Tameness for the distribution of sums of Markov random variables

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Abstract

This paper studies the spectral properties of a class of probability measures on the circle. The key aim is to describe the local structure of the maximal ideal space on the $L$-subspaces generated by the measures and hence the spectral properties of these measures. In particular we give a necessary and sufficient condition for such measures to belong to $M_{M}(\mathbf{T})$.

1. Introduction

Within the measure algebra $M_{M}(\mathbf{T})$ of regular Borel measures on the circle group, we describe the spectral properties of a particular class of probability measures. The measures in question are of the form

$$d\mu = \prod_{n=1}^{\infty} \left( 1 + \frac{a_n + b_n}{2} r_{n+1}(x) + \frac{a_n - b_n}{2} r_n(x) r_{n+1}(x) \right) d\lambda,$$

where $r_n$ denotes the $n$th Rademacher function, $\lambda$ is Lebesgue measure on $[0,1]$ and the sequences $(a_n)$ and $(b_n)$ are in the interval $[-1,1]$. In fact we shall consider a somewhat more general class of measures than these by replacing Rademacher functions by corresponding functions for $r$-adic expansions of the members of $[0,1]$. Measures of this kind have been studied by Bisbas and Karanikas in $[BK1]$ and $[BK2]$. There they were shown to be singular when

$$\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

diverges and, for all $n$,

$$|a_n - b_n| < C < 2.$$

In fact it can be easily seen, from the results of $[BK1]$ and $[BK2]$, that, when (3) is assumed, (2) is both necessary and sufficient. There also a formula was given which estimates the minimum Hausdorff dimension of sets of positive $\mu$ measure in terms of the entropy of the Markov chain. We shall refer to these measures as Markov measures.

As we shall see later, these measures can be thought of as being generated by a Markov chain. In fact they are sums of the form $\sum_{n=1}^{\infty} 2^{-n} X_n$, where $(X_n)$ is a (non-homogeneous) Markov chain taking values 0, 1. Billingsley$[B]$ has defined such measures; in fact his definition is less restrictive on the type of stochastic process.

Tameness is a property of measures akin to, but in some senses sharper than, an ergodic condition. In fact, on the way to proving it we shall show that these measures are ergodic for the action of the group of elements of the circle of the form
Tameness says that, for the measure in question, the values of the complex homomorphisms of the Banach algebra $M(T)$ on the space of measures $\nu$ absolutely continuous with respect to $\mu$ are of the form
\[ \nu \rightarrow a\hat{\nu}(n), \]
where $\hat{\nu}$ is the Fourier–Stieltjes transform of the measure $\nu$ at the integer $n$ and $a$ is a complex number in the unit disc. If the only possible choices of $a$ are 0 and 1 then it is a consequence of a theorem of Taylor\[T\] that $\mu$ is in the radical; that is, $\|\mu^n\|^{1/n}$ tends to zero in the quotient algebra $M(T)/L^1(T)$. Roughly speaking this means that the powers of $\mu$ approach absolutely continuous measures. From the point of view of the study of the spectral theory of measures, such measures are trivial. The same conclusion holds (for the same reason) if the set of allowable $a$’s does not contain any number inside the unit disc except for 0. Thus non-trivial tame measures have at least one $a$ satisfying $0 < |a| < 1$ and by a simple argument then all elements of the unit disc are allowable.

Tameness was first exhibited by J. L. Taylor\[T\] for measures in the group $\Pi \mathbb{Z}(3)$ which is an infinite product of copies of $\mathbb{Z}(3)$. He used this result to prove that the Silov boundary of the algebra of $M(\Pi \mathbb{Z}(3))$ is a proper subset of the maximal ideal space. The same result was obtained more generally by Johnson\[J\] by showing the existence of tame measures on a sufficiently rich class of locally compact abelian group. The measures considered by Taylor and Johnson were infinite convolutions of discrete measures. Brown and Moran\[BM1\] were able to show that large classes of such infinite convolutions are tame and subsequently Brown\[BR\] showed that most Riesz product measures are also tame. Tameness carries with it a lot of information about the behaviour of the measure under convolution both with itself and with discrete measures. We refer the reader to [BM1] and to our final section for more details, but in particular it implies that the measure in question exhibits the Wiener–Pitt phenomenon. It also implies that these measures satisfy purity laws of various kinds, thus, for example, they are either in the radical or singular to every measure in the radical. We describe some of the consequences of tameness in the final section of the paper.

From a spectral viewpoint, tame measures are the simplest genuinely singular measures. It is of interest then to generate examples of such measures which lie outside the classes already known.

Our main aim in this paper is to exhibit a large subclass of the class of Markov measures which are tame. In addition we shall give necessary and sufficient conditions for these measures to belong to the class $M_0(T)$ of measures whose Fourier–Stieltjes transform vanishes at infinity.

