c$.
2. If Q is an interval, then $Q = [q, 1]$. Hence $a \leq b \Rightarrow a \wedge c \leq b \wedge c \Rightarrow a \wedge c \wedge q \leq b \wedge c \wedge q$. Hence

$$a \wedge^Q c = [a \wedge c \wedge q, a \wedge c] \leqslant_1 [b \wedge c \wedge q, b \wedge c] = b \wedge^Q c$$

On the other hand, if $\left(\forall c \in X : a \wedge^Q c \leqslant_1 b \wedge^Q c\right)$, then taking $c = 1$ we get

$$[a \wedge q, a] \leqslant_1 [b \wedge q, b] \Rightarrow a \leq b.$$

The same result follows for $\leqslant_2$, since this is equivalent to $\leqslant_1$ over a class of intervals. $\square$

Remark 17.3. Note the double implications in the above proposition. They show that $\wedge^Q$ has a stronger-than-monotonicity property (when $1 \in Q$).

From the above propositions we can state the following proposition which give conditions for $\wedge^Q$ to be a hyper-t-norm.

Proposition 17.11 *Let $Q \subseteq \mathbf{P}(X)$ with $1 \in Q$.*

1. *If Q is a filter then $\wedge^Q$ is a hyper-t-norm (with respect to the order $\leqslant_2$).*
2. *If Q is an interval, then $\wedge^Q$ is a hyper-t-norm (with respect to the orders $\leqslant_1$, $\leqslant_2$).*

Proof. This is a simple consequence of Propositions 17.7, 17.9, 17.10. □

In the rest of the paper we will always assume that Q is (at least) a filter and that $1 \in Q$. Here are some additional properties of $\wedge^Q$ regarding order.

Proposition 17.12 *For all $a, b \in X$ we have: $\max(a \wedge^Q b) = a \wedge b$.*

Proof. Indeed, $x \in a \wedge^Q b \Rightarrow x = a \wedge b \wedge q \leq a \wedge b \wedge 1 = a \wedge b$. And $a \wedge b = a \wedge b \wedge 1 \in a \wedge^Q b$. □

Proposition 17.13 *For all $a, b \in X$ we have: $a \wedge^Q b = a \wedge^Q a \Leftrightarrow a \leq b$.*

Proof. Assume $a \leq b$. Then $a \wedge^Q b = \cup_{q \in Q} a \wedge b \wedge q = \cup_{q \in Q} a \wedge q = \cup_{q \in Q} a \wedge a \wedge q = a \wedge^Q a$. Conversely, if $a \wedge^Q b = a \wedge^Q a$, then by Proposition 17.12

$$a \wedge b = \max\left(a \wedge^Q b\right) = \max\left(a \wedge^Q a\right) = a \wedge a = a \Rightarrow a \leq b. \square$$

17.3.2 The $\vee^P$ hyper-t-conorm

In completely analogous manner to that of the previous section, we define a hyperoperation which, under suitable restrictions, has the properties of a multivalued t-conorm.

Definition 17.4 *Let P be some subset of X and define the hyperoperation $\vee^P$ as follows:*

$$\forall a, b \in X : a \vee^P b \doteq \{a \vee b \vee p' : p \in P\}.$$

Remark 17.4. We could also write $a \vee^P b = a \vee b \vee P'$.

Proposition 17.14 *Take any set $P \subseteq X$. Then, for all a, b, c we have*

$$a \vee^P b = b \vee^P a; \quad \left(a \vee^P b\right) \vee^P c = a \vee^P \left(b \vee^P c\right).$$

Proposition 17.15 *We have: $\left(\forall a : a \in a \vee^P 0\right) \Leftrightarrow (1 \in P)$.*

Proposition 17.16 *Take any $P \subseteq X$ such that $1 \in P$. Then we have the following*

1. *$(P$ is a convex sublattice$) \Leftrightarrow \left(\forall a, b \in X : a \vee^P b$ is a convex sublattice$\right)$;*
2. *$(P$ is a filter$) \Rightarrow \left(\forall a, b \in X : a \vee^P b$ is a convex sublattice$\right)$;*
3. *$(P$ is an interval$) \Leftrightarrow \left(\forall a, b \in X : a \vee^P b$ is an interval$\right)$.*

Proposition 17.17 *If $P = [p, 1]$ then $a \vee^P b = [a \vee b, a \vee b \vee p']$ $(\forall a, b \in X)$.*

Proposition 17.18 *Take any $P \subseteq X$ such that $1 \in P$.*

1. *If P is a convex sublattice, then $\leqslant_2$ is an order on $\left\{a \vee^P b\right\}_{a,b \in X}$.*
2. *If P is an interval, then $\leqslant_1, \leqslant_2$ are orders on $\left\{a \vee^P b\right\}_{a,b \in X}$.*

