Book on Gibbs Phenomenon

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Chapter 1

How to reduce Gibbs ripples for the Shannon and Meyer’s wavelet sampling series

Abstract. It is well known that most multiresolution analyses induce sampling expansions for all continuous functions (see [10]). It was shown by H. T. Shim and G. G. Walter (see [11]) that the Gibbs phenomenon occurs for most sampling series based on wavelets. The magnitude of the ripple is shown to depend on the value assigned to the function at the jump location, whenever the location of the jump is a dyadic rational. Explicit calculations are given (see [1]) for the magnitude and the location of the maximum Gibbs ripple for the Shannon series of discontinuous functions, for all possible values at the jump location. Further calculations are made of the ripple, for sampling series associated with Meyer wavelets. It is shown that the Gibbs phenomenon can be significantly reduced for certain classes of Meyer’s sampling functions. These results are trivially valid as well for all dyadic rational points by translation. Functions with jump discontinuities at points which are not dyadic rationals need special treatment. A condition characterizing the Gibbs phenomenon at such points is also given.

Introduction. Gibbs phenomenon exists for many integral representations (see [3], [6], [7], [8], [9]) and for many series representations as well. The presence of this phenomenon is undesirable, since it is related to the behavior of the series approximating a discontinuous function \( f \) at a jump location \( t \), implying non-uniform approximation at \( t \), so it is important to examine ways to reduce or even avoid it. In [4] and [5], Gibbs phenomenon has been shown to exist for Fourier interpolation. Since most multiresolution analyses induce sampling expansions (see G. G. Walter’s book in [10]), Gibbs phenomenon for wavelet sampling expansions has been examined in [1], [9] and [11].

In this chapter we resume results about Gibbs phenomenon on wavelet sampling expansions. We see that this phenomenon always exists for Shannon and most Meyer’s sampling series and we see how it can be reduced.

The chapter is organized as follows: In section 1.1 we see how the classical Shannon sampling theorem is emerged from the more general setting of sampling theory in shift invariant reproducing kernel Hilbert spaces. We discuss wavelet sampling series arising from reproducing kernel Hilbert spaces and we give a variety of examples including the Shannon sampling expansion.

In section 1.2 we give necessary and sufficient conditions for the existence of Gibbs phenomenon on wavelet sampling expansions. It turns out that in the case where the location of the jump discontinuity is a dyadic rational, Gibbs phenomenon (if it exists), depends on the value of \( f \).
assigned at the jump location. So, we have the flexibility to define for any $\alpha \in [0, 1]$:

$$f(t) = \alpha f(t - 0) + (1 - \alpha) f(t + 0)$$

and to examine the magnitude and location of the maximum Gibbs ripples for every value of $\alpha$.

In section 1.3 we examine Gibbs phenomenon for the classical Shannon sampling expansion for all values of $\alpha$. The magnitude and the location of the maximum Gibbs ripple is calculated for every $\alpha$ and the specific value of $\alpha$ for which the minimum magnitude of the maximum Gibbs ripple is obtained, is calculated as well.

Finally, in section 1.4 sufficient conditions are given for the existence of Gibbs phenomenon on Meyer’s sampling expansions and examples of Meyer’s sampling formulas with small Gibbs ripples are presented.

The figures below illustrate Gibbs phenomenon for several $\alpha$’s in cases of Shannon and Meyer’s wavelet sampling expansions:

![Figure 1.1: Gibbs phenomenon of Shannon sampling expansion of the unitstep function for various values of $\alpha$. Black line corresponds to the value $\alpha = 0.25$. Dash line corresponds to the value $\alpha = 0.5$. Dot line corresponds to the value $\alpha = 0.75$.](image1)

![Figure 1.2: Gibbs phenomenon of Shannon sampling expansion of the unitstep function for $\alpha \approx 0.365$. The ripple is incon siderable on the right hand side of the jump discontinuity.](image2)
1.1 From Whittaker-Shannon-Kotelnikov sampling theorem to wavelet sampling theorems

Let $C(\mathbb{R})$ be the space of continuous functions on the real line $\mathbb{R}$ and let $L_2(\mathbb{R})$ be the space of square integrable functions on $\mathbb{R}$, the classical Whittaker-Shannon-Kotelnikov sampling theorem (WSK), states that any band-limited function $f$ belonging in the classical Paley Wiener space:

$$PW_\pi = \{ f \in L_2(\mathbb{R}) : \hat{f}(\omega) = 0, \ |\omega| > \pi \},$$

where $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$, $\omega \in \mathbb{R}$ is the Fourier transform of a complex valued function $f$, can be reconstructed from its samples $\{f(n)\}_{n \in \mathbb{Z}}$ ($\mathbb{Z}$ is the set of all integers), via the formula:

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) \text{sinc}(t - n), \ t \in \mathbb{R},$$

where $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ and the convergence is uniform on $\mathbb{R}$.

**Definition 1.1** (see [12] or [13]) A closed subspace $U$ of $L_2(\mathbb{R})$ is a reproducing kernel Hilbert space (RKHS) with reproducing kernel (RK) $K(t, x)$, $t, x \in \mathbb{R}$, if:

(i) for every $x$, the function $K(t, x)$ as a function of $t$ belongs in $U$,

(ii) for every $t, x \in \mathbb{R}$ and $f \in U$ there holds: $f(t) = < f, K(t, \cdot) >$, where $< \cdot, \cdot >$ is the usual inner product of $L_2(\mathbb{R})$.

If a reproducing kernel exists, then it is unique. As for its existence, it is known that $U$ is a RKHS if and only if for any $f \in U$ the point evaluation functional $< f, K(t, \cdot) > \rightarrow f(t)$ is continuous. Obviously, if $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis of $U$ and if $K(t, x)$ is its RK, then $K(t, x) = \sum_{n=-\infty}^{\infty} e_n(x)e_n(t)$. It is easy to see that in a RKHS, convergence in norm implies pointwise convergence. Moreover, the convergence is uniform in those intervals of $\mathbb{R}$ where $K(t, t)$ is uniformly bounded.
CHAPTER 1. HOW TO REDUCE GIBBS RIPPLES FOR THE SHANNON AND MEYER’S WAVELET SAMPLING SERIES

Definition 1.2 (see [13]) A basis \( \{S_n\}_{n \in \mathbb{Z}} \) of a reproducing kernel Hilbert space \( U \) with reproducing kernel \( K(t,x) \) is called a sampling basis if there exists a sequence of real numbers \( \{t_n\}_{n \in \mathbb{Z}} \) such that for any \( f \in U \) we have:

\[
f(t) = \sum_{n=-\infty}^{\infty} f(t_n)S_n(t),
\]

where the series converges in norm and hence uniformly on intervals of \( \mathbb{R} \), in case where \( K(t,t) \) is uniformly bounded.

Proposition 1.1 [13] Let \( \{S_n\}_{n \in \mathbb{Z}} \) be a basis of a reproducing kernel Hilbert space \( U \) with reproducing kernel \( K(t,x) \), then \( \{S_n\}_{n \in \mathbb{Z}} \) is a sampling basis if and only if its biorthonormal basis \( \{S^*_n\}_{n \in \mathbb{Z}} \) is given by

\[
S^*_n(t) = K(t,t_n)
\]

and the sequence \( \{t_n\} \) satisfies (1.1). Notice that two bases \( \{S_n\}, \{S^*_n\} \) are biorthonormal, if \( <S_n,S^*_k>=\delta_{n,k} \).

It is known (see [12]) that every basis of a Hilbert space possesses a unique biorthonormal basis.

We list some important properties of sampling bases:

(P1) An orthonormal basis \( \{S_n\} \) of a RKHS is also a sampling basis if and only if it is generated from the RK \( K(t,x) \) via the relation:

\[
S_n(t) = K(t,t_n)
\]

and the sequence \( \{t_k\} \) satisfies (1.1).

(P2) If \( \{S_n\}_{n \in \mathbb{Z}} \) is a sampling basis satisfying (1.1), then:

\[
S_n(t_k) = \delta_{n,k}.
\]

(P3) If a sampling basis \( \{S_n\} \) of a closed subspace \( U \) of \( L_2(\mathbb{R}) \) is generated by the integer translates of a single function \( S(t) \), i.e. \( S_n(t) = S(t-n) \), then \( S(t) \) is called the sampling function of \( U \).