2. Preliminaries

First we provide the description of the measures of the form given in (1) in terms of a Markov chain. Define $P^{(0)}$ to be the (row) vector $(\frac{1}{2}, \frac{1}{2})$ and $P^{(n)} = (p_{ij}^{(n)})$ to be the $2 \times 2$ matrix
\[ P^{(n)} = \begin{pmatrix} \frac{1+a_n}{2} & \frac{1-a_n}{2} \\ \frac{1+b_n}{2} & \frac{1-b_n}{2} \end{pmatrix}. \]
Let \( E_{n,k}(x) \) be the interval of the form \([k/2^n, (k+1)/2^n)\) containing \( x \). Then define a measure \( \nu \) on \([0,1]\) by
\[
\nu(E_{n,k}(x)) = p^{(0)}_{\varepsilon(x)} \prod_{j=1}^{n-1} p^{(j)}_{\varepsilon_j(x), \varepsilon_{j+1}(x)},
\]
where \( \varepsilon_j(x) \) is the \( j \)th digit in the binary expansion of \( x \). This measure is well-defined by the Theorem of Carathéodory and it is relatively straightforward to check that it coincides with \( \mu \) as defined in equation (1).

It is now clear how to generalise the ideas to bases other than 2. Let \( r \) be an integer not less than 2 and let
\[
P^{(0)} = (p^{(0)}_i), \quad i \in \{0, 1, \ldots, r-1\}
\]
be the vector of length \( r \)
\[
P^{(0)} = \left( \frac{1}{r}, \ldots, \frac{1}{r} \right).
\]
Suppose, also, that we have a (not necessarily homogeneous) Markov chain with state space \( \{0, 1, \ldots, r-1\} \) given by the sequence of \( r \times r \) transition matrices
\[
P^{(n)} = (p^{(n)}_{ij}), \quad i,j \in \{0, 1, \ldots, r-1\}, \quad n = 1, 2, \ldots.
\]
It will be convenient to use the notation
\[
P^{(n,m)} = P^{(n)}, P^{(n+1)} \ldots P^{(m)}
\]
and \( E_{n,k}(x) \) for the interval
\[
\left[ \frac{k}{r^n}, \frac{k+1}{r^n} \right) \quad (n = 1, 2, \ldots, k = 0, 1, \ldots, r^n - 1)
\]
in the \( n \)th generation of the \( r \)-adic partition of \([0,1]\) which contains \( x \in \mathbb{T} = [0,1) \). We write the \( r \)-adic expansion of \( x \) as
\[
x = \sum_{j=1}^{\infty} \varepsilon_j(x) r_j, \quad \varepsilon_j(x) \in \{0, 1, \ldots, r-1\}.
\]
Again by the Carathéodory Theorem the measure \( \mu \) is determined by its values on \( E_{n,k} \) and these are now defined to be
\[
\mu(E_{n,k}(x)) = p^{(0)}_{\varepsilon(x)} \prod_{j=1}^{n-1} p^{(j)}_{\varepsilon_j(x), \varepsilon_{j+1}(x)}.
\]
It is clear that the measure is well-defined and is a probability measure on the circle group. We shall need some other descriptions of measures of this type.