Proposition 17.19 *Take any $P \subseteq X$ with $1 \in P$.*

1. *If P is a filter, then we have:* $a \leq b \Leftrightarrow \left(\forall c \in X : a \vee^P c \leqslant_2 b \vee^P c\right).$
2. *If P is an interval, then (for $i = 1, 2$) we have:*

$$a \leq b \Leftrightarrow \left(\forall c \in X : a \vee^P c \leqslant_i b \vee^P c\right).$$

Proposition 17.20 *Let $P \subseteq \mathbf{P}(X)$ with $1 \in P$.*

1. *If P is a filter then $\vee^P$ is a hyper-t-conorm (with respect to the order $\leqslant_2$).*
2. *If P is an interval, then $\vee^P$ is a hyper-t-conorm (with respect to the orders $\leqslant_1$, $\leqslant_2$).*

Proposition 17.21 *For all $a, b \in X$ we have:* $\mathtt{min}\!\left(a \vee^P b\right) = a \vee b.$

Proposition 17.22 *For all $a, b \in X$ we have:* $a \vee^P b = a \vee^P a \Leftrightarrow a \leq b.$

17.3.3 Further Properties of $\wedge^Q$ and $\vee^P$

The following proposition establishes that $\wedge^Q$ and $\vee^P$ have a weak form of distributivity.

Proposition 17.23 *For all $a, b, c \in X$ we have*

$$\left(a \wedge^Q \left(b \vee^P c\right)\right) \cap \left(\left(a \wedge^Q b\right) \vee^P \left(a \wedge^Q c\right)\right) \neq \emptyset, \qquad (17.1)$$

$$\left(a \vee^P \left(b \wedge^Q c\right)\right) \cap \left(\left(a \vee^P b\right) \wedge^Q \left(a \vee^P c\right)\right) \neq \emptyset. \qquad (17.2)$$

Proof. Since $1 \in Q$ and $1 \in P$, then

$$a \wedge (b \vee c) = a \wedge (b \vee c \vee 1') \wedge 1 \in a \wedge^Q \left(b \vee^P c\right). \qquad (17.3)$$

Also

$$(a \wedge b) \vee (a \wedge c) = (a \wedge b \wedge 1) \vee (a \wedge c \wedge 1) \vee 1' \in \left(a \wedge^Q b\right) \vee^P \left(a \wedge^Q c\right). \quad (17.4)$$

Finally,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \vee c). \qquad (17.5)$$

From (17.3), (17.4) and (17.5) we obtain (17.1); (17.2) is proved dually. $\square$

Here are some more properties related to order.

Proposition 17.24 *For all $a, b \in L$ we have:* $x \in a \vee^P b, \ y \in a \wedge^Q b \Rightarrow y \leq x.$

Proof. True, since $y \leq \mathtt{max}\left(a \wedge^Q b\right) = a \wedge b \leq a \vee b = \mathtt{min}\left(a \vee^P b\right) \leq x.$ $\square$

Proposition 17.25 *For all $a, b \in L$ we have:* $\left(a \vee^P b\right) \cap \left(a \wedge^Q b\right) \neq \emptyset \Rightarrow a = b.$

Proof. Take some $x \in \left(a \vee^P b\right) \cap \left(a \wedge^Q b\right)$; then for some $p \in P, q \in Q$ we have

$$a \wedge b \geq a \wedge b \wedge q = x = a \vee b \vee p' \geq a \vee b \Rightarrow a = a \wedge b = a \vee b = b.\square$$

We also have a sort of "reduction property".

Proposition 17.26 *For all $a, x, y \in L$ we have:*

$$\left(a \vee^P x = a \vee^P y \text{ and } a \wedge^Q x = a \wedge^Q y\right) \Rightarrow x = y.$$

Proof.

$$a \vee^P x = a \vee^P y \Rightarrow \mathtt{max}\left(a \vee^P x\right) = \mathtt{max}\left(a \vee^P y\right) \Rightarrow a \vee x = a \vee y \quad (17.6)$$

$$a \wedge^Q x = a \wedge^Q y \Rightarrow \mathtt{max}\left(a \wedge^Q x\right) = \mathtt{max}\left(a \wedge^Q y\right) \Rightarrow a \wedge x = a \wedge y \quad (17.7)$$

From (17.6), (17.7) and a standard property of distributive lattices we get $x = y$. $\square$

Proposition 17.27 *For every $'$ and Q, we obtain the* de Morgan triple $\left(\wedge^Q, \vee^Q, '\right)$. *In other words, $\vee^{Q'}$ is a hyper-t-conorm, $\wedge^Q$ is a hyper-t-norm and the following analogs of de Morgan's laws hold:*

$$\left(a \wedge^Q b\right)' = a' \vee^Q b' \text{ and } \left(a \vee^Q b\right)' = a' \wedge^Q b'$$

Proof. Straightforward. $\square$

So far we have assumed that P, Q are filters. Let us now strengthen this assumption; for the rest of this section we will asume that P, Q are intervals of the form $P = [p, 1]$, $Q = [q, 1]$ and we prove several additional results. It will be convenient to introduce a new notation.