(P4) If \( S \) is a sampling function of \( U \), then:

\[
\sum_{n \in \mathbb{Z}} S(\omega + 2\pi n) = 1, \ a.e \ on \ [0, 2\pi].
\]

(P5) If \( S \) is a sampling function of \( U \), then:

\[
\sum_{n \in \mathbb{Z}} S(t-k) = 1, \ t \in [0,1).
\]

It is shown (see [12]) that \( PW_{\pi} \) is a RKHS with reproducing kernel \( K(t,x) = sinc(t-x) \), whose orthonormal basis \( \{sinc(-n), n \in \mathbb{Z}\} \) and its sampling basis coincide. As a consequence, \( PW_{\pi} \) is a shift invariant space, i.e. it is a closed subspace of \( L_2(\mathbb{R}) \) generated by the integer translates of the function \( \phi(t) = sinc(t) \).
Although the classical WSK is of great theoretical interest, it can hardly be carried out with all precision in the various applications. First, only a finite number of sampled values can be used in practice, so the representation of a signal \( f \) by the finite sums is provided with a truncation error that decreases rather slowly, since the sinc function behaves like \( O(|t|^{-1}) \) for \( t \to \infty \). Second, when \( f \) is a continuous non-bandlimited function in \( L^2(\mathbb{R}) \), the formula

\[
    f(t) = \lim_{n \to \infty} \sum_{n \in \mathbb{Z}} f(n/W) \text{sinc}(Wt-n), \quad W > 0, \quad t \in \mathbb{R},
\]

holds under restrictive conditions upon \( f \), since the sinc function is not in \( L^1(\mathbb{R}) \) (the space of absolutely integrable functions defined on \( \mathbb{R} \)). For example, continuity of \( f \) does not suffice.

To overcome these difficulties we consider shift invariant spaces \( V_S \), as the closure of the linear span of a basis \( \{S(Wt-n), \; n \in \mathbb{Z}\} \), \( W > 0 \), where \( S \in L^2(\mathbb{R}) \). So, several authors studied sampling series of the form:

\[
    (T^S_W)(t) = \sum_{n \in \mathbb{Z}} f(n/W) S(Wt-n), \quad W > 0, \quad t \in \mathbb{R} \tag{1.2}
\]

and examined conditions upon \( S \) so that the sampling series (1.2) exists for each uniformly continuous and bounded function \( f \). Butzer, Ries and Stens in [2] proved the following:

**Proposition 1.2** If \( f : \mathbb{R} \to \mathbb{C} \) is a complex valued function bounded on \( \mathbb{R} \) and if \( S \in L^1(\mathbb{R}) \cap C(\mathbb{R}) \), then the following assertions are equivalent:

(i) \((T^S_W)(t) \to f(t), \; W \to \infty, \) at each point \( t \) where \( f \) is continuous.

(ii) \( \sum_{k=-\infty}^{\infty} S(t-k) = 1, \; t \in \mathbb{R} \).

Famous examples of shift invariant RKHS are the wavelet subspaces emerged from a multiresolution Analysis of \( L^2(\mathbb{R}) \).

We recall (see [10]) that a multiresolution analysis (MRA) of \( L^2(\mathbb{R}) \) is a nested sequence \( \{V_m \subseteq V_{m+1}, \; m \in \mathbb{Z}\} \) of closed subspaces in \( L^2(\mathbb{R}) \) satisfying:

(i) \( \bigcup_m V_m = L^2(\mathbb{R}) \),

(ii) \( \bigcap_m V_m = \{0\} \),

(iii) \( f(t) \in V_0 \iff f(2^m t) \in V_m \), for all \( m \in \mathbb{Z} \),

(iv) there exists a function \( \phi \) called scaling function, such that for any \( m \in \mathbb{Z} \), the set \( \{2^{m/2} \phi(2^m \cdot - n), \; n \in \mathbb{Z}\} \) is an orthonormal basis of \( V_m \).

Sampling theory in wavelet subspaces has been intensively studied in recent years. Gilbert Walter used the theory of RKHS to prove a sampling theorem for the wavelet subspaces \( V_m \). He imposed the following assumptions on the scaling function \( \phi \):
(a) \( \phi \) is a continuous bounded functions on \( \mathbb{R} \): \( \sup_{t \in [0,1)} \sum_{n=-\infty}^{\infty} |\phi(t-n)| < \infty \). This condition ensures that each subspace \( V_m \) is a subset of \( L_2(\mathbb{R}) \cap C(\mathbb{R}) \). Additionally, \( V_m \) becomes a RKHS with reproducing kernel:

\[
K_m(t, x) = 2^m \sum_{k=-\infty}^{\infty} \phi(2^m t - k)\overline{\phi(2^m x - k)}.
\]

(b) Let for any \( \omega \in \mathbb{R} \):

\[
\phi^*(\omega) = \sum_{n=-\infty}^{\infty} \phi(n)e^{-in\omega} \neq 0.
\]

This condition ensures that the sequence \( \{K_m(n/2^m, \cdot), n \in \mathbb{Z}\} \) is a basis of \( V_m \), having a unique biorthonormal basis \( \{S_n(2^m \cdot), n \in \mathbb{Z}\} \). Since \( K_m(n/2^m, \cdot) = K_m(0, \cdot - n/2^m) \), the sampling basis \( S_n(2^m \cdot) \) satisfies \( S_n(2^m \cdot) = S(2^m \cdot - n) \).

Then, it is easy to derive the following sampling theorem on wavelet subspaces:

**Theorem 1.1** [10] Under the assumptions (a)-(b) on the scaling function \( \phi \), there exists a unique sampling basis \( \{S(2^m \cdot - n), n \in \mathbb{Z}\} \) of \( V_m \) in the sense that any continuous function \( f \in V_m \) can be written as:

\[
f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2^m}\right)S(2^m t - n), \quad t \in \mathbb{R},
\]

where the convergence is uniform on compact intervals of \( \mathbb{R} \). The function \( S(t) \in V_0 \) is the sampling function of \( V_0 \) derived by its Fourier transform via the relation:

\[
\hat{S}(\omega) = \sum_{n \in \mathbb{Z}} \hat{\phi}(\omega) e^{-in\omega}, \quad \omega \in \mathbb{R}.
\]

We present some examples of wavelet sampling expansions:

**Example 1:** The Shannon sampling function. The scaling function \( \phi(t) = \text{sinc}(t) \) and the sampling function \( S(t) \) coincide.

![Figure 1.4: The Shannon sampling function \( S(t) = \text{sinc}(t) \).](image-url)
1.1. FROM WHITTAKER-SHANNON-KOTELNIKOV SAMPLING THEOREM TO WAVELET SAMPLING THEOREMS

Example 2: The Haar sampling function. The scaling function \( \phi(t) = \chi_{[0,1)}(t) \), (\( \chi(t) \) is the characteristic function on interval 1) and the sampling function \( S(t) \) coincide.

Example 3: The Meyer sampling function. The Meyer’s scaling function \( \phi(t) \) is defined via its Fourier transform:

\[
\hat{\phi}(\omega) = \begin{cases} 
1, & |\omega| \leq \frac{2\pi}{3}, \\
\cos\left[\frac{\pi}{2} \nu\left(\frac{3|\omega|}{2\pi} - 1\right)\right], & \frac{2\pi}{3} < |\omega| < \frac{4\pi}{3}, \\
0, & |\omega| \geq \frac{4\pi}{3},
\end{cases}
\]

where \( \nu(\omega) \) is a real valued function satisfying \( \nu(\omega) + \nu(1-\omega) = 1, \omega \in [0,1] \).

Example 3.1 Let \( \delta \in (1/2, 1] \), we define:

\[
\nu(\omega) = \begin{cases} 
\frac{2\pi}{\pi} \arccot\left[\frac{\omega - \delta}{1-\omega}\right], & 1/2 \leq \omega \leq \delta, \\
1, & \omega > \delta,
\end{cases}, \quad \omega \in [1/2, 1],
\]

and we use the equality \( \nu(\omega) = 1 - \nu(1-\omega) \) to extend \( \nu(\omega) \) on \( [0,1/2) \). We derive the following Meyer’s sampling function:

\[
S(t) = \frac{3 \sin(\pi t) \sin(2\pi (\delta - 0.5) t/3)}{2\pi^2 (\delta - 0.5) t^2}.
\]

Figure 1.5: The Meyer’s sampling function for \( \nu(\omega) \) as in example (3.1) for \( \delta = 0.75 \).