Let \( (P^{(n)}) \) be the sequence of transition matrices of a Markov chain on the state space \( \{0, 1, \ldots, r-1\} \). Suppose that \( \rho_n \) is a continuous probability measure concentrated on \([0, r^{-n+1})\) and write
\[
\rho_n = \tau^{n}_{0} + \tau^{1}_n + \cdots + \tau^{r-1}_n,
\]
where \( \tau^{i}_n \) (\( i = 0, \ldots, r-1 \)) is concentrated on \( \left[ \frac{i}{r^n}, \frac{i+1}{r^n} \right) \) and we assume that \( \tau^{i}_n \) has mass \( 1/r \) for each \( i \). Define measures
\[
\tau^{i}_{n-1} = \sum_{j=0}^{r-1} p^{(n-1)}_{ij} \tau^{i}_n \delta \left( \frac{i}{r^{n-1}} \right), \quad i = 0, 1, \ldots, r-1,
\]
where \( \delta \left( \frac{i}{r^{n-1}} \right) \) is the point mass at \( \frac{i}{r^{n-1}} \).
where $\delta(i/r^{n-1})$ is the point mass at $i/r^{n-1}$ and $\ast$ is the symbol for convolution of measures, and note that $\tau_{n-1}^i$ is a measure on $[i/r^{n-1}, (i+1)/r^{n-1})$ which, because the matrix $P^{(n)}$ is stochastic, has mass $1/r$. Now define
\[
\rho_{n-1} = T_{n-1}(\rho_n) = \sum_{i=0}^{r-1} \tau_{n-1}^i.
\]
It is clear that $\rho_{n-1}$ has mass 1 at $[0, r^{-n+2}]$. It is convenient to rewrite these measures in terms of products of vector and matrix valued measures. Let $\mathbf{t}_n$ be the column vector of measures $(\tau_n^0, \tau_n^1, \ldots, \tau_n^{r-1})^T$, and let $Q^{(n-1)}$ be the matrix of measures $(\rho_n^{(n-1)} \delta(i/r^{n-1}))$. Then, using (5) and writing $E$ for the row vector of length $r$ all of whose entries are 1,
\[
\mathbf{t}_{n-1} = Q^{(n-1)} \ast \mathbf{t}_n
\]
and $\rho_{n-1} = E \mathbf{t}_{n-1}$. In a similar way we define $\rho_{n-2}, \ldots, \rho_2, \rho_1$ and $T_{n-2}, \ldots, T_1$ such that
\[
T_{k-1}(\rho_k) = \rho_{k-1}, \quad k = 2, 3, \ldots, n-1.
\]
Let
\[
v_n = T_1(T_2(\ldots(T_{n-1}(\rho_n))\ldots));
\]
thus, in the measure-valued matrix picture,
\[
v_n = EQ^{(1)} \ast EQ^{(2)} \ast \cdots \ast EQ^{(n-1)} \ast \mathbf{t}_n = EQ^{(1,n-1)} \ast \mathbf{t}_n.
\]
Clearly $v_n$ is a probability measure and has mass one on the interval $[0, 1]$. Observe, as in (4), that, for $m \geq n$,
\[
v_m(E_n, x) = p^{(0)}_{n, n} \prod_{j=1}^{n-1} p^{(j)}_{n, j}(x),
\]
for any $n$ and any $x$. Here we may take $\rho_n$ to be any continuous probability measure on the interval $[0, r^{-n+1}]$ which assigns equal mass to each of the intervals $[ir^{-n}, (i+1)r^{-n})$ where $i = 0, 1, \ldots, r-1$, but for definiteness we take it initially to be Lebesgue measure on $[0, r^{-n+1})$. In view of (6), the weak * limit of the sequence $(v_n)$ exists and is equal to $\mu$. Once $\mu$ is defined in this way we may choose $\rho_n$ to be the weak* limit of the sequence whose $m$th term is
\[
\lim_{n \to \infty} T_n(T_{n+1}(\ldots(T_{n+m-1}(\rho_{n+m-1})\ldots)) = EQ^{(n+m-1)} \ast \mathbf{t}_{n+m}.
\]
This measure has the required properties for $\rho_n$ and we now have the identity
\[
\mu = T_1(T_2(\ldots(T_{n-1}(\rho_n))\ldots) = EQ^{(1,n-1)} \ast \mathbf{t}_n.
\]
Yet another description of these measures is as the distributions of sums of random variables
\[
X = \sum_{n=1}^{\infty} X_n, \quad r^n,
\]
where $(X_n)$ is a Markov chain with state space $\{0, 1, \ldots, r-1\}$ and the transition probabilities are given by the matrices $P^{(n)}$. We shall not use this representation in this paper. However it does show that these measures arise in a very natural way.

We shall, throughout the paper, make the added assumption that the Markov chain is ergodic in the sense that the products $P^{(N,n)}$ become stationary as $n \to \infty$ for
Markov random variables

all $N$, by which we mean that the difference of any pair of row vectors of this product matrix tends to 0. Within the condition of ergodicity we shall add an extra (non-triviality) hypothesis that the limit of any convergent subsequence of row vectors contains at least two non-zero terms at positions $i_0$ and $i_1$ and that for these indices $\liminf p_{i_0}^{(n)}$, $\liminf p_{i_1}^{(n)}$, and $\liminf p_{i_0, i_1}^{(n)}$ are all positive. We call measures generated according to (7) by an ergodic Markov chain of this kind ergodic Markov measures. It is these measures which will be the object of our attention here. There are many criteria which guarantee ergodicity in the sense that we have defined it. One such is that, for some $\varepsilon > 0$,

$$\varepsilon \leq p_{ij}^{(n)} \leq 1 - \varepsilon,$$

(see [S], p. 116). Other criteria are also found there.

3. Tameness

Before proving tameness, we shall show that $\mu$ is ergodic for the action of the subgroup of the circle generated by the elements of the sequence $(r^{-n})$. We call this group $D$.

**Proposition 3.1.** Let $F$ be any Borel set of $T$ which is invariant under the action of $D$. Then $\mu(F)$ is either 0 or 1.