Notation 17.2 *When $Q = [q, 1]$ we will denote $\wedge^Q$ by $\wedge_q$; when $P = [p, 1]$ (and $P' = [0, p]$) we will denote $\vee^P$ by $\vee_p$.*

The following proposition establishes that $\wedge_q$ and $\vee_p$ have a weak form of distributivity.

Proposition 17.28 *For all $p, q \in X$ and for all $a, b, c \in X$ we have*

$$a \wedge_q (b \vee_p c) \subseteq (a \wedge_q b) \vee_p (a \wedge_q c), \qquad a \vee_p (b \wedge^q c) \subseteq (a \vee_p b) \wedge_q (a \vee^p c).$$

Proof. Ít is straightforward to show that

$$a \wedge_q (b \vee_p c) = [a \wedge (b \vee c) \wedge q, a \wedge (b \vee c \vee p')] \quad (17.8)$$

$$(a \wedge_q b) \vee_p (a \wedge_q c) = [(a \wedge b \wedge q) \vee (a \wedge c \wedge q), (a \wedge b) \vee (a \wedge c) \vee p']. \quad (17.9)$$

Also

$$a \wedge (b \vee c) \wedge q = (a \wedge b \wedge q) \vee (a \wedge c \wedge q) \quad (17.10)$$

$$a \wedge (b \vee c \vee p') = (a \wedge b) \vee (a \wedge c) \vee (a \wedge p') \leq (a \wedge b) \vee (a \wedge c) \vee p' \quad (17.11)$$

and (17.8) – (17.11) show that the first part of the proposition holds; the second part is proved dually. $\square$

Proposition 17.29 *For all $p_1, p_2, q_1, q_2 \in X$ and $a, b, c \in X$ the following properties hold.*

$$(a \wedge_{q_1} b) \wedge_{q_2} c = (a \wedge_{q_1} b) \wedge_{q_2} c = a \wedge_{q_1 \wedge q_2} b \wedge_{q_1 \wedge q_2} c. \tag{17.12}$$

$$(a \vee_{p_1} b) \vee_{p_2} c = (a \vee_{p_1} b) \vee_{p_2} c = a \vee_{p_1 \vee p_2} b \vee_{p_1 \vee p_2} c. \tag{17.13}$$

Proof. Straightforward. $\square$

17.4 The $\rightarrow^P$ S-hyper-implication

We can also introduce an additional hyperoperation, the *hyper-implication*. This is a straightforward generalization of Boolean and fuzzy implications. Recall that in Boolean logic we can define the implication $\rightarrow$ as follows

$$\forall a, b : a \rightarrow b \doteq a' \vee b. \tag{17.14}$$

Several other equivalent definitions can be used. In fuzzy logic we generalize (17.14) by introducing the class of S-implications: given a t-conorm S (and a negation $'$) we define

$$\forall a, b : a \rightarrow b \doteq a'Sb. \tag{17.15}$$

We can define a hyper-implication (i.e. a multi-valued implication) by using $\vee^P$ in place of S in (17.15). In other words, we introduce the following hyper-implication, denoted by $\rightarrow^P$ and defined as follows

$$a \rightarrow^P b \doteq a' \vee^P b.$$

This hyper-implication has several interesting properties (analogous to these of the classical implication).

Proposition 17.30 *Given a filter $P \subseteq X$ such that $1 \in P$, we have for every a, b, c the following.*

1. $a \leq b \Rightarrow \left((a \rightarrow^P c) \geqslant_2 (b \rightarrow^P c) \text{ and } (c \rightarrow^P a) \leqslant_2 (c \rightarrow^P b) \right),$
2. $1 \in \left(0 \rightarrow^P a \right)$ and $a \in \left(1 \rightarrow^P a \right),$
3. $\left(a \rightarrow^P b \right) = \left(b' \rightarrow^{P'} a' \right).$

Proof. This is fairly straightforward.

1. Assume $a \leq b$. Then $a' \geq b'$ and, by monotonicity of $\vee^P$ we have $\left(a \rightarrow^P c \right) = a' \vee^P c \geqslant_2 b' \vee^P c = \left(b \rightarrow^P c \right)$. Similarly, if $a \leq b$ then $\left(c \rightarrow^P a \right) = c' \vee^P a \leqslant_2 c' \vee^P b = \left(c \rightarrow^P b \right)$.
2. $\mathtt{min}\left(0 \rightarrow^P a \right) = \mathtt{min}\left(0' \vee^P a \right) = 1 \vee a = 1$; hence $1 \in \left(0 \rightarrow^P a \right)$. Also $1 \rightarrow^P a = 1' \vee^P a = \cup_{p \in P} [0 \vee a, 0 \vee a \vee p']$. Since $0 \in P'$, $a = 0 \vee a \vee 0 \in \left(1 \rightarrow^P a \right)$.
3. $\left(a \rightarrow^P b \right) = a' \vee^P b = (b')' \vee^P a' = \left(b' \rightarrow^{P'} a' \right). \square$

The above properties are similar to the ones established in [9] and other places about (uni-valued) implications.