Example 3.2 Let

\[
\nu(\omega) = \begin{cases} 
\frac{2\pi}{\pi} \arccot\left[\frac{1}{2(\omega-1)}\right], & 0 \leq \omega \leq 1/2, \\
\frac{2\pi}{\pi} \arccot\left[\frac{1}{(1-2(\omega-1)^2)}\right], & 1/2 < \omega \leq 1.
\end{cases}
\]

In this case the Meyer’s sampling function is: \( S(y) = 36 \frac{\sin(\pi y) \sin^2(\pi y/6)}{\pi^2 y^2} \).
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Figure 1.6: The Meyer’s sampling function for $\nu(\omega)$ as in example (3.2).

Example 4 The Franklin’s sampling function. The scaling function $\phi(t)$ satisfies:

$$\hat{\phi}(\omega) = \sin^2(\omega/2) \left( 1 - \frac{1}{3} \sin^2(\omega/2) \right)^{-1/2}$$

and

$$S(\omega) = \frac{\hat{\phi}(\omega)}{\phi^*(\omega)} = \left( \frac{\sin(\omega/2)}{\omega/2} \right)^2,$$

whose inverse transform is the original hat function $S(t) = 1 - |t|, t \in [-1, 1], S(t) = 0$ elsewhere.

Example 5 The Daubechies sampling function. The Daubechies wavelets with support on $[0, 3]$ are defined by the solution of the dilation equations:

$$\phi(t) = \sqrt{2} \sum_{k=0}^{3} c_k \phi(2t - k),$$

where $c_0 = \nu(\nu - 1)/D$, $c_1 = (1 - \nu)\nu/D$, $c_2 = (\nu + 1)/D$, $c_3 = \nu(\nu + 1)/D$ and $D = \sqrt{2}(\nu^2 + 1), \nu \in \mathbb{R}$.

The sampling formula for $\nu < 0$ is:

$$S(t) = \frac{2\nu}{\nu - 1} \sum_{n=0}^{\infty} \left( \frac{1 + \nu}{1 - \nu} \right)^n \phi(t - n + 1).$$

1.2 Gibbs Phenomenon for Wavelet Sampling Expansions

Let $f \in L_2(\mathbb{R})$ be a continuous function on $\mathbb{R} \setminus \{t\}$ with a jump discontinuity at $t$, satisfying $f(t - 0) \leq f(t) \leq f(t + 0)$. We assume that $t$ is a dyadic rational $t = \lambda 2^{-\mu}, \mu \in \mathbb{N}, \lambda \in \mathbb{Z}$ and that for some $0 \leq \alpha \leq 1$ we have

$$f(t) = \alpha f(t - 0) + (1 - \alpha) f(t + 0). \quad (1.3)$$

We consider the following approximation of $f$ in terms of its sampling series $T_m f$:

$$T_m f(x) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2^m}\right) S(2^m x - n),$$

provided that such a wavelet sampling formula exists under the conditions imposed in Theorem 1.1.
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**Definition 1.3** The sampling expansion $T_m f(x)$ exhibits Gibbs Phenomenon on the right hand side of $t$ (resp., on the left hand side) if there is a $y > 0$ (resp., $y < 0$) such that for all but possibly finitely many $m$, we have

$$T_m f(t + y/2^m) > f(t + 0) \quad \text{(resp., } T_m f(t + y/2^m) < f(t - 0))\).$$

The part of the curve for which $T_m f(x) > f(x)$ (or $T_m f(x) < f(x)$) is called the **Gibbs ripple** of $f$. The part of the ripple that contains the highest overshoots (undershoots) of $T_m f(x)$ is called the **maximum ripple** of $f$.

**Theorem 1.2** [1] or [11] The sampling expansion $T_m f(x)$ exhibits Gibbs Phenomenon on the right side (on the left side) of the discontinuity point $t = \lambda 2^{-\mu}$, $\mu \in \mathbb{N}$, $\lambda \in \mathbb{Z}$, if and only if for some $y > 0$ (or $y < 0$), we have

$$G_\alpha^r(y) = \alpha S(y) + \sum_{k > 0} S(y + k) < 0 \quad \text{(or } G_\alpha^l(y) = (1 - \alpha) S(y) + \sum_{k < 0} S(y + k) < 0\),

where $\alpha$ is defined in (1.3). The function $G_\alpha^r(y)$, $(G_\alpha^l(y))$ is called **Gibbs function** on the right (on the left) side of the discontinuity point.

**Proof.** Let $f$ be defined above, we denote a continuous function $g$ as

$$g(x) = \begin{cases} f(x) - f(t - 0), & x < t, \\ 0, & x = t = \lambda 2^{-\mu}. \\ f(x) - f(t + 0), & x > t \end{cases}$$

Take $m \geq \mu$. Since $g(t) = 0$, we get:

$$T_m f(x) = T_m g(x) + f(t - 0) \sum_{n < 2^m t} S(2^m x - n)$$

$$+ f(t + 0) \sum_{n > 2^m t} S(2^m x - n) + f(t) S(2^m(x - t)).$$

(1.4)

Let $t_{y,m} = t + y 2^{-m}$, where $y \in \mathbb{R}$, we examine two cases:

**Case 1.** Let $y > 0$, we apply $x = t_{y,m}$ in (1.4). Let $k = 2^m t - n$, then:

$$T_m f(t_{y,m}) = T_m g(t_{y,m}) + f(t - 0) \sum_{k > 0} S(y + k) + f(t + 0) \sum_{k < 0} S(y + k) + f(t) S(y).$$

Since $\sum_{k < 0} S(y + k) = 1 - \sum_{k \geq 0} S(y + k)$ (see property (P5) in section 1.1), we use (1.3) in the equality above and we get:

$$T_m f(t_{y,m}) = T_m g(t_{y,m}) + f(t + 0) - (f(t + 0) - f(t - 0))(\alpha S(y) + \sum_{k > 0} S(y + k)).$$

As $m \to \infty$, the term $T_m g(t_{y,m})$ vanishes and we have:

$$T_m f(t_{y,m}) \to f(t + 0) - (f(t + 0) - f(t - 0))G_\alpha^r(y),$$

so, the result follows.
Case 2. Let \( y < 0 \). We work as in case 1 to deduce:
\[
T_m f(t_{y,m}) = T_m g(t_{y,m}) + f(t - 0) + (f(t + 0) - f(t - 0))((1 - \alpha)S(y) + \sum_{k < 0} S(y + k)).
\]
As \( m \to \infty \), we get: \( T_m f(t_{y,m}) \to f(t - 0) + (f(t + 0) - f(t - 0))G^d_\alpha(y) \). □

If the jump discontinuity is not a dyadic rational \( t \), then (1.4) becomes:
\[
T_m f(x) = T_m g(x) + f(t - 0) \sum_{n < 2^m t} S(2^m x - n) + f(t + 0) \sum_{n > 2^m t} S(2^m x - n).
\]

If \( \sum_{n < 2^m t} S(2^m t - n) \) is not constant for any \( t \in \mathbb{R} \), the limit \( \lim_{m \to \infty} T_m f(t) \) does not exist. Indeed, if we set \( x = t_{y,m} = t + y/2^m \) in the equality above and proceed as in the proof of theorem 1.2, we get:

(i) for \( y > 0 \) and \( \mu \in \mathbb{Z} \),
\[
T_m f(t_{y,m}) = T_m g(t_{y,m}) + f(t + 0) - (f(t + 0) - f(t - 0)) \sum_{\mu \geq 0} S(\beta + y + \mu),
\]

(ii) for \( y < 0 \) and \( \mu \in \mathbb{Z} \),
\[
T_m f(t_{y,m}) = T_m g(t_{y,m}) + f(t - 0) + (f(t + 0) - f(t - 0)) \sum_{\mu < 0} S(\beta + y + \mu),
\]

where \( \beta = 2^m t - \lfloor 2^m t \rfloor \in (0,1) \). As \( m \to \infty \), the term \( T_m g(t_{y,m}) \) becomes negligible, while \( \sum_{\mu \geq 0} S(\beta + y + \mu) \) oscillates.