**Proof.** It is enough to show that any real-valued Borel function which is invariant under $D$ is constant almost everywhere with respect to $\mu$. Let $g$ be such a function. We shall show that, for any interval of the form

$$F = F(\xi_1, \xi_2, \ldots, \xi_n) = \{x : \xi_i(x) = \xi_i, \quad i = 1, 2, \ldots, n\},$$

we have

$$\int_F g(x) \, d\mu(x) = \mu(F) \int_T g(x) \, d\mu(x).$$

By the $D$-invariance of $g$ and (7),

$$\int_F g(x) \, d\mu(x) = p^{(0)}_{i_1} p^{(1)}_{i_1} \cdots p^{(n-1)}_{i_{n-1}} = \prod_{j=n+1}^{N} P^{(0)} \cdot (a^{(N+1)}_0, \ldots, a^{(N+1)}_{N-1})^T = \mu(F) \int_T g(x) \, d\rho_{N+1}(x), \quad (8)$$

where $a^{(N+1)}_i = \int g \, d\tau_{N+1}^i$. Further, by the ergodic property of the Markov chain, $\prod_{j=n+1}^{N} P^{(0)}$ becomes stationary as $N \to \infty$. In particular, $EP^{(n+1,N)} \sim EP^{(1,N)}$. Together with (8), this yields

$$\int_F g(x) \, d\mu(x) = \lim_{N \to \infty} \mu(F) \cdot EP^{(1,N)} (a^{(N+1)}_0, \ldots, a^{(N+1)}_{N-1})^T = \mu(F) \int_T g(x) \, d\mu(x)$$

as claimed.
Recall that a probability measure $\mu$ is tame, see [GMG], if for each generalized character $\psi$ of $M(\mathbb{T})$ there is an $a \in \mathbb{C}$ and a continuous character $\gamma(t)$ on $\mathbb{T}$ such that the local coordinate, $\psi_\mu \in L^\infty(\mu)$, at $\mu$ of $\psi$ satisfies

$$\psi_\mu(t) = a_\gamma(t) \quad a.e. \, d\mu.$$ 

Our main theorem follows.

**Theorem 1.** Let $\mu$ be an ergodic Markov measure as in (4). Then $\mu$ is tame.

**Proof.** Let $f$ be any generalized character of $M(\mathbb{T})$. Replacing $f$ by $\gamma f$ for some $\gamma \in \mathbb{T}$, we may assume that $\mu f_\# \neq 0$, for otherwise $f$ is identically 0 and there is nothing to prove. The map $\eta$ from $D$ to $C$

$$\eta: d \to f_\#(d): D \to C,$$

is a complex homomorphism of $D$. It will be enough to show that this is continuous, since, if so, it extends to a member $\gamma$ of $\mathbb{T}$ and then $\gamma f_\#$ is a $D$-invariant function and so constant (= $a$, say) almost everywhere with respect to $\mu$ by the preceding result. As a result $f_\# = a_\gamma, a.e. \mu$ and we are done. There remains the proof of continuity of $\eta$ and this is given in the next two lemmas.

Before proving the continuity of $\eta$ we develop some notation. First observe that (7) implies that the Gelfand transform, also denoted by $\mu_\#$, satisfies

$$\mu_\#(f) = \frac{1}{\pi} \int f(	au_{\lambda n} f) \Gamma_n^T \Gamma_n d\mu.$$ 

We write

$$\Delta_n = \frac{1}{\pi} \int f_\#(d) \Gamma_n^T \Gamma_n d\mu,$$

where $\eta_n = \eta(i\tau^{-n})$ and $\Gamma_n$ is the column vector

$$\Gamma_n = (\tau_{\lambda n}^0(f), \tau_{\lambda n}^1(f), \ldots, \tau_{\lambda n}^{r-1}(f))^T.$$ 

Thus, for all $N$,

$$\hat{\mu}(f) = \frac{1}{\pi} \prod_{j=1}^N \Delta_j \Gamma_N \neq 0. \quad (10)$$

**Lemma 3.1.** Let, for $\lambda = 0, 1, \ldots, r-1$ and $n = 1, 2, \ldots,$

$$a_0^\lambda = 1,$$

$$a_{\lambda n+1}^\lambda = \sum_{i=0}^{r-1} p_{i\lambda}^{(n)} a_i^\lambda z_n^\lambda.$$ 

Then

$$A = \lim_{n \to \infty} \sum_{\lambda = 0}^{r-1} |a_n^\lambda|$$

exists and is non-zero. Furthermore, there are at least two values of $\lambda$ for which

$$\lim_{n \to \infty} |a_n^\lambda| > 0.$$ 

**Proof.** By (11), for each $\lambda$ and any $n = 1, 2, \ldots,$

$$|a_{\lambda n+1}^\lambda| \leq \sum_{i=0}^{r-1} p_{i\lambda}^{(n)} |a_n^\lambda|.$$
and, since
\[
\sum_{\lambda=0}^{r-1} p_{\lambda}^{(n)} = 1,
\]
\[
\sum_{\lambda=0}^{r-1} |a_{\lambda+1}^n| \leq \sum_{\lambda=0}^{r-1} \sum_{i=0}^{r-1} p_{\lambda}^{(n)} |a_i^n| = \sum_{i=0}^{r-1} |a_i^n|
\]
and so from (10) for some \(A > 0\),
\[
\lim_{n \to \infty} \sum_{\lambda=0}^{r-1} |a_{\lambda+1}^n| = A.
\]

Suppose that there is a subsequence of \((a_{\lambda_k}^n), (a_{m_k}^n)\) which tends to 0 as \(k \to \infty\), where \(\lambda\) is \(i_0\) or \(i_1\). Then
\[
\sum_{i+\lambda} |a_{m_k}^n| \to A, \quad k \to \infty.
\]
Now choose a subsequence \((n_k)\) of \((m_k)\) such that the products \(P^{(n_k, n_{k-1})}\) converge to a matrix with rows of the form \((u_{n_k}, u_{n_k}, \ldots, u_{n_k+1}^n)\) where \(u_{n_k}\) and \(u_{n_k+1}\) are non-zero. This is a consequence of the ergodic property of the Markov chain. We let \(\epsilon = \min(u_{i_0}, u_{i_1})\).