17.5 The (P, Q)-superlattice

As we have already mentioned in Section 17.1, $\wedge$ has several special properties, as compared to other t-norms. Among all these properties, we believe the most special is that $\wedge$ is both monotone and "order-generating". In other words, while by definition every t-norm T satisfies

$$\text{monotonicity:} \qquad \forall a, b, c : a \leq b \Rightarrow aTc \leq bTc \qquad (17.16)$$

the only t-norm that satisfies

$$\text{order generation:} \qquad \forall a, b : a \leq b \Leftrightarrow aTb = a. \qquad (17.17)$$

is $T = \wedge$. The order generated by $\wedge$ can then be defined as follows

$$a \leq b \text{ iff } a \wedge b = a. \qquad (17.18)$$

Similar things hold for the t-conorm $\vee$. The algebra $(X, \leq, \wedge, \vee)$ is a *lattice*; now we will show that the *hyper*algebra $(X, \leq, \wedge^Q, \vee^P)$ is a *superlattice*, i.e. the multivalued analog of a lattice. First consider the following general definition.

Definition 17.5 *Given a poset $(U, \sqsubseteq)$ and hyperoperations $\sqcap$ and $\sqcup$ mapping $U \times U$ to $\mathbf{P}(U)$, the structure $(U, \sqsubseteq, \sqcap, \sqcup)$ is called a* superlattice *iff the following conditions hold for all $a, b, c \in U$.*

G1. $a \in (a \sqcup a)$, $a \in (a \sqcap a)$.
G2. $a \sqcup b = b \sqcup a$, $a \sqcap b = b \sqcap a$.
G3. $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$, $(a \sqcap b) \sqcap c = a \sqcap (b \sqcap c)$.
G4. $a \in (a \sqcup b) \sqcap a$, $a \in (a \sqcap b) \sqcup a$.
G5. $a \sqsubseteq b \Rightarrow b \in a \sqcup b$, $a \in a \sqcap b$.
G6. $b \in a \sqcup b \Leftrightarrow a \in a \sqcap b \Leftrightarrow a \sqsubseteq b$.

   The similarity ("order generation") between $\sqcap$ and $\wedge$, as well as between $\sqcup$ and $\vee$ is especially seen in properties G5, G6.
   We now show that the hyperalgebra $(X, \wedge^Q, \vee^P)$ is a superlattice, in the sense of Definition 17.5. To this end we must prove the following auxiliary proposition.

Proposition 17.31 *Take any $P, Q \subseteq X$ such that $1 \in Q$ and $1 \in P$. Then $(X, \leq, \wedge^Q, \vee^P)$ is a superlattice, i.e. it satifies*

S1. $a \in (a \vee^P a)$, $a \in (a \wedge^Q a)$.
S2. $a \vee^P b = b \vee^P a$, $a \wedge^Q b = b \wedge^Q a$.
S3. $(a \vee^P b) \vee^P c = a \vee^P (b \vee^P c)$, $(a \wedge^Q b) \wedge^Q c = a \wedge^Q (b \wedge^Q c)$.
S4. $a \in (a \vee^P b) \wedge^Q a$, $a \in (a \wedge^Q b) \vee^P a$.
S5. $a \leq b \Rightarrow b \in a \vee^P b$, $a \in a \wedge^Q b$.
S6. $b \in a \vee^P b \Leftrightarrow a \in a \wedge^Q b \Leftrightarrow a \leq b$.

Proof. S1 is straightforward. S2 and S3 were proved in Propositions 17.5, 17.14. For S4, we know that

$$a \in [(a \vee b) \wedge a \wedge 1, (a \vee b \vee 0) \wedge a] \subseteq [a \vee b, a \vee b \vee 0] \wedge^Q a \subseteq \left(a \vee^P b\right) \wedge^Q a;$$

and $a \in \left(a \wedge^Q b\right) \vee^P a$ is proved similarly. For S5, $a \leq b \Rightarrow a \vee b = b$ and so $[b, b] = [a \vee b, a \vee b \vee 0] \subseteq a \vee^P b$. Conversely, if $b \in a \vee^P b$, then $a \vee b \leq b \leq a \vee b \vee 0$ and so $b = a \vee b$, $a \leq b$. Hence we have proved $a \leq b \Leftrightarrow \left(b \in a \vee^P b\right)$. Similarly we can prove $a \leq b \Leftrightarrow \left(a \in a \wedge^Q b\right)$. For S6, if $b \in a \vee^P b$, then (for some $p \in P$) we have $b = a \vee b \vee p' \geq a$; similarly for $a \in a \wedge^Q b$. $\square$

The superlattice $\left(X, \leq, \wedge^Q, \vee^P\right)$ has been studied in [20, 21], under the name of "(P, Q)-superlattice". A case of special interest is the (Q', Q)-superlattice (i.e. using $P = Q$).

