Classical Control Theory:

A Course in the Linear Mathematics of Systems and Control

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Preface

This book originates in sets of notes for introductory courses in *"Classical Control Theory"* that I have taught to advanced undergraduate students in Electrical Engineering and Applied Mathematics in England and the U.S. over a number of years.

I believe that the subject of Classical Control Theory, despite its 'traditional' appellation, contains mathematical techniques crucial to an understanding of System Theory in general. This fact is in danger of being obscured in the surfeit of design examples and leisurely expositions of the theoretical aspects that seems to be the norm in the subject. The present text is an attempt to remedy this perceived imbalance: it is a succinct presentation of the underlying *linear system approach* as it relates to the subject of linear control theory. Even though one cannot do full justice to the remarkable bits of theory that have enriched the subject over the years (such as the foundations of the Routh-Hurwiz criterion and its recent refinements), this book makes an attempt to reveal the beauty and power of the mathematical tools that form the foundations of system theory.

An important feature of this book is the employment of the unifying perspective of **linear mathematics**. By this, we mean linear differential equations and, through the Laplace transform, systems of linear equations with polynomial coefficients.[1] The core of this approach is the breaking up of systems into 'modes': we freely switch between rational functions and the corresponding time functions and use the Partial Fractions Expansion in conjunction with the Laplace transform as our main tool. Since it is frequently the case that students have not seen the Laplace transform before,

[1]This perspective is actually making a comeback in system theory: starting with the polynomial system description of the '80's and leading up to some thorough algebraic approaches of more recent vintage. Naturally, in an introductory course such as this, we can only hint at some of the interesting aspects of this topic.

an account of the relevant theory is given in a preparatory chapter. One can argue that *transform techniques* constitute the main contribution of Applied Mathematics to general mathematical culture.

The teaching of Control theory at an elementary level tends to put emphasis on the Engineering applications. Indeed, the majority of Control courses, in this country and abroad, take just such an Engineering perspective, emphasizing the practical design examples—which can no doubt be of interest to students and help explain the significance and widespread applicability of the control theory viewpoint. This is also reflected in the available textbooks, which tend to be of considerable bulk, perhaps reflecting a trend in undergraduate teaching internationally. In my experience this is not an approach that suits everybody and many students used to more concise and elegant presentations of material are put off by it. The present text will hopefully appeal to them.

Our presentation of the standard topics of classical control (Routh-Hurwitz, root locus and the Nyquist plot) is from a modern perspective and we try to give enough of a flavour of the mathematics involved to keep the student interested. In a sense, it is true for this author, as no doubt for many other authors, that this is the kind of Control course that one would have liked to have had as a student. I hope that the attempt at conceptual clarity outweights the slightly more challenging mathematical level adopted.

The texts makes liberal use of Examples to illustrate the theory. A few Exercises are sprinkled throughout the text to help reinforce the theory, with more complete Problem sets at the end of each chapter. A final section contains a collection of Exercises that draw as a rule from more than one chapter. We have used *Matlab*, as well as graphics packages such as Micrographics Designer and Corel Draw to produce the figures.

Chapter 1

Introduction: Systems and Control

1.1 Systems and their control

From the viewpoint of *system theory*, the role of a scientist or engineer is to study the behaviour of **systems** of various origins: mechanical, electrical, chemical, biological or mixes of these. A system, for our purposes, is then very much a *black box*, with connections to the outside world roughly divided into **inputs** and **outputs**, see Figure 8.1. A system typically consists of a number of inter-connected **sub-systems**. These inter-connections are shown in a **block diagram** involving the system components and a *'graph'* tracing the inputs and outputs through the various blocks, as in Figure 1.2.