Let \(F_n = (|a_{n_k}^0|,\ldots,|a_{n_k}^{r-1}|)\) and \(H_{\lambda}\) be a column vector with all entries 1 except in position \(\lambda\) where we have 0. It is not difficult to see that, for any \(k = 1, 2, \ldots\) and \(n \leq n_k - 1\),
\[
\sum_{i+\lambda} |a_{n_k}^i| \leq F_n P^{(n_k, n_{k-1})} H_{\lambda},
\]
and so
\[
\limsup_{k \to \infty} \sum_{i+\lambda} |a_{n_k}^i| \leq \left(\sum_{i=0}^{r-1} |a_i^n|\right) (u_{i_0}, u_{i_1}, \ldots, u_{n_k+1}) H_{\lambda}.
\]
Because \(\lambda\) is one of \(i_0\) and \(i_1\), letting \(n\) tend to infinity, we obtain
\[
\limsup_{k \to \infty} \sum_{i+\lambda} |a_{n_k}^i| \leq A(1 - \epsilon) < A
\]
which contradicts (12). This completes the proof.

We need the following lemma to finish the proof of the theorem.

**Lemma 3.** If \(f\) is a generalized character such that \(\hat{f}(\hat{f}) \neq 0\), then the restriction, \(\eta\), of \(f\) to the group \(D\) extends to a continuous character on \(T\).

**Proof.** For any \(\lambda = 0, \ldots, r-1\) and \(n = 1, 2, \ldots\), let
\[
a_{\lambda}^n = |a_{\lambda}^n| e^{i\theta_{n, \lambda}},
\]
where \(\theta_{n, \lambda} \in [0, 2\pi]\). Then we have, using (11),
\[
\sum_{\lambda=0}^{r-1} |a_{\lambda+1}^n| = \sum_{\lambda=0}^{r-1} \sum_{j=0}^{r-1} p_{\lambda j}^{(n)} a_{\lambda}^j z_{\lambda}^j = \sum_{\lambda=0}^{r-1} \sum_{j=0}^{r-1} |p_{\lambda j}^{(n)}| |a_{\lambda}^j| e^{i\theta_{n, j} z_{\lambda}^j}.
\]
(13)

Choose, using the previous lemma, two values \(\lambda = i_0, i_1\) such that \(\lim_{n \to \infty} |a_{\lambda}^n| > 0\), and note that, also by that lemma, both sides of (13) tend to \(A \neq 0\). There is no loss of generality in choosing \(i_0 = 0\). Using the ergodic property of the Markov chain and Lemma 3.1 we have that, for \(j = 1, 2, \ldots, r-1\) and \(\lambda = i_0, i_1\),
\[
p_{\lambda j}^{(n)} |a_{\lambda}^j| (e^{i\theta_{n, j} z_{\lambda}^j} - 1) \to 0 \quad \text{as} \quad n \to \infty.
\]
(14)
Since

\[ \left| a_{n+1}^\lambda \right| = \left| \sum_{j=0}^{r-1} p_{j,k}^\lambda |a_n^j| e^{i\theta_n,j} z_n^j \right| \]

we have, using (11) and (13), for any \( \lambda = 0, 1, \ldots, r-1, \)

\[ \left| a_{n+1}^\lambda \right| - \sum_{j=0}^{r-1} p_{j,k}^\lambda |a_n^j| \to 0, \quad \text{as} \quad n \to \infty. \]  \hspace{1cm} \text{(15)}

Moreover, for any \( \lambda, \)

\[ \left| a_{n+1}^\lambda \right| e^{i\theta_n,\lambda} - \left( \sum_{j=0}^{r-1} p_{j,k}^\lambda |a_n^j| e^{i\theta_n,j} z_n^j \right) e^{i\theta_n, \lambda} = 0 \]
and so from (14) and (15) we have

\[ \left| a_{n+1}^\lambda \right| \left( e^{i\theta_n,\lambda} - 1 \right) \to 0 \quad \text{as} \quad n \to \infty, \quad \lambda = i_0, i_1. \]

In particular,

\[ (\theta_{n+1, i_1} - \theta_{n, i_1}) \to 0 \mod 2\pi, \quad (\theta_{n+1, i_0} - \theta_{n, i_0}) \mod 2\pi \quad \text{as} \quad n \to \infty \]
and using the ergodic property of the Markov chain with (14) we obtain

\[ z_n^{i_1} \to 1 \quad \text{as} \quad n \to \infty. \]
It follows that the character \( \eta^{i_1} \) of \( D \) is continuous. Evidently then \( \eta \) is also continuous and we are finished.