17.6 The Boolean Case

We now turn to a special case which yields additional results. Up to this point we have assumed that the lattice $(X, \leq, \wedge, \vee, ')$ is de Morgan; now we make the stronger assumption that it is *Boolean*. In other words we assume that $\forall a \in X : a \wedge a' = 0$ and $a \vee a' = 1$. Furthermore in this section (and in the rest of the paper) we assume that $P = [p, 1]$ $(P' = [0, p'])$, $Q = [q, 1]$. We will also use again the notation $\wedge_q$, $\vee_p$ and $\rightarrow_p$ for the implication. Under the interval assumption we can obtain additional properties of $\wedge_q$ and $\vee_p$.

The $(X, \wedge_q, \vee_p)$ hyperalgebra can be characterized as a *weak* Boolean superlattice. By this we mean that it is a *weakly distributive* and *hypercomplemented* superlattice, i.e. in addition to S1-S6 the following properties hold

$$\text{weak distributivity} : a \wedge_q (b \vee_p c) \subseteq (a \wedge_q b) \vee_p (a \wedge_q c),$$
$$a \vee_p (b \wedge_q c) \subseteq (a \vee_p b) \wedge_q (a \vee_p c),$$
$$\text{hypercomplementation} : 1 \in (a \vee_p a'), \quad 0 \in (a \wedge_q a').$$

These properties hold in the general de Morgan case; now we see additional properties for the Boolean case.

Proposition 17.32 *For every $a, b, c \in X$, $i = 1, 2$ and $p \in X$ we have the following:*

1. $a \leq b \Rightarrow (a \rightarrow_p c) \geqslant_i (b \rightarrow_p c)$ *and* $a \leq b \Rightarrow (c \rightarrow_p a) \leqslant_i (c \rightarrow_p b)$,
2. $1 \in (0 \rightarrow_p a)$ *and* $a \in (1 \rightarrow_p a)$,
3. $(a \rightarrow_p b) = (b' \rightarrow_p a')$,
4. $1 \in (a \rightarrow_p a)$ *and* $a \in (1 \rightarrow_p a)$.
5. $\left(a \rightarrow^P (b \rightarrow_p c)\right) = (b \rightarrow_p (a \rightarrow_p c))$ *(strong exchange property);*

Proof. Parts 1 and 2 are proved exactly as in Proposition 17.30, except that now we use the order $\leqslant_1$. For 4 we have $(a \to_p a) = [a' \vee a, a' \vee a \vee p'] = [1, 1]$; for the second part we have $(1 \to_p a) = [1' \vee a, 1' \vee a \vee p'] = [a, a \vee p']$; the second part is proved similarly. For 5 (exchange property) we have

$$(a \to_p (b \to_p c)) = a' \vee_p [b' \vee c, b' \vee c \vee p'] = [a' \vee b' \vee c, a' \vee b' \vee c \vee p']$$
$$(b \to_p (a \to_p c)) = b' \vee_p [a' \vee c, a' \vee c \vee p'] = [a' \vee b' \vee c, a' \vee b' \vee c \vee p']$$

which are obviously equal. $\square$

Furthermore in the Boolean case the implication $\to_p$ has Modus Ponens, Modus Tollens and syllogistic reasoning properties.

Proposition 17.33 *For every $a, b, c \in X$ and $p, q \in X$ we have the following*

$$Modus\ Ponens : a \wedge_q b \subseteq a \wedge_q (a \to_p b) \qquad (17.19)$$
$$Modus\ Tollens : a' \wedge_q b' \subseteq b' \wedge^p (a \to_p b) \qquad (17.20)$$
$$Syllogistic\ Reasoning : (a \to_p b) \wedge_q (b \to_p c) \leqslant_2 (a \to_p c) \qquad (17.21)$$

Proof. For (17.19) we note that

$$a \wedge_q b = [a \wedge b \wedge q, a \wedge b]$$
$$a \wedge_q (a \to_p b) = a \wedge_q [a' \vee b, a' \vee b \vee p'] = [a \wedge (a' \vee b) \wedge q, a \wedge (a' \vee b \vee p')].$$