Theorem 1.3 [1] If the jump location \( t \neq \lambda 2^{-\mu}, \lambda \in \mathbb{Z}, \mu \in \mathbb{N} \), then the sampling expansion \( T_m f(x) \) exhibits Gibbs Phenomenon on the right (on the left) if and only if there is at least one \( y > 0 \) (or \( y < 0 \)) such that, for all except finitely many \( m \) we have
\[
G^r(y, \beta) = \sum_{\mu \geq 0} S(\beta + y + \mu) < 0, \quad \text{or} \quad G^l(y, \beta) = \sum_{\mu < 0} S(\beta + y + \mu) < 0.
\]

Proposition 1.3 If the sampling function \( S(x) \) of an MRA of \( L_2(\mathbb{R}) \) satisfies \( S(x) \geq 0 \) for any \( x \in \mathbb{R} \), then the sampling expansion \( T_m f(x) \) does not exhibit Gibbs phenomenon.

Proof. Immediate consequence of theorems 1.2 and 1.3. □

Remark 1 As a consequence of Proposition 1.3, Haar and Franklin’s sampling expansions do not exhibit Gibbs phenomenon. The same is also true for Daubechies sampling series for \( -1 \leq \nu < 0 \), since Shim and Walter in [11] proved that Daubechies sampling function is non negative for \( -1 \leq \nu < 0 \) (for \( \nu < -1 \) they also proved that Gibbs phenomenon exists).

Proposition 1.4 If \( f : \mathbb{R} \to \mathbb{R} \) is a bounded function and if \( u \in V_0 \cap C(\mathbb{R}) \cap L_1(\mathbb{R}) \) satisfies:
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(i) \( u(t) \geq 0 \) for any \( t \in \mathbb{R} \),
(ii) \( \sum_{k=-\infty}^{\infty} u(t-k) = 1, \ t \in \mathbb{R}, \)

then \( T_m f(t) = \sum_{n \in \mathbb{Z}} f(n/2^m) S(2^m t - n) \to f(t), \ m \to \infty, \) at each point \( t \) where \( f \) is continuous and \( T_m f(t) \) does not exhibit Gibbs phenomenon.

Proof. Combine Propositions 1.2 and 1.3. □

Remarks

2 If \( S \) is an even function, then for any \( y < 0 \) we have \( G^r_\alpha(y) = G^r_{1-\alpha}(-y) \); thus, for any \( \alpha \) in \([0,1]\), it suffices to deal with \( G^r_\alpha(y) \), where \( y > 0 \).

3 Since \( \sum_{k \in \mathbb{Z}} S(k+1/2) = 1 \) (see property (P5) in section 1.1), for all even sampling functions we have:

\[
S(1/2) + S(3/2) + S(5/2) + \ldots = 1/2. \quad (1.5)
\]

1.3 Gibbs Phenomenon for Shannon Sampling Expansions

In Theorem 1.4 of this section, we prove that Gibbs Phenomenon exists for the classical Shannon wavelet sampling approximation:

\[
T_m f(t) = \sum_{n \in \mathbb{Z}} f(n/2^m \text{sinc}(2^m t - n)) \quad (1.6)
\]

of a function \( f \) satisfying: (a) \( f \) has a jump discontinuity at a dyadic rational \( t = \lambda/2^\mu, \ \lambda \in \mathbb{Z}, \ \mu \in \mathbb{N} \) and (b) \( f(t) = \alpha f(t-0) + (1-\alpha) f(t+0), \ \alpha \in [0,1]. \)

In Theorems 1.5 and 1.6 we find the magnitude and location of the maximum Gibbs ripple for any value of \( \alpha \).

In Theorem 1.7 we detect the range of values \( \alpha \in [0.335, 0.378] \) for which the minimum magnitude of the maximum Gibbs ripple is obtained.

Theorem 1.2 and remark 2 state that Gibbs Phenomenon exists for the classical Shannon sampling theorem on the right side (on the left side) of the discontinuity point, if and only if for some \( y > 0 \) we have:

\[ G^r_\alpha(y) < 0 \quad (\text{or} \ G^r_{1-\alpha}(y) < 0), \]

where \( G^r_\alpha(y) \) is defined in theorem 1.2.

We use the following:

Notations:

1: \( A_k(y) = y^k \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(y+n)^k}, \ k = 1,2,\ldots \) and \( y \geq 0. \)

\( A_k(y) \) has the following properties (for a proof see [1]):
The Gibbs function $G_r^\alpha(y)$ on the right side of the jump discontinuity corresponding to Shannon’s sampling expansion is:

$$G_r^\alpha(y) = \frac{\sin(\pi y)}{\pi y} \left( \alpha - y \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{y+k} \right) = \frac{\sin(\pi y)}{\pi y} (\alpha - A_1(y)),$$

where $A_1(y)$ is as in notation 1 and $\alpha \in [0, 1]$.

**Theorem 1.4** (see [1]) or [11]) Let $f(t) = \alpha f(t-h) + (1-\alpha)f(t+h)$, $\alpha \in [0, 1]$ be the value assigned to a function $f$ at a jump location $t$, where $t$ is a dyadic rational, then the Shannon wavelet sampling expansion (1.6) exhibits Gibbs Phenomenon for every value of $\alpha$.

**Proof.** It suffices to find one $y > 0$ such that for any $\alpha$: $G_r^\alpha(y) < 0$. We apply $y = 3/2$ and $y = 5/2$ on $G_r^\alpha(y)$ as in theorem 1.2 and we use (1.5) to get:

$$G_r^\alpha(3/2) = \alpha S(3/2) + 1/2 - S(1/2) - S(3/2) < 0 \iff \alpha > 0.356194,$$

$$G_r^\alpha(5/2) = \alpha S(5/2) + 1/2 - S(1/2) - S(3/2) - S(5/2) < 0 \iff \alpha < 0.4038079. \quad \Box$$

In order to find the maximum Gibbs ripples we need to find the lower minima of the function $G_r^\alpha(y)$.

**Theorem 1.5** (see [1]) Let $A_k(y)$, $G_r^\alpha(y)$ be as in notations 1 and 2 respectively and let $\rho_1$ be as in (C4). If $\alpha \in [0, 1/2)$, then the lower minima of $G_r^\alpha(y)$ are in at least one of the intervals of Matrix 1:

| Matrix 1 |
|-----------------|-----------------|
| Values of $\alpha \in [0, 1/2]$ | Location of lower minima |
| $0 \leq \alpha < A_1(1) \approx 0.306853$ | $(\rho_1, 1), (2, 3)$ |
| $A_1(1) \leq \alpha < A_1(2) \approx 0.386294$ | $(1, \rho_1), (2, 3), (4, 5)$ |
| $A_1(2) \leq \alpha < 0.5$ | $(1, 2)$ |
1.3. GIBBS PHENOMENON FOR SHANNON SAMPLING EXPANSIONS

**Proof.** We can write $G^n_r(y) = \frac{\sin(\pi y)}{\pi} G^n_1(y)$, where $G^n_1(y) = \frac{2 - A_1(y)}{y}$, then we use (C6) for $k = 2$, to get:

$$-\frac{1}{y^2} (\alpha - A_2(y)) = (G^n_1)'(y) \begin{cases} > 0, & y > \rho_2 \\ = 0, & y = \rho_2 \\ < 0, & y < \rho_2 \end{cases}$$

(1.8)

where $\rho_2$ is the unique positive real satisfying $A_2(\rho_2) = \alpha$. We use (C6) to deduce that $G^n_r(y) < 0$ for all $y$ in the intervals of a class $F = \{I_k : k = 0, 1, \ldots, \}$:

$$I_k = \begin{cases} (k, k + 1), & \text{whenever } k \text{ is odd and } k < \lfloor \rho_1 \rfloor \\ (k, \rho_1), & \text{whenever } k = \lfloor \rho_1 \rfloor \text{ is odd} \\ (\rho_1, k + 1), & \text{whenever } k = \lfloor \rho_1 \rfloor \text{ is even} \\ (k, k + 1), & \text{whenever } k \text{ is even and } k > \lfloor \rho_1 \rfloor \end{cases}$$

We split $F$ into three classes of intervals:

**class 1:** $J = \{J_k \in F : y \in J_k \text{ and } 0 \leq y < \rho_1 \}$. If $\lfloor \rho_1 \rfloor = 0$, then $J = \emptyset$. Let $J_0 \in J$ be the interval where $G^n_r(y)$ attains its lower minimum. If $\lfloor \rho_1 \rfloor = 1$ or $\lfloor \rho_1 \rfloor = 2$, the class $J$ contains only the intervals $(1, \rho_1)$ or $(1, 2)$. If $\lfloor \rho_1 \rfloor \geq 3$, the class $J$ is the following:

$$J = \{J_k = (k, k + 1) : k \text{ is odd and } k < \lfloor \rho_1 \rfloor \} \cup \{(\lfloor \rho_1 \rfloor, \rho_1), \text{where } \lfloor \rho_1 \rfloor \text{ is odd} \}$$

Using (C5), (C6) and (1.8), we observe that $G^n_r(y)$ is positive and decreasing for $y < \rho_1$, thus for any $y \in J_k$, there is always a $y_0 = y - (k - 1) \in (1, 2)$, such that $G^n_r(y_0) < G^n_r(y)$. The same is also true for $y \in (\lfloor \rho_1 \rfloor, \rho_1)$ (take $y_0 = y - (\lfloor \rho_1 \rfloor - 1) \in (1, \rho_1 - \lfloor \rho_1 \rfloor + 1)$). Therefore,

$$J_0 = \begin{cases} (1, 2), & \lfloor \rho_1 \rfloor \geq 2 \\ (1, \rho_1), & \lfloor \rho_1 \rfloor = 1 \end{cases}$$

**class 2:** $L = \{L_k \in F : y \in L_k \text{ and } y \geq \lfloor \rho_2 \rfloor + 1 \}$. Let $L_0 \in L$ be the interval where $G^n_r(y)$ attains its lower minimum. Since $\rho_1 \leq \lfloor \rho_1 \rfloor + 1 \leq \lfloor \rho_2 \rfloor + 1$ (see (C5)), we have:

$$L = \{L_k = (k, k + 1) : k \text{ is even and } k \geq \lfloor \rho_2 \rfloor + 1 \}$$

For $y \geq \lfloor \rho_2 \rfloor + 1$, $G^n_r(y)$ is negative and increasing, thus for any $y \in L_k$, there is always a $y_0 = y - (k - \lfloor \rho_2 \rfloor - 2) \in L_{\lfloor \rho_2 \rfloor + 2}$ provided that $\lfloor \rho_2 \rfloor$ is even (respectively, if $\lfloor \rho_2 \rfloor$ is odd, then $y_0 = y - (k - \lfloor \rho_2 \rfloor - 1) \in L_{\lfloor \rho_2 \rfloor + 1}$), such that $G^n_r(y_0) < G^n_r(y)$. Hence,

$$L_0 = \begin{cases} L_{\lfloor \rho_2 \rfloor + 2}, & \lfloor \rho_2 \rfloor \text{ is even} \\ L_{\lfloor \rho_2 \rfloor + 1}, & \lfloor \rho_2 \rfloor \text{ is odd} \end{cases}$$

**class 3:** $Q = \{Q_k \in F : y \in Q_k \text{ and } \rho_1 \leq y < \lfloor \rho_2 \rfloor + 1 \}$.

As a consequence, the lower minima of $G^n_r(y)$ are in at least one of the intervals of the class $M$:

$$M = \{J_0, \ L_0\} \cup Q.$$ 

We now examine the following cases.
CHAPTER 1. HOW TO REDUCE GIBBS RIPPLES FOR THE SHANNON AND MEYER’S WAVELET SAMPLING SERIES

Case 1. Let $[\rho_1] = 0$, i.e. $0 \leq \rho_1 < 1$, then by (C2) and (C4): $A_1(0) \leq A_1(\rho_1) < A_1(1) \nleq 0 \leq \alpha < A_1(1) \approx 0.306853$. One may see that $0 \leq [\rho_2] \leq 2$, hence the class $M$ contains the intervals of Matrix II:

<table>
<thead>
<tr>
<th>$[\rho_2]$</th>
<th>$= 0$</th>
<th>$= 1$</th>
<th>$= 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class M</td>
<td>$K = {(\rho_1,1)}$,</td>
<td>$K = {(\rho_1,1)}$,</td>
<td>$K = {(\rho_1,1), (2,3)}$,</td>
</tr>
<tr>
<td>$L_0=(2,3)$</td>
<td>$L_0=(2,3)$,</td>
<td>$L_0=(4,5)$,</td>
<td></td>
</tr>
</tbody>
</table>

Notice that whenever $[\rho_2]=2$ and $y \in (4,5)$ (see fourth column of matrix II), by (1.8) and (C6) we have $G_{\alpha}^u(y) > 1/\pi G_{\alpha}^u(4)$, so $G_{\alpha}^u(5/2) - G_{\alpha}^u(y) < \frac{2}{5\pi}(\alpha-A_1(5/2)) - \frac{1}{4\pi}(\alpha-A_1(4)) < 0$; so the lower minima belong in the class $K$. To calculate $A_1(4)$ or $A_1(5/2)$ we use either

$$\sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{2j-1} = \log 2$$

or

$$\sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{2j-1} = \pi 4.$$

Figure 1.7: Gibbs ripple on the right hand side of the jump discontinuity of the unitstep function for Shannon’s sampling expansion for $0 \leq \alpha < A_1(1) \approx 0.306853$. Black line corresponds to the value $\alpha = 0.1$. Dash line corresponds to the value $\alpha = 0.2$. Dot line corresponds to the value $\alpha = 0.25$. We see that the maximum overshoot appears either for $y \in (\rho_1,1)$ or for $y \in (2,3)$, where $\rho_1$ is defined in (C4).

Case 2. Let $[\rho_1] = 1$, i.e. $1 \leq \rho_1 < 2$, then $A_1(1) \leq \alpha < A_1(2) \approx 0.386294$. In this case $2 \leq [\rho_2] \leq 4$, hence the class $M$ contains the intervals of Matrix III:

<table>
<thead>
<tr>
<th>$[\rho_2]$</th>
<th>$= 2$</th>
<th>$= 3$</th>
<th>$= 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class M</td>
<td>$J_0 = (1,\rho_1)$,</td>
<td>$J_0 = (1,\rho_1)$,</td>
<td>$J_0 = (1,\rho_1)$,</td>
</tr>
<tr>
<td>$K = {(2,3)}$,</td>
<td>$K = {(2,3)}$,</td>
<td>$K = {(2,3), (4,5)}$,</td>
<td></td>
</tr>
<tr>
<td>$L_0=(4,5)$,</td>
<td>$L_0=(4,5)$,</td>
<td>$L_0=(6,7)$,</td>
<td></td>
</tr>
</tbody>
</table>

Whenever $[\rho_2]=4$ and $y \in (6,7)$ (see fourth column of Matrix III), then we work as in case 1 to get that: $G_{\alpha}^u(9/2) - G_{\alpha}^u(y) < 0$, so $M = J_0 \cup K$. 


1.3. GIBBS PHENOMENON FOR SHANNON SAMPLING EXPANSIONS

Figure 1.8: Gibbs ripple on the right hand side of the jump discontinuity of the unitstep function for Shannon’s sampling expansion for $0.306853 < \alpha < A_1(2) \approx 0.386294$. Black line corresponds to the value $\alpha = 0.35$. Dot line corresponds to the value $\alpha = 0.37$. We see that the maximum overshoot appears either for $y \in (1, \rho_1)$ or for $y \in (2, 3)$ or for $y \in (4, 5)$.

Case 3. Let $[\rho_1] \geq 2$, that is $0.386294 \leq \alpha < 0.5$. In this case $[\rho_2] \geq 4$. For any $y \in L_0$ or $y$ in any of the intervals of the class $K$ we have

$$G'_\alpha(3/2) - G'_\alpha(y) \leq - \frac{2}{3\pi}(\alpha - A_1(3/2)) - \frac{1}{\pi \rho_2}(\alpha - A_1(\rho_2))$$

$$= - \left( \frac{2}{3\pi} + \frac{1}{\pi \rho_2} \right) A_1(2) + \frac{2}{3\pi} A_1(3/2) + \frac{1}{\pi \rho_2} A_1(\rho_2) < 0.$$  

(If $[\rho_2] \geq 4$, we use (1.7) to get a bound for $A_1(\rho_2)$). Hence the lower minima are located in $(1, 2)$.

□

Figure 1.9: The Gibbs ripple $G'_\alpha(y)$ on the right of the jump location $t = 0$ of the unitstep function for $0.386294 \leq \alpha < 0.5$. Black line corresponds to the value $\alpha = 0.4$. Dot line corresponds to the value $\alpha = 0.49$. We see that the maximum overshoot appears for $y \in (1, 2)$.