4. Measures in \( M_q(T) \)

Recall that a measure \( \mu \) belongs to \( M_q(T) \) if its Fourier–Stieltjes coefficients \( \hat{\mu}(n) \) tend to 0 as \( |n| \to \infty \). The aim of this section is to characterise the ergodic Markov measures which belong to \( M_q(T) \). The result generalises the theorem proved by Blum and Epstein\[BE\] (cf. Brown and Moran\[BM2\]). They studied the case when the Markov chain is replaced by a sequence of independent random variables. Before stating the theorem, we introduce the notation \( A_r \) for the \( r \times r \) matrix all of whose entries are 1.

**Theorem 2.** Let \( \mu \) be an ergodic Markov measure. The \( \mu \) belongs to \( M_q(T) \) if and only if \( P^{(n)} \to A_r \) as \( n \to \infty \).

Its proof involves a sequence of lemmas and is based on the criterion of Rajchman. According to this (see [Z] II, pp. 144–145), a necessary and sufficient condition for a measure \( \mu \) to be in \( M_q(T) \) is that, for the characteristic function \( \chi_I \) of any interval \( I \subset [0, 1], \)

\[ \lim_{|n| \to \infty} \sum_{T} \chi_I(nT^\mu) d\mu(x) = \lambda(I) \hat{\mu}(0). \]

It will be convenient to re-express this criterion in terms of generalisations of the Rademacher functions, defined as follows:

For each \( i = 0, 1, \ldots, r-1 \) \( (r > 2) \) we define the sequence of functions

\[ R^i_n(x) = 1 - r \delta_{n(x), i}, \quad n = 1, 2, \ldots, \quad x \in [0, 1]. \]
Markov random variables

where $\delta$ is the Kronecker delta. For $r = 2$ this coincides with the usual sequence at Rademacher functions $r_n(x)$. The following lemma is now a simple consequence of the Rajchman criterion.

**Lemma 4.3.** Let $\mu$ be an ergodic Markov measure. If $\mu \in M_0(T)$ then, for $i, j \in \{0, \ldots, r - 1\}$

(1) $\lim_{n \to \infty} \int_T R^i_n(t) d\mu(t) = 0$, and

(2) $\lim_{n \to \infty} \int_T R^i_n(t) R^j_{n+1}(t) d\mu(t) = 0$.

**Proof.** We merely have to remark that the functions $R^i_n$ and $R^i_n R^j_{n+1}$ are step functions whose Lebesgue integrals are 0.

**Lemma 4.4.** Let $\mu$ be an ergodic Markov measure. Then, for $i, j \in \{0, \ldots, r - 1\}$ and $m < n$,

$$\int_T R^i_m(x) R^j_n(x) d\mu(x) = P^{(m, n - 1)} I_i E T,$$

$$\int_T R^i_m(x) d\mu(x) = P^{(m, n - 1)} I_i E T,$$

where $I_i$ (respectively $I_j$) is the identity matrix in all positions except position $(i, i)$ (respectively $(j, j)$) where 1 is replaced by $1 - r$.

**Proof.** This is done by direct calculation. The reader is referred to [BK1] and [BK2] for details.

**Lemma 4.5.** Let $(P^{(n)})$ be an ergodic Markov chain and $\mu$ the corresponding ergodic Markov measure. Suppose that $P^{(n)}$ does not tend to $A_r$. Then there is some subsequence $(n_k)$ and some $i, j \in \{1, \ldots, r\}$ such that one of the following limits exists and is not 0:

$$\lim_{k \to \infty} \int_T R^i_{n_k}(t) d\mu(t), \quad \lim_{k \to \infty} \int_T R^i_{n_k}(t) R^j_{n_k+1}(t) d\mu(t), \quad \lim_{k \to \infty} \int_T R^i_{n_k+1}(t) d\mu(t).$$

**Proof.** Let $(n_k)$ be a sequence such that for all $i$, all of the sequences in (16) converge. To obtain a contradiction we assume that they all converge to 0. Further we assume, by taking a subsequence if necessary, that $P^{(n_k)}$ converges to a matrix $P = (p_{ij})$ other than $A_r$ and that $P^{(0, n_k - 1)}$ converges to some matrix of the form $(s_0, s_1, \ldots, s_{r-1})$. Now, by Lemma 4.4,

$$\int_T R^i_{n_k}(t) d\mu(t) \sim (s_0, \ldots, s_{r-1}) I_i E T$$

and so, since, for all $i = 0, 1, \ldots, r - 1$,

$$\lim_{k \to \infty} \int_T R^i_{n_k} d\mu(t) = 0,$$

we have

$$s_i = \frac{1}{r} i = 0, \ldots, r - 1.$$
Next, also by Lemma 4.4,

$$\int_T R_{n_k}^i(t) R_{n_k+1}^i(t) \, d\mu(t) = P^{(s, n_k-1)} R_{n_k}^i \sim (s_0, \ldots, s_{r-1}) R_{n_k}^i \mu(t).$$