Now, $a \wedge (a' \vee b) \wedge q = (a \wedge a' \wedge q) \vee (a \wedge b \wedge q) = a \wedge b \wedge q$ and $a \wedge (a' \vee b \vee p') = (a \wedge a') \vee (a \wedge b) \vee (a \wedge p') = a \wedge (b \vee p') \geq a \wedge b$ which complete the proof of (17.19). We omit the proof of (17.20), which is similar. Regarding (17.21), it follows $(a \to_p b) = [a' \vee b, a' \vee b \vee p']$, $(b \to_p c) = [b' \vee c, b' \vee c \vee p']$, $(a \to_p c) = [a' \vee c, a' \vee c \vee p']$. Also, with $f = (a' \vee b) \wedge (b' \vee c) \wedge q$ and $g = (a' \vee c \vee p') \wedge (b' \vee c \vee p')$, we have

$$[a' \vee b, a' \vee b \vee p'] \wedge_q [b' \vee c, b' \vee c \vee p'] = [f, g].$$

Now

$$(a' \vee b) \wedge (b' \vee c) = (a' \wedge b') \vee (b \wedge b') \vee (a' \wedge c) \vee (b \wedge c) \leq a' \vee c;$$

since the first term in the middle expression above is less than a', the second is 0, the third term is less than a' and the last less than c. Also

$$(a' \vee b \vee p') \wedge (b' \vee c \vee p') = x \vee y \vee z$$

where

$$x = (a' \wedge b') \vee (b \wedge b') = (a' \wedge b') \leq a'$$
$$y = (a' \wedge c) \vee (b \wedge c) \vee (p' \wedge c) \leq c$$
$$z = (p' \wedge b') \vee (a' \wedge p') \vee (b \wedge p') \vee (p' \wedge p') \leq p'$$

hence $(a' \vee b \vee p') \wedge (b' \vee c \vee p') = a' \vee c \vee p'$ from which follows (17.21). $\square$

The $\rightarrow_p$ implication also induces an "order-like" relationship.

Proposition 17.34 *For all $a, b, p \in X$ we have:* $1 \in (a \rightarrow_p b) \Leftrightarrow a \vee p' \leq b \vee p'$

Proof. On the one hand, $1 \in (a \rightarrow_p b) = [a' \vee b, a' \vee b \vee p']$ implies that $1 = a' \vee b \vee p'$. Then

$$a = (a' \vee b \vee p') \wedge a = (b \vee p\,') \wedge a \Rightarrow a \leq b \vee p' \Rightarrow a \vee p' \leq b \vee p'.$$

Conversely,

$$a \vee p' \leq b \vee p' \Rightarrow a' \vee a \vee p' \leq a' \vee b \vee p' \Rightarrow 1 \leq a' \vee b \vee p' \Rightarrow 1 \in (a \rightarrow_p b).\,\square$$

This motivates us to define the relations $\leq_p, =_p$.

Definition 17.6 *For every $p \in X$ we define $\leq_p$ and $=_p$ as follows:*

$$a \leq_p b \Leftrightarrow a \vee p' \leq b \vee p', \qquad a =_p b \Leftrightarrow a \vee p' = b \vee p'.$$

Proposition 17.35 *For all $a, b, p \in X$ we have*

$$a \leq_p b \Leftrightarrow a \wedge p \leq b \wedge p, \qquad a =_p b \Leftrightarrow a \wedge p = b \wedge p. \tag{17.22}$$

Proof. In the one direction

$$a \leq_p b \Rightarrow a \vee p' \leq b \vee p' \Rightarrow (a \vee p') \wedge p \leq (b \vee p') \wedge p \Rightarrow$$
$$(a \wedge p) \vee (p' \wedge p) \leq (b \wedge p) \vee (p' \wedge p) \Rightarrow a \wedge p \leq b \wedge p.$$

Conversely,

$$a \wedge p \leq b \wedge p \Rightarrow (a \wedge p) \vee p' \leq (b \wedge p) \vee p' \Rightarrow$$
$$(a \vee p') \wedge (p \vee p') \leq (b \vee p') \wedge (p \vee p) \Rightarrow a \vee p' \leq b \vee p' \Rightarrow a \leq_p b.$$

This proves the first part of (17.22); the second part is proved similarly. $\square$

Proposition 17.36 *The relation $\leq_p$ is a* preorder *(i.e. it is reflexive and transitive) and $=_p$ is the natural equivalence indiced from $\leq_p$.*

Proof. Straightforward (also see [7]). $\square$

Let us now establish some properties of the equivalence relation $=_p$.