A modern *jet engine* (see Figure 1.3) is an example of a complex mechan-

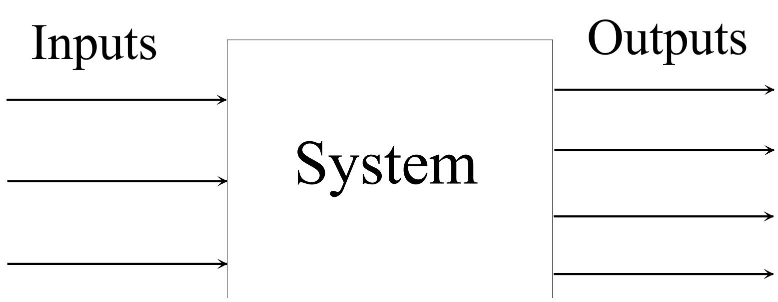


Figure 1.1: A system as a black box

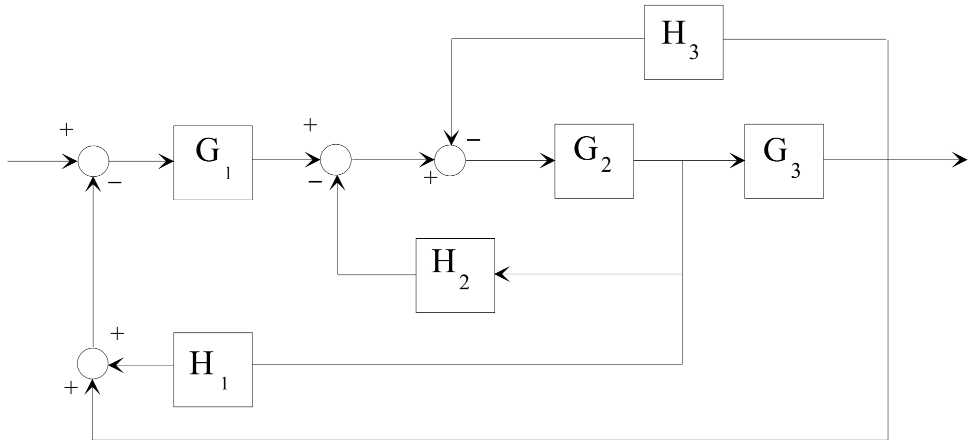


Figure 1.2: Block diagram of a system

ical/chemical machine that is challenging to model accurately. The number of variables that must be used to give an adequate description is large. Some of these can be measured, such as fan speed, pressure and temperature at accessible stages of the engine; some cannot be measured directly, such as thrust and stall margins. Yet from the point of view of performance, it is crucial that all variables stay within reasonable bounds. In system theory, these variables are called **state variables**. A first step in analyzing a system is then to derive an *adequate* [1] **model**, usually in the form of a system of differential or difference equations.

The object of **control theory** is to make systems perform specific tasks by using suitable *control actions*. When the pilot pushes the throttle, for example, the action is translated into a control signal that makes the engine increase its power output by a specified amount. Thus, even though the controller of a system is occasionally only a small part of the overall system, its role is crucial. From the control designer's viewpoint, the model of a system is given as a single block and the task is then to design a controller.

Feedback Suppose you are standing in front of a long corridor and someone challenges you to walk the length of it blindfolded and without touching the walls. You take a good look, position yourself, then the blindfold is put on. Now even though you think your mental impression of the corridor is good,

[1] A definition of *'adequate'* already involves subtle theoretical control considerations, such as controllability and minimality.

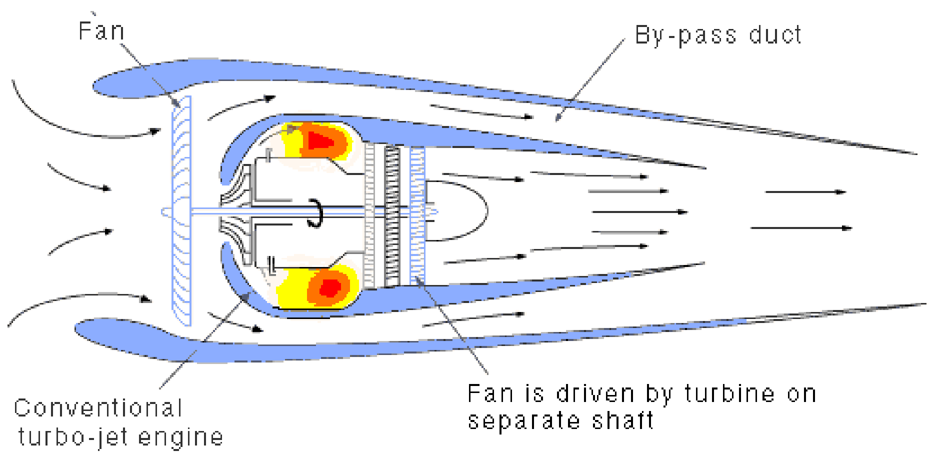


Figure 1.3: Schematic diagram of a turbofan jet engine. (Source: Natural History Museum)

it is very likely that your path will veer from the straight line that you have envisaged and probably end up banging your head on a sidewall.