We use (17) to obtain, for all $i, j = 0, \ldots, r-1$,

$$\int_T R_{n_k}^i(t) R_{n_k+1}^i(t) \, d\mu(t) = r p_{ij} - \sum_{k=0}^{r-1} p_{kj}. \tag{18}$$

Since, for all $i, j$,

$$\lim_k \int_T R_{n_k}^i(t) R_{n_k+1}^i(t) \, d\mu(t) = 0,$$

we have from (18) that

$$p_{ij} = \frac{1}{r} \sum_{k=0}^{r-1} p_{kj};$$

in other words, the limit matrix $P$ is stationary. Finally, we observe that

$$\int_T R_{n_k+1}^i(t) \, d\mu(t) = P^{(s, n_k-1)} P^{(s_{n_k})} \sim (s_0, \ldots, s_{r-1}) P \mu(t).$$

where $s_i = 1/r$ and $P$ is stationary. Consequently, using the fact that

$$\lim_{k \to \infty} \int_T R_{n_k+1}^i(t) \, d\mu(t) = 0,$$

for all $j$, we find that

$$\int_T R_{n_k+1}^j(t) \, d\mu(t) \sim 1 - r p_{0j}$$

and so $p_{0j} = 1/4, j = 0, \ldots, r-1$. This yields

$$\lim_{k \to \infty} P^{(s_{n_k})} = A_r$$

and a contradiction.

We note that it is possible to obtain another proof of this part of Theorem 2 using Lyons criterion that $\mu$ belongs to $M(T)$ if and only if it annihilates $W$-sets (see [L1] and [L2]). The proof uses techniques similar to those in theorems 1 and 2 of [BK2] and we omit it.

There remains the proof that the condition is sufficient for $\mu$ to belong to $M(T)$. This is contained in the following two lemmas.

**Lemma 4.6.** Let $\mu$ be an ergodic Markov measure and suppose that the transition matrices satisfy $P^{(s_{n_k})} \to A_r$ as $n \to \infty$. If there is a sequence $(\gamma_n)$ of characters such that $\hat{\mu}(\gamma_n) \to c \neq 0$, then there is such a sequence where $\gamma_n(d) \to 1$ for all $d \in D$.

**Proof.** By taking a subsequence, if necessary, we may assume that $(\gamma_n)$ converges in the weak$^*$ topology on $L^\infty(\mu)$. Its limit must be the local coordinate of some generalised character and so, by tameness, must be of the form $a \gamma$ for some character $\gamma$ of $T$ and some $a \neq 0$ in the unit disc. Now we replace $(\gamma_n)$ by $(\gamma \gamma_n)$ to obtain a sequence, also called $(\gamma_n)$, which converges weak$^*$ to $a$. We shall show that
then some subsequence is of the form \( \mu \) to both \( d \) to show this for a few comments about the structure of theorems for the behaviour of ergodic Markov measures. To do this we need to make matrices satisfy \( P(d) \) to \( x \) (using the fact that \( \nu \) is absolutely continuous with respect to \( \delta(d) * \mu \)) and on the other hand to \( a \). Thus \( z = 1 \) and \( \gamma_n(d) \rightarrow 1 \) as \( n \rightarrow \infty \).

It will be sufficient then to show that \( \delta(d) * \mu \) for all \( d \in D \). In fact, it is enough to show this for \( d = r^n \) for \( n \) sufficiently large since the set of \( d \in D \) on which \( \gamma_n(d) \rightarrow 1 \) is clearly a subgroup of \( D \). However, since \( P^{(n)} \rightarrow A_r \), all of the entries in \( P^{(n)} \) are eventually positive, and now the form of \( \mu = EQ^{(1,n-1)} * t_n \) given in (7) reveals that this is the case. This completes the proof of the lemma.

Before giving the final lemma, we remark that, because of Lemma 4, if

\[
\hat{\mu}(\gamma_n) \rightarrow c \neq 0
\]

then some subsequence is of the form \( \gamma_n(t) = \exp 2\pi i m_k r^k \), where \( r \) does not divide \( m_k \) and \( r \) tends to infinity. Thus to complete the proof of the theorem it will be enough to show that \( \hat{\mu}(\gamma_n) \rightarrow 0 \) as \( k \rightarrow \infty \). This lemma shows that this is the case.

**Lemma 4.7.** Let \( \mu \) be an ergodic Markov measure and suppose that the transition matrices satisfy \( P^{(n)} \rightarrow A_r \) as \( n \rightarrow \infty \). For any sequence \( (m_n) \) of integers not divisible by \( r \),

\[
\hat{\mu}(m_n r^n) \rightarrow 0
\]
as \( n \rightarrow \infty \).