To examine the case $\alpha \geq 0.5$, we need the first two derivatives of $G'_\alpha(y)$:

$$\left(G'_\alpha(y)\right)' = \frac{\cos(\pi y)}{y}(\alpha - A_1(y)) - \frac{\sin(\pi y)}{\pi y^2}(\alpha - A_2(y))$$

$$\left(G'_\alpha(y)\right)'' = - \frac{\sin(\pi y)}{y} \Omega_\alpha(y) - \frac{2\cos(\pi y)}{y^2}(\alpha - A_2(y)),$$

where $\Omega_\alpha(y) = \pi(\alpha - A_1(y)) - \frac{2}{\pi y^2}(\alpha - A_3(y))$. 

Theorem 1.6 (see [1]) Let $A_k(y)$, $G_{\alpha}^r(y)$ be as in notations 1 and 2 respectively. Let also $\rho_1$ be as in (C4), if $\alpha \in [0.5, 1]$, then:

(i) $G_{\alpha}^r(y)$ has a unique lower minimum in $(1, 1.5)$;

(ii) $\max_{0.5 \leq \alpha \leq 1} \{ \min_{y \in (1, 1.5)} G_{\alpha}^r(y) \} = \min_{y \in (1, 1.5)} G_{0.5}^r(y)$.

Proof.

(i): By (C1) we have $\alpha - A_k(y) \geq 1/2 - A_k(y) > 0$ and so for any $y \geq 0$ we have $(G_{\alpha}^r)'(y) < 0$ (see 1.8) and $G_{\alpha}^r(y) < 0$ on $(\mu, \mu + 1)$ for odd $\mu$. Let $y_0 \in (1, 1.5)$, then the sequence $\{ G_{\alpha}^r(y_0 + 2\mu) \}_{\mu=0}^\infty$ is increasing and negative, hence the lower minima of $G_{\alpha}^r(y)$ are located in $(1, 2)$. For any $y \in [1.5, 2]$, we can see that $(G_{\alpha}^r)'(1) < 0$ and $(G_{\alpha}^r)'(y) > 0$. If $y \in [1, 1.5]$ and if $\alpha > A_1(2)$, then by (1.7) we have

$$\Omega_{\alpha}(y) > (\pi - \frac{2}{\pi y^2}) \alpha - \pi y \left[ \frac{1}{y + 1} - \frac{3}{4y + 2} + \frac{1}{4y + 3} \right] + \frac{2}{\pi} y \left[ \frac{3}{4} \left( \frac{1}{y + 1} \right)^3 - \frac{1}{4} \left( \frac{1}{y + 2} \right)^3 \right] > 0,$$

thus $(G_{\alpha}^r)''(y) > 0$ and so we have a unique minimum in $(1, 1.5)$.

Figure 1.10: Gibbs ripple on the right hand side of the jump discontinuity of the unitstep function for Shannon’s sampling expansion for $0.5 \leq \alpha \leq 1$. Black line corresponds to the value $\alpha = 0.5$. Dash line corresponds to the value $\alpha = 0.75$. Dot line corresponds to the value $\alpha = 1$. We see that the maximum overshoot appears for $y \in (1, 1.5)$. The minimum of the maximum overshoots appears for $\alpha = 0.5$.

(ii): For any fixed $y \in (1, 1.5)$ the function $G_{\alpha}^r(y)$ is negative and decreasing with respect to $\alpha$, thus if $\alpha_1 < \alpha_2$ and if $G_{\alpha_1}^r(y_0)$, $G_{\alpha_2}^r(y_1)$ are the unique lower minima of $G_{\alpha}^r(y)$, we have

$$G_{\alpha_1}^r(y_1) < G_{\alpha_2}^r(y_0) < G_{\alpha_1}^r(y_0).$$

□

Theorem 1.7 (see [1])

(iii) If $\alpha \in [A_1(2), 0.5)$, then $G_{\alpha}^r(y)$ has a unique lower minimum in $(1, 1.5)$.

(iv-1) If $A_1(1) \leq \alpha < 0.335278$, then the lower minima of $G_{\alpha}^r(y)$ are in $(2, 3)$.

(iv-2) If $0.378847 < \alpha < A_1(2)$, then $G_{\alpha}^r(y)$ has a unique lower minimum in $(1, 1.5)$.

(v) The minimum of the highest overshoots of $G_{\alpha}^r(y)$ is obtained for some $\alpha \in [0.335, 0.378]$. 

Figure 1.10: Gibbs ripple on the right hand side of the jump discontinuity of the unitstep function for Shannon’s sampling expansion for $0.5 \leq \alpha \leq 1$. Black line corresponds to the value $\alpha = 0.5$. Dash line corresponds to the value $\alpha = 0.75$. Dot line corresponds to the value $\alpha = 1$. We see that the maximum overshoot appears for $y \in (1, 1.5)$. The minimum of the maximum overshoots appears for $\alpha = 0.5$.
1.3. GIBBS PHENOMENON FOR SHANNON SAMPLING EXPANSIONS

Proof.

(iii): The lower minima of $G^r_\alpha(y)$ are located in (1,2) and since $|\rho_1| \geq 2$, by (C5) and (C6) we have $\alpha - A_k(y) > 0$ in (1,2). The rest follow as in the proof of Proposition 2(ii).

(iv-1): Let $|\rho_1| = 1$ and $|\rho_2| = 2$ (see Matrix III). We use (1.8) to get the following:

if $y \in (4,5)$, then

$$G^r_\alpha(5/2) - G^r_\alpha(y) < \frac{2}{5\pi} (\alpha - A_1(5/2)) - \frac{1}{4\pi} (\alpha - A_1(4)) < 0 \iff \alpha < 0.35,$$

while for $y \in (1,\rho_1)$ we have,

$$G^r_\alpha(5/2) - G^r_\alpha(y) < \frac{2}{5\pi} (\alpha - A_1(5/2)) + \frac{1}{\pi} (\alpha - A_1(1)) < 0 \iff \alpha < 0.335278.$$

Thus whenever $0.306853 \leq \alpha < 0.335278$ the lower minima are located in (2,3).

(iv-2): Working as in (iii) for $|\rho_1| = 1$ and $|\rho_2| = 3, 4$, we obtain that for $0.378847 \leq \alpha < 0.386294$ the lower minima are in $(1,1.5) \subset (1,\rho_1)$. For the uniqueness of the minimum we follow the proof given in theorem 1.6, part(ii).

(v): Combining theorem 1.6 part(i) with theorem 1.7 part (iii) and (iv-2) we get:

$$\max_{0.378847 \leq \alpha \leq 1} \left\{ \min_{y \in (1,1.5)} G^r_\alpha(y) \right\} = \min_{y \in (1,1.5)} G^r_\alpha(\rho_1,1.5) = G^r_{0.378847}(y).$$

For $\alpha \leq 0.335278$, the lower minima of $G^r_\alpha(y)$ appear in $(\rho_1,1)$ or in (2,3) (see Matrix I and (iii)). Obviously, for any $y \in (\rho_1,1)$ or $y \in (2,3)$, $G^r_\alpha(y)$ is a negative and increasing function with respect to $\alpha$. Let $\alpha_1 < \alpha_2$ and $G^r_{\alpha_1}(y_{0,\kappa})$, $G^r_{\alpha_2}(y_{1,\lambda})$ be the lower minima of $G^r_\alpha(y)$, where $y_{0,\kappa} \in (\rho_1,1)$ and $y_{1,\lambda} \in (2,3)$. Then,

$$G^r_{\alpha_1}(y_{0,\kappa}) < G^r_{\alpha_1}(y_{1,\lambda}) < G^r_{\alpha_2}(y_{1,\lambda});$$

the same inequality holds if $y_{0,\kappa} \in (2,3)$ and $y_{1,\lambda} \in (\rho_1,1)$. Thus,

$$\max_{0 \leq \alpha \leq 0.335278} \left\{ \min_{y \in ((\rho_1,1),(2,3))} G^r_\alpha(y) \right\} = \min_{y \in (2,3)} G^r_{0.335278}(y). \quad \Box$$

Remarks:

4 Let $0 \leq \alpha \leq 0.335278$. If $\alpha$ increases, the maximum ripple decreases.

5 For $0.378847 \leq \alpha \leq 0.5$, whenever $\alpha$ increases, the highest overshoots increase, while the highest undershoots decrease.