What made your task difficult was the lack of **feedback**: sensory information telling you whether you are on course or not. (Ashby's book [2] is a nice exposition of the main ideas of feedback, even though it is by now quite dated.)

The number of examples where feedback is present is endless: driving a car, thermostatic controllers for heating systems (or your own body's temperature control called homeostasis), even the Bank of England's adjustments of the basic rate in response to economic conditions.

The history of *feedback* is almost as long as that of science itself. There is feedback at work in the *water clock* of the ancient Greeks. The flyball governor of James Watt (1769) was of such interest that J.C. Maxwell became one of the originators of control theory with his paper "On Governors" (1868).

The basic control arrangement, shown in Figure 1.4 involves measuring the outputs of a given system and comparing them with desirable responses to form the *error signal*. The errors are then fed into the **controller** (which can be a mechanical or fully electronic system) to produce the control signal, which is finally applied to the system through an **actuator**. The crucial feature is the **closed loop** that feeds the output back to the system.

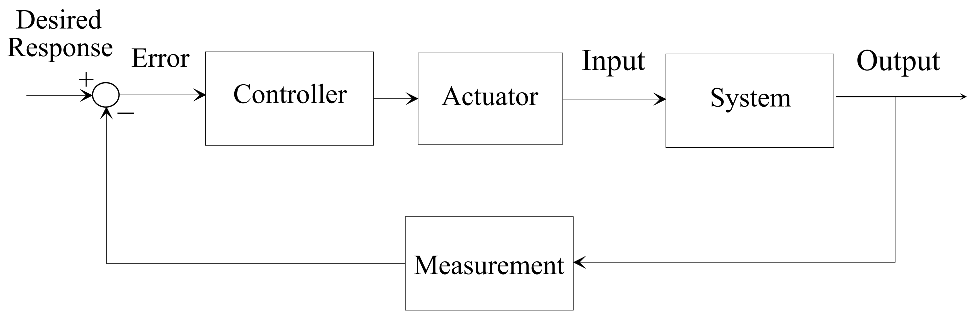


Figure 1.4: A Standard Control System

1.2 The Development of Control Theory

Even though the principle of feedback control was intuitively understood since antiquity, the theory of control has more recent origins.

During the Second World War, theoretical research was initiated into problems of significance for the war effort, such as accurate automatic gun pointing. This involved two aspects: predicting the trajectory of the target and steering the weapon in its direction fast and with small overshoot. This research involved engineers as well as pure scientists and led to the development of the new theories of Control Systems and Signal Analysis. Norbert Wiener's *Cybernetics* is a fascinating account of the early steps in understanding feedback systems by one of its main pioneers and popularizers (Figure 1.5.) The decades after the war witnessed an acceleration in the understanding and development of control techniques based on *linear mathematics* in the transform or *frequency domain*: the Laplace transform and the resulting modal analysis using Nyquist, Bode and Nichols diagrams and the Root Locus approach. This research was largely carried out in large laboratories in the U.S., such as the Bell Telephone Laboratory, where Bode, Nyquist and Black worked.[2]

Applications of these control design methodologies were primarily in the process industry and involved by-and-large **single-input, single-output** control of systems that were adequately modeled as *linear*. In the 60's, a new area of application stimulated control research: the aeronautics and aerospace industry demanded fast, accurate control of aeroplanes and rockets. Since these systems had a faster transient response structure and since

[2]We shall be learning what most of these are in this course, see Chapters 4 and **??**.