**Proof.** Write \( \gamma_n(t) = \exp 2\pi i m_n r^n \), and recall from (9) and (10) that

\[
\hat{\mu}(\gamma_n) \rightarrow \lim E \sum_{N \rightarrow \infty} \Delta_{n}^n \Pi_{N} \Gamma_{N}
\]

where

\[
\Delta_{n}^n = Q^{(k)}(\gamma_n)
\]

and \( \Gamma_{N} = t_{N}(\gamma_n) \) is a column vector all of whose entries have absolute value not exceeding 1. Observe that, for \( k \leq n \), \( \gamma_n((i r)^{k-1}) = 1 \) and so \( \Delta_{n}^n = P^{(k)} \). Since \( P^{(n)} \rightarrow A_r \),

\[
\hat{\mu}(\gamma_n) \rightarrow \lim E \sum_{N \rightarrow \infty} \prod_{k = n+1}^{N} \Delta_{n}^n \Gamma_{N} \sim E \Delta_{n+1}^n \Phi_n,
\]

where again \( \Phi_n \) is a column vector whose terms do not exceed 1 in absolute value. For the same reason, the \((j,k)\) term in \( \Delta_{n+1}^n \) is approximately of the form \( r^{-1} \exp(2\pi i m_n j r^{-1}) \). In consequence,

\[
\hat{\mu}(\gamma_n) \sim \frac{1}{r} \sum_{j=0}^{r-1} \exp(2\pi i m_n j r^{-1}) (1, 1, \ldots, 1) \Phi_n = 0.
\]

This completes the proof of the lemma and hence of Theorem 2.

5. **Consequences**

In this final section we briefly indicate the consequences of the previous two theorems for the behaviour of ergodic Markov measures. To do this we need to make a few comments about the structure of \( M(T) \), the convolution algebra of finite Borel
measures on the circle group. An $L$-subspace in $M(\mathbf{T})$ is a closed linear subspace $L$ which has the property that if $\mu$ is in $L$ and $v$ is absolutely continuous with respect to $\mu$ then $v$ is also in $L$. Examples of such are the space $L^1(\mathbf{T})$ of all absolutely continuous (with respect to Lebesgue measure) measures, the radical $L^{1/2}(\mathbf{T})$, being the intersection of the maximal ideals of $M(\mathbf{T})$ containing $L^1(\mathbf{T})$, the space $M_v(\mathbf{T})$ of measures whose Fourier–Stieltjes transforms vanish at infinity, the space $M_d(\mathbf{T})$ of continuous measures and the space $M_a(\mathbf{T})$ of discrete measures. All of these have additional algebraic structure. All but the last are ideals in $M(\mathbf{T})$ (and so $L$-ideals) and $M_d(\mathbf{T})$ is an $L$-algebra. An $L$-ideal is called a radical $L$-ideal if it is the intersection of all maximal ideal which contain it. It is straightforward to see that the intersection of all maximal ideas containing a given $L$-ideal is a radical $L$-ideal. Of the $L$-ideals listed above, only $L^1(\mathbf{T})$ is not a radical $L$-ideal.

We state the following result, which, as far as we know, has not previously been enunciated.

**Proposition 5.2.** Let $\mu$ be a tame measure and let $I$ be a radical $L$-ideal. Then either $\mu \in I$ or $\mu \perp I$.

**Proof.** Suppose that $\mu \notin I$. Then there is some generalized character $f$ such that $\hat{\mu}(f) \neq 0$ where $\hat{\mu}(f) = 0$ for all $v \in I$. Now $f_\rho = a\gamma$ for some $a \neq 0$ and some $\gamma \in \hat{T}$. Now if $\rho$ is absolutely continuous with respect to $\mu$, there is some measure $\sigma = \gamma \cdot \rho$, for instance) absolutely continuous with respect to $\rho$ for which $\hat{\sigma}(f) \neq 0$. It follows that $\mu \perp I$.

From this result and the tameness of ergodic Markov measures, we can deduce that these measures satisfy several ‘purity’ theorems. For example such measures are either purely discrete or purely continuous. They either belong entirely to the radical or are singular to it and they belong entirely to $M_d(\mathbf{T})$ or are singular to it. Other radical $L$-ideals can be defined using Raikov systems (see, for example, [GMG, 5.2]) and corresponding results, which we shall not state here, are then possible.

Another consequence of tameness is a strong independent power property. A measure is of this kind if $\delta(x) \ast \mu^n \perp \mu^m$ unless $n = m$ for all $x \in \mathbf{T}$. If $\mu$ is a continuous tame measure not in the radical then it has strongly independent powers. Since to be in the radical a measure must be in $M_d(\mathbf{T})$, any ergodic Markov measure which is not in $M_d(\mathbf{T})$ must have strongly independent powers. Note that the ergodic property forces continuity of the measure.

**REFERENCES**


Markov random variables