Notation 17.3 *For every $p \in X$ we denote by $\overline{a}^p$ the class of a under $=_p$, i.e. $\overline{a}^p = \{x : a \vee p' = x \vee p'\}$.*

Proposition 17.37 *For every $a, p \in X$, $\overline{a}^p$ is an interval. More specifically $\overline{a}^p = [a \wedge p, a \vee p']$.*

Proof. If $x \in \overline{a}^p$ then clearly $x \le x \vee p' = a \vee p'$. Also

$$x \vee p' = a \vee p' \Rightarrow (x \vee p') \wedge p = (a \vee p') \wedge p \Rightarrow x \wedge p = a \wedge p \Rightarrow x \ge a \wedge p.$$

Hence

$$\overline{a}^p \subseteq [a \wedge p, a \vee p']. \tag{17.23}$$

On the other hand, let us show that $\overline{a}^p$ is a convex sublattice. Indeed, $\overline{a}^p$ is a sublattice: take any $x, y \in \overline{a}^p$ then

$$\left. \begin{array}{l} a \vee p' = x \vee p' \\ a \vee p' = y \vee p' \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} a \vee p' = (x \vee p') \vee (y \vee p') = (x \vee y) \vee p' \\ a \vee p' = (x \vee p') \wedge (y \vee p') = (x \wedge y) \vee p' \end{array} \right.$$

and so $x \vee y, x \wedge y \in \overline{a}^p$. Further, take any $z \in [x \wedge y, x \vee y]$, then

$$(x \wedge y) \vee p' \le z \vee p' \le x \vee y \vee p' \Rightarrow$$
$$a \vee p' = (x \vee p') \wedge (y \vee p') \le z \vee p' \le (x \vee p') \vee (y \vee p') = a \vee p' \Rightarrow$$
$$z \vee p' = a \vee p'$$

and $(x \wedge y) \wedge p \le z \wedge p \le (x \vee y) \wedge p \Rightarrow$

$$a \wedge p = (x \wedge p) \wedge (y \wedge p) \le z \wedge p \le (x \wedge p) \vee (y \wedge p) = a \wedge p \Rightarrow z \wedge p = a \wedge p \tag{17.24}$$

So $z \in \overline{a}^p$ and $\overline{a}^p$ is a convex sublattice. And $a \wedge p$ and $a \vee p'$ belong to $\overline{a}^p$, hence

$$[a \wedge p, a \vee p'] \subseteq \overline{a}^p. \tag{17.25}$$

Combining (17.24) with (17.25) we get the desired result. $\square$

Proposition 17.38 *For all $p, a, b, c \in X$ we have:*

$$\overline{a}^p = \overline{b}^p \Rightarrow \left\{ \begin{array}{l} \overline{a \vee c}^p = \overline{b \vee c}^p \\ \overline{a \wedge c}^p = \overline{b \wedge c}^p \end{array} \right.$$

Proof. We only prove the first part (the second is proved similarly). We have

$$\overline{a}^p = \overline{b}^p \Rightarrow [a \wedge p, a \vee p'] = [b \wedge p, b \vee p'] \Rightarrow \left\{ \begin{array}{l} a \wedge p = b \wedge p \\ a \vee p' = b \vee p' \end{array} \right.$$

Now

$$a \vee p' = b \vee p' \Rightarrow a \vee c \vee p' = b \vee c \vee p' \tag{17.26}$$

and

$$\left. \begin{array}{l} b \wedge p = a \wedge p \\ c \wedge p = c \wedge p \end{array} \right\} \Rightarrow (a \wedge p) \vee (c \wedge p) = (b \wedge p) \vee (c \wedge p) \Rightarrow (a \vee c) \wedge p = (b \vee c) \wedge p. \tag{17.27}$$

From (17.26) and (17.27) we get

$$[(a \vee c) \wedge p, (a \vee c) \vee p'] = [(b \vee c) \wedge p, (b \vee c) \vee p'] \Rightarrow \overline{a \vee c}^p = \overline{b \vee c}^p. \square$$

We will prove a Proposition similar to Proposition 17.38 for $\vee_p$ and $\wedge_p$, but first we need the following.

Definition 17.7 *The* set *of classes $\overline{A}^p$ is defined as follows $\overline{A}^p = \{\overline{x}^p : x \in A\}$.*

Proposition 17.39 *For all $p, a, b \in X$ we have:*

$$\overline{a \vee_p b}^p = \{[(a \vee b) \wedge p, (a \vee b) \vee p']\}, \qquad \overline{a \wedge_q b}^q = \{[(a \vee b) \wedge q, (a \vee b) \vee q']\} \tag{17.28}$$

i.e. each of $\overline{a \vee_p b}^p, \overline{a \wedge_q b}^q$ contains a single class.