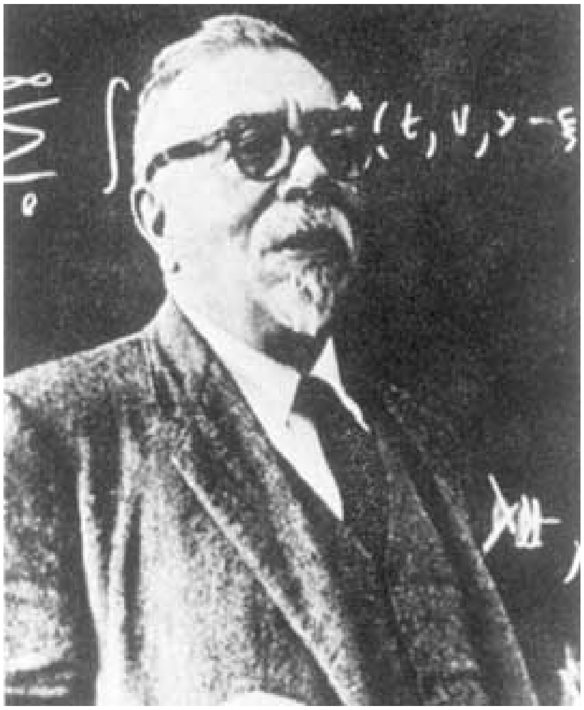


Figure 1.5: Norbert Wiener

the decoupling assumptions used previously did not hold (roughly, each variable was significantly affected by more than one other variable), this led to the development of **multivariable control systems**. Furthermore, since efficiency and optimality were paramount, the theory of **optimal control** flourished, its foundations going back to the nineteenth century *Calculus of Variations*[3] and drawing on important Soviet research on optimality conditions (Pontryagin and his school.)

The result was that the more direct **state space** approach gained ground, leading to an improved understanding of concepts such as **controllability, observability, minimality, system zeros** etc. In recent years, attempts to combine the two approaches (frequency domain and state space) have led to a more balanced perspective that draws profitably on both viewpoints.

These successes established the discipline of (Linear) Control Theory both as a branch of Engineering and as modern Applied Mathematics. Still, the vast majority of real systems is **nonlinear**. In the past, the nonlinearities

[3]Itself a generalization of Classical Mechanics.

were dealt with by essentially patching together linear regimes (e.g. **gain scheduling**.) Nonlinear Control theory is a more recent creation, one that involves quite a different set of mathematical tools. It must be said that the applications of nonlinear control design are still few, theory running far ahead of engineering practice.

In this course, we shall present the fundamentals of *Linear Control*, mainly from the frequency domain point of view.

1.3 The Aims of Control

The main benefit of **feedback control** is perhaps intuitively clear: the information fed back helps keep us on course, in the presence of *uncertainties or perturbations*. Note that in a 'perfect world', feedback is not strictly necessary since, provided our control plan is accurate enough (remember the dark corridor), we can steer towards the target with no information feedback. This goes by the name of **open-loop control** and played an important role in the history of control theory. I believe the notion is of less use now.

Here is a list of the most common desirable benefits associated with the use of (feedback) control:

Path Following We can use control to make the system outputs follow a desired path (e.g. in Robotics.)

Stabilization Make an inherently unstable system *stable* (as in some *fly-by-wire* aircraft.)

Disturbance rejection Reduce or eliminate the effect of undesirable inputs to the system (random or deterministic.)

Performance improvement The overall closed-loop system often has better transient characteristics (pole placement.)

Robustness to plant uncertainties Real systems are not known exactly; even though their models are only approximate, control can ensure that performance is minimally affected by these uncertainties.

1.4 Exercises

1. Think up as many systems as you can where control is used (at least ten.)

   These can be mechanical, chemical, biological or of any nature whatever.

2. Read in, for example, Dorf, about a number of practical control design problems.

Chapter 2

Linear Differential Equations and The Laplace Transform

2.1 Introduction

2.1.1 First and Second-Order Equations

A number of situations of practical interest involve a *dynamic* variable[1], say x, whose rate of change is proportional to the present value of x, so that

$$\frac{dx}{dt} = kx,$$

for some real k. If x_0 is the starting value, the solution is then

$$x(t) = x_0 e^{kt}, \quad t \geq 0,$$

which, for $k < 0$ decays exponentially and, for $k > 0$, grows exponentially, in principle without bound. We say that for $k < 0$, the system is **stable**, while for $k > 0$ it is **unstable**. If $k = 0$, $x(t) = x_0$. Also, the larger $|k|$ is, the faster the transient (to zero in the stable case or to *infinity* in the unstable one.)

An even larger number of practical systems is adequately modeled by the second-order equation

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0 \frac{dx}{dt} + \omega_0^2 x = 0, \tag{2.1}$$

[1] i.e. one changing over time

where $\omega_0 > 0$ is called the *natural frequency* and ζ the *damping ratio* of the system; it is usually taken to be non-negative, $\zeta \geq 0$, but **negative** damping is also possible, for instance in electronic circuits that include *negative resistance*.