6 Whenever $\alpha \in [0.335278, 0.378847]$, the minimum of the maximum Gibbs ripple is obtained. A numerical evaluation shows that for $\alpha \approx 0.365$ we have the smallest maximum overshoot taken at $y \approx 4.49$, and is approximately $0.57\%$.\[\]
1.4 Gibbs Phenomenon for Meyer Sampling Expansions

The scaling function \( \phi(y) \) of Meyer’s wavelet is given by its Fourier transform in Example 3 of section 1.1 and the sampling function satisfies \( \hat{S}(\omega) = \frac{\hat{\phi}(\omega)}{\phi'(\omega)} \). The Meyer’s sampling function is an even function satisfying:

\[
S(y) = \frac{\sin(\pi y)}{\pi y} - \frac{2}{\pi} \sin(\pi y) \int_{0}^{\pi/3} \hat{S}(\omega + \pi) \sin(\omega y) d\omega.
\] (1.9)

To derive (1.9) we use the property (P4) in section 1.1: \( \hat{S}(\omega) + \hat{S}(\omega + 2\pi) + \hat{S}(\omega - 2\pi) = 1 \), \( |\omega| \leq \frac{2\pi}{3} \) and we apply (P4) in the Fourier inversion formula of \( S \). The sampling function \( S \) and the scaling function \( \phi \) do not coincide. Since (1.9) involves the Shannon sampling function, in order to examine whether \( T_m f(x) \) shows Gibbs Phenomenon in the Meyer’s case, we use the following:

**Notation 3.**

(i) \( G_{r,M}(y) \) is the Gibbs function on the right hand side of the discontinuity point corresponding to the Meyer’s sampling function (1.9)

(ii) \( G_{r,F}(y) \) is the Gibbs function on the right hand side of the discontinuity point corresponding to the Shannon’s sampling function as in section 1.3.

We use the following Proposition for our calculations:

**Proposition 1.5** Let \( \hat{S}(\omega) \) be an even function of bounded variation in \( L_1(\mathbb{R}) \). Let \( F(\omega) = \sum_{n \in \mathbb{Z}} \hat{S}(\omega + 2n\pi) \) and \( \int_{0}^{\pi} \left| \frac{F(\omega) - F(-\omega)}{\omega} \right| d\omega < \infty \), then for any \( y \) in a bounded interval of \( \mathbb{R} \) we have:

\[
\sum_{n > 0} S(y + n) = \frac{1}{2} + \frac{1}{2\pi} \int_{0}^{\infty} \hat{S}(\omega) \frac{\sin(\omega (y + 1/2))}{\sin(\omega/2)} d\omega.
\]

**Proof.**

\[
\sum_{n > 0} S(y + n) = \frac{1}{2\pi} \lim_{N \to \infty} \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}} \hat{S}(\omega + 2k\pi) e^{i(\omega + 2k\pi)y} \frac{e^{i\omega} - e^{i\omega(N+1)}}{1 - e^{i\omega}} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{S}(\omega) e^{i\omega y} \frac{e^{i\omega}}{1 - e^{i\omega}} d\omega + \frac{1}{4\pi} \lim_{N \to \infty} \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}} \hat{S}(\omega + 2k\pi) e^{i(\omega + 2k\pi)y} \frac{\cos((N + 1/2)\omega)}{i \sin(\omega/2)} d\omega
\]

\[
+ \frac{1}{4\pi} \lim_{N \to \infty} \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}} \hat{S}(\omega + 2k\pi) e^{i(\omega + 2k\pi)y} \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)} d\omega.
\]

The second term tends to zero by the well known Lemma of Riemann-Lebesgue, while the third term tends to \( 1/2 \) as an application of Jordan’s Theorem, thus:

\[
\sum_{n > 0} S(y + n) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{S}(\omega) e^{i\omega y} \frac{e^{i\omega}}{1 - e^{i\omega}} d\omega.
\]

Since \( e^{i\omega y} \frac{e^{i\omega}}{1 - e^{i\omega}} = -e^{i\omega(y + 1/2)} \frac{1}{2i \sin(\omega/2)} \), and since \( S \) is an even function the result follows. \( \square \)
By Theorem 1.2, \( G_{\alpha,M}(y) = \alpha S(y) + \sum_{n>0} S(y + n) \), so we use Proposition 1.5 to get:

\[
G_{\alpha,M}(y) = \alpha S(y) + \frac{1}{2} - \frac{1}{2\pi} \int_0^{\pi/3} \hat{S}(\omega) \frac{\sin(\omega(y + 1/2))}{\sin(\omega/2)} d\omega. \tag{1.10}
\]

By substitution of (1.9) in (1.10) we calculate:

\[
G_{\alpha,M}(y) = G_{\alpha,F}(y) - \frac{2\alpha}{\pi} \sin(\pi y) \int_0^{\pi/3} \hat{S}(\omega + \pi) \sin(\omega/2) d\omega
+ \frac{1}{2} \sin(\pi y) \int_0^{\pi/3} \hat{S}(\omega + \pi) \frac{1 - \alpha}{\cos(\omega/2)} d\omega. \tag{1.11}
\]

First we examine whether \( S \) is positive on \( \mathbb{R} \).

**Proposition 1.6** All sampling functions of Meyer type satisfy \( S(y) < 0 \) for some \( y \in \mathbb{R} \).

**Proof.** We apply \( y = 1/2 \) or \( y = 3/2 \) in (1.9) and we use \( 0 \leq \hat{S}(\omega) \leq 1 \) to get

(i) \( 0.466 \approx 2(\sqrt{3} - 1)/\pi \leq S(1/2) \leq 2/\pi \approx 0.6366 \),

and

(ii) \( S(3/2) = -\frac{2}{3\pi} + \frac{2}{\pi} \int_0^{\pi/3} \hat{S}(\omega + \pi) \sin(3\omega/2) d\omega \geq -\frac{2}{3\pi} + \frac{4}{\pi} \int_0^{\pi/3} \hat{S}(\omega + \pi) \sin(\omega/2) d\omega \)

\[
= -\frac{2}{3\pi} + 2 \left( \frac{2}{\pi} - S(1/2) \right).
\]

Thus,

\[
S(3/2) \geq \frac{4}{\pi} - \frac{2}{3\pi} - 2S(1/2). \tag{1.12}
\]

For any \( S(1/2) \in [0.4638, 0.6366] \), there exists an integer \( k_0 \geq 2 \), such that \( S(k_0 + 1/2) < 0 \).

By (1.5) this is obvious for \( S(1/2) > 1/2 \). For \( S(1/2) \leq 1/2 \), we use (1.12) and we have

\( S(1/2) + S(3/2) \geq \frac{4}{\pi} - \frac{2}{3\pi} - S(1/2) > 1/2 \), thereby obtaining the result by applying (1.5) again.

\( \square \)

**Theorem 1.8** [1] Let \( f(t) = \alpha f(t - 0) + (1 - \alpha)f(t + 0) \), \( \alpha \in [0,1] \) be the value assigned to a function \( f \) at a jump location \( t \), where \( t \) is a dyadic rational, then the Meyer’s wavelet sampling expansion exhibits Gibbs Phenomenon in the following cases:

(M1) For any \( \alpha > A_1(1) \approx 0.306853 \), where \( A_1(y) \) is as in notation 1 and
(M2) For any $\alpha \in [0, 0.12459)$.

Proof.

(M1): Let $0.306853 \approx \alpha_1(1) < \alpha < 0.5$, then the Shannon's Gibbs function $G_{\alpha,F}^r(y) < 0$ for any $y \in (1, \rho_1)$ or in $(1, 2)$ (see theorem 1.4, Matrix 1), where $\rho_1$ is as in (C4). For such $y$'s the integrand in (1.11) is positive, so $G_{\alpha,M}^r(y) < 0$.

Let $\alpha \geq 1/2$ and $y = 1.5$, then the Shannon's Gibbs function $G_{\alpha,F}^r(y) < 0$ for any $y \in (1, 2)$ (see theorem 1.5). Let $\alpha = 1.5$, then the integrand in (1.11) is negative for $\omega \in [\arccos(\frac{\alpha}{2 - 2\alpha}), \frac{\pi}{3}]$ or $\omega \in [0, \pi/3]$, whenever $\alpha \in (1/2, 2/3)$ or $\alpha \geq 2/3$ respectively.