Proof. We only prove the first part of (17.28):

$$\overline{a \vee_p b}^p = \{\overline{x}^p : a \vee b \leq x \leq a \vee b \vee p'\} = \{[x \wedge p, x \vee p'] : a \vee b \leq x \leq a \vee b \vee p'\}$$

Take any x such that $\overline{x}^p \in \overline{a \vee_p b}^p$. Then $a \vee b \leq x \leq a \vee b \vee p'$ and so $a \vee b \vee p' \leq x \vee p' \leq a \vee b \vee p' \Rightarrow a \vee b \vee p' = x \vee p'$. Hence

$$(a \vee b) \wedge p \leq x \wedge p \leq (a \vee b \vee p') \wedge p = ((a \vee b) \wedge p) \vee (p' \wedge p) = (a \vee b) \wedge p$$

which implies $(a \vee b) \wedge p = x \wedge p$. Hence

$$\overline{x}^p = [x \wedge p, x \vee p'] = [(a \vee b) \wedge p, (a \vee b) \vee p'] = \overline{a \vee b}^p$$

and the proof is complete. $\square$

Now we prove the analog of Proposition 17.38.

Proposition 17.40 *For all $p, a, b, c \in X$ we have:*

$$\overline{a}^p = \overline{b}^p \Rightarrow \begin{cases} \overline{a \vee_p c}^p = \overline{b \vee_p c}^p \\ \overline{a \wedge_p c}^p = \overline{b \wedge_p c}^p \end{cases}.$$

Proof. Take any $\overline{x}^p \in \overline{a \vee_p c}^p, \overline{y}^p \in \overline{b \vee_p c}^p$. Then, by the previous Proposition,

$$\overline{x}^p = [(a \vee c) \wedge p, (a \vee c) \vee p'], \qquad \overline{y}^p = [(b \vee c) \wedge p, (b \vee c) \vee p'] \tag{17.29}$$

also $\overline{a}^p = \overline{b}^p \Rightarrow [a \wedge p, a \vee p'] = [b \wedge p, b \vee p']$ and so

$$\left\{ \begin{array}{l} a \wedge p = b \wedge p \\ a \vee p' = b \vee p' \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (a \vee c) \wedge p = (b \vee c) \wedge p \\ (a \vee c) \vee p' = (b \vee c) \vee p' \end{array} \right\}. \tag{17.30}$$

From (17.29) and (17.30) we obtain the first part of the proposition; the second part is proved similarly. $\square$

Proposition 17.41 *For all $p, a, b, c \in X$ we have:*

$$\left. \begin{array}{l} \overline{a \vee c}^p = \overline{b \vee c}^p \\ \overline{a \wedge c}^p = \overline{b \wedge c}^p \end{array} \right\} \Rightarrow \overline{a}^p = \overline{b}^p \quad and \quad \left. \begin{array}{l} \overline{a \vee_p c}^p = \overline{b \vee_p c}^p \\ \overline{a \wedge_{p'} c}^p = \overline{b \wedge_{p'} c}^p \end{array} \right\} \Rightarrow \overline{a}^p = \overline{b}^p.$$

Proof. We have $\overline{a \vee c}^p = \overline{b \vee c}^p \Rightarrow$

$$\left. \begin{array}{l} (a \vee c) \wedge p = (b \vee c) \wedge p \\ (a \vee c) \vee p' = (b \vee c) \vee p' \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (a \wedge p) \vee (c \wedge p) = (b \wedge p) \vee (c \wedge p) \\ (a \vee p') \vee (c \vee p') = (b \vee p') \vee (c \vee p') \end{array} \right. .$$
$$(17.31)$$

Similarly, $\overline{a \wedge c}^p = \overline{b \wedge c}^p \Rightarrow$

$$\left. \begin{array}{l} (a \wedge c) \wedge p = (b \wedge c) \wedge p \\ (a \wedge c) \vee p' = (b \wedge c) \vee p' \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (a \wedge p) \wedge (c \wedge p) = (b \wedge p) \wedge (c \wedge p) \\ (a \vee p') \wedge (c \vee p') = (b \vee p') \wedge (c \vee p') \end{array} \right. .$$
$$(17.32)$$

From the first parts of (17.31), (17.32) we get $a \wedge p = b \wedge p$ and from the second parts $a \vee p' = b \vee p'$, which together imply $\overline{a}^p = \overline{b}^p$. $\square$

17.7 Conclusion

We have introduced two crisp *hyperoperations*, $\wedge^Q$ and $\vee^P$, which are natural multi-valued generalizations of the t-norm $\wedge$ and the t-conorm $\vee$. The hyperoperations depend on the sets Q and P. In the special case $Q = [q, 1]$, $P = [0, p]$, we have the hyperoperations denoted as $\wedge_q$ and $\vee_p$ Clearly, the new hyperoperations have a great potential for applications to computational intelligence, where they can extend the concepts and procedures of fuzzy reasoning.

  We intend to further pursue our research especially in the following directions: a formulation of the orders $\leqslant_1, \leqslant_2$ in terms of the zeta function discussed in [13] and secondly an in-depth study of the multi-valued implication $\rightarrow^P$ along the lines of [9].

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