Now we integrate only on the previous segments and we use $\hat{S}(\omega) \leq 1$ to obtain that $G_{\alpha,M}^r(3/2) < 0$ for $1/2 < \alpha < 0.8399$. These values of $\alpha$ are solutions of the inequality

$$G_{\alpha,F}^r(3/2) = \frac{1}{\pi} \int_{K(\alpha)}^{\pi/3} \frac{(1 - \alpha) \sin(2\omega) - \alpha \sin(\omega)}{\cos(\omega/2)} d\omega < 0,$$

where $K(\alpha)$ is either $\arccos(\frac{\alpha}{2 - 2\alpha})$ or 0. Recall that $G_{\alpha,F}^r(3/2) = -(2\alpha)/(3\pi) + 1/2 - 2/\pi + 2/(3\pi)$. Let now $\alpha \geq 0.8399$. If $S(3/2) < 0$, by (1.10) we have $G_{\alpha,M}^r(3/2) < 0$, thus it suffices to examine the case $S(3/2) \geq 0$. We apply $y = 3/2$ in (1.9) and we get:

$$\frac{1}{\pi} \int_0^{\pi/3} \hat{S}(\omega + \pi) \sin(3\omega/2) d\omega = \frac{1}{2} \left(S(3/2) + \frac{2}{3\pi}\right),$$

(1.13)

Now we apply $y = 5/2$ in (1.11) and we use the fact that the numerator $(1 - \alpha) \sin(3\omega) - \alpha \sin(\omega)$ inside the integrand is negative for all values of $\alpha > 0.8399$. Using elementary trigonometric calculations, the integrand can be written as $((1 - \alpha) \sin(3\omega) - \alpha \sin(\omega))/\cos(\omega/2) = -\frac{19}{20} \frac{\pi}{\sin(3\omega/2)} + $ (negative term), so:

$$G_{\alpha,M}^r(5/2) = G_{\alpha,F}^r(5/2) - \frac{19\alpha}{20\pi} \int_0^{\pi/3} \hat{S}(\omega + \pi) \sin(3\omega/2) d\omega + ($ a negative term$)$$

and we use (1.13) to get:

$$G_{\alpha,M}^r(5/2) = G_{\alpha,F}^r(5/2) - \frac{19\alpha}{20\pi} \frac{1}{2} \left(S(3/2) + \frac{2}{3\pi}\right) < 0 \text{ for any } \alpha > 0.8399.$$

(M2): Let $\alpha \in [0, 0.306853)$. We consider $y = 3/2$, then we use (1.5) to deduce:

$$G_{\alpha,M}^r(3/2) = \alpha S(3/2) + 1/2 - S(1/2) - S(3/2).$$

(1.14)

Take $S(3/2) \geq 0$, it is easy to see that $G_{\alpha,M}^r(3/2) < 0$ for any $\alpha \in [0, 1]$, whenever $1/2 < S(1/2)$. In case where $1/2 \geq S(1/2) \geq 0.4660$ (see also Proposition 1.6 part (i)), we use (1.12) and the bounds on $S(1/2)$ to deduce that $S(3/2) > 0.061$ so (1.14) yields that whenever $\alpha \in [0, 0.306853]$ we have $G_{\alpha,M}^r(3/2) < 0$.

Let now $S(3/2) < 0$. By (1.12) we easily deduce that $S(1/2) > 1/2$. The relation (1.14) implies that we have to consider $S(3/2) \leq 1/2 - S(1/2)$, else $G_{\alpha,M}^r(3/2) < 0$ for any $\alpha \in [0, 1]$. Then we have

$$G_{\alpha,M}^r(3/2) < 0 \iff \alpha > 1 - \frac{S(0.5) - 0.5}{-S(1.5)} \text{ and } G_{\alpha,M}^r(0.5) < 0 \iff \alpha < 1 - \frac{0.5}{S(0.5)}.$$

(1.15)
It is elementary to see that there is at least one \( \alpha \in [0, 1] \) satisfying both inequalities in (1.15) if and only if
\[
-2 S(1/2) \lfloor S(1/2) - 1/2 \rfloor \leq S(3/2),
\]
and this implies that Gibbs Phenomenon exists for any \( \alpha \). In this case, we observe that \( S(1/2) \leq 0.571162 \) are the solutions of the inequality
\[
-2 S(0.5) (S(0.5) - 0.5) < \frac{4}{\pi} - \frac{2}{3\pi} - 2 S(0.5),
\]
which by (1.12) is less than \( S(3/2) \), thus (1.16) holds. It suffices to examine \( S(1/2) > 0.571162 \). Then by (1.15) we have Gibbs Phenomenon for
\[
\alpha < 0.12459 \approx 1 - \frac{0.5}{0.571162} < 1 - \frac{0.5}{S(0.5)}. \quad \square
\]

**Proposition 1.7** In the following cases, the sampling expansion of Meyer exhibits Gibbs Phenomenon for any \( \alpha \in [0, 1] \):

(M3) There is at least one \( y \in (1, 1.5] \) such that \( S(y) \geq 0 \);
(M4) \(-2 S(0.5) \lfloor S(0.5) - 0.5 \rfloor < S(3/2) < 0;\)
(M5) \([S(5/2) - S(3/2)] [S(0.5) - 0.5] \geq S(3/2)^2 \), where \( S(3/2) \leq 0.5 - S(0.5) \) and \( S(5/2) > 0.5 - S(3/2) - S(0.5) \).

**Proof.**

(M3): If \( \alpha > A_1(1) \), we apply (M1). For \( \alpha \leq A_1(1) \) we apply (1.10).

(M4) is an immediate consequence of the proof of (M2) (see also (1.15)).

(M5): We apply \( y = 5/2 \) in theorem 1.2 and we use (1.5) to get
\[
G_{\alpha,M}^r (5/2) < 0 \Leftrightarrow \alpha < 1 - \frac{1/2 - S(3/2) - S(1/2)}{S(5/2)}.
\]

It is easy to see that there is at least one \( \alpha \) satisfying both the above inequality and the first inequality in (1.15) if and only if (M5) holds, hence Gibbs Phenomenon exists for any \( \alpha \). \( \square \)

**Remarks**

7 There exists sampling function of Meyer type that does not satisfy conditions (M3)-(M5), but shows Gibbs Phenomenon for any \( \alpha \in [0, 1] \) (see Example (3.2) in section 1.1). We don’t know yet any sampling function that does not show Gibbs Phenomenon for some \( \alpha \in [0.12459, 0.306853] \).

8 For \( \alpha \leq 0.375847 \), nice candidates for reducing largest Meyer’s overshoots compared to Shannon’s are those satisfying \( S(3/2) > 0 \), but not conditions (M3)-(M5). If \( \alpha > 0.5 \), the highest overshoots may increase or decrease according to the shape of \( \hat{S} \) (see (1.11)).
We present some examples based on these remarks.

**Example (5.1):** Consider the Meyer sampling function of Example 3.1. By (M4) or (M5), the sampling formula obtained from this family of sampling functions exhibits Gibbs Phenomenon for any $\alpha$. For $\delta = 0.5$ it coincides with Shannon’s sampling function and for $\delta > 0.5$ gives more successful approximation of the Dirac function. Finally we should note that for $\delta = 0.75$ and $\alpha \approx 0.33$ we have very low Gibbs overshoot.

![Figure 1.11: Gibbs function for the Meyer sampling expansion corresponding to example 3.1 with $\delta = 0.75$ of the unitstep function for various values of $\alpha$. Black line corresponds to the value $\alpha = 0.33$. Dot line corresponds to the value $\alpha = 0.13$. Notice that the maximum ripples on the right hand side of the discontinuity are inconsiderable.](image)

**Example (5.2):** Gibbs phenomenon for Meyer’s sampling function of Example 3.1 with $\delta = 0.49$.

![Figure 1.12: Gibbs function for the Meyer sampling expansion corresponding to example 3.1 with $\delta = 0.49$ of the unitstep function for various values of $\alpha$. Black line corresponds to the value $\alpha = 0.4$. Dot line corresponds to the value $\alpha = 0.6$. Notice that the maximum ripples on the right hand side of the discontinuity are inconsiderable for $\alpha = 0.6$, while maximum ripples on the left hand side of the discontinuity are inconsiderable for $\alpha = 0.4$.](image)
Bibliography