The method for finding the solution of 2.1 for initial conditions x_0 and $\dot{x}_0$ involves the **auxilliary equation**

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0, \tag{2.2}$$

a second-order polynomial equation that has, therefore, two roots. We shall start calling the roots of auxilliary equations **poles**; this term will then generalize to equations of higher order.

Because of the special form that we used to write equation 2.1, the two roots are

$$s = \omega_0(-\zeta \pm \sqrt{\zeta^2 - 1}).$$

When there is no damping, $\zeta = 0$, we get the *harmonic oscillator* solution at the natural frequency ω_0:

$$x(t) = x_0 \cos(\omega_0 t) + \frac{\dot{x}_0}{\omega_0} \sin(\omega_0 t).$$

For small damping, $0 < \zeta < 1$, we have a pair of complex poles with nonzero imaginary part,

$$s = \omega_0(-\zeta \pm i\sqrt{1 - \zeta^2}),$$

and the solution is a sinusoidal wave of frequency

$$\omega = \omega_0\sqrt{1 - \zeta^2}$$

modulated by the exponential envelope $\pm e^{-\zeta\omega_0 t}$:

$$x(t) = e^{-\zeta\omega_0 t}(x_0 \cos(\omega t) + \frac{\dot{x}_0 + \zeta\omega_0 x_0}{\omega} \sin(\omega t)) \tag{2.3}$$

This is the **underdamped** case.

At the critical damping $\zeta_c = 1$, the solution is

$$x(t) = x_0 e^{-\omega_0 t} + (\dot{x}_0 + \omega_0 x_0) t e^{-\omega_0 t}.$$

Finally, the *overdamped* case gives two real poles

$$s = \zeta\omega_0(-1 \pm \sqrt{1 - \frac{1}{\zeta^2}})$$

and hence a solution that is a sum of real exponentials.

Exercise 2.1. Which configuration of poles of the general second-order system

$$\ddot{x} + a\dot{x} + bx = 0$$

is not possible for the system 2.1 above? *(Hint: Note the positive x term in 2.1.)*

It is very helpful to consider how the *poles* move in the complex plane $\mathbb{C}$ as the damping ζ increases from zero. The harmonic oscillator corresponds to poles on the imaginary axis; underdamped motion corresponds to a pair of complex conjugate poles, while critical damping means a double (by necessity) real pole. The overdamped poles move on the real axis and in opposite directions as ζ increases (see Figure 2.1.) One of the main topics of this

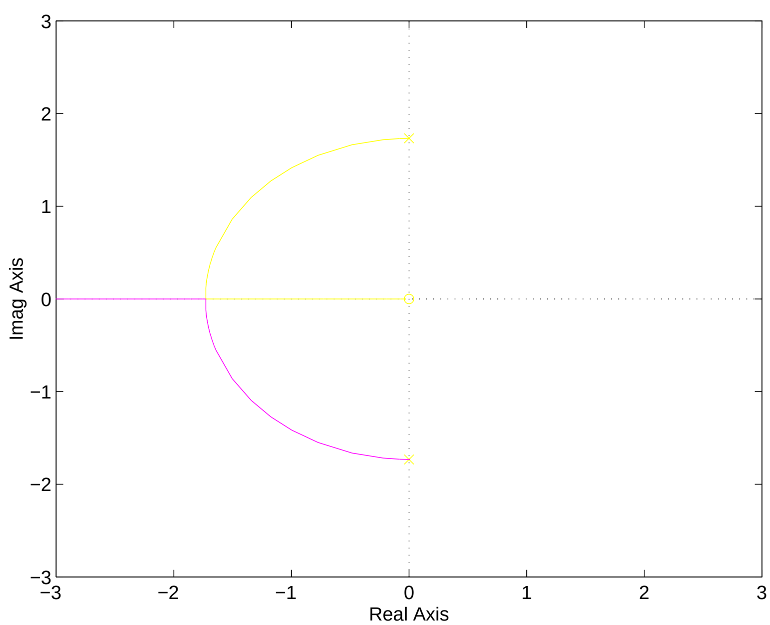


Figure 2.1: The movement of poles as a function of the damping ratio ζ

course, the *root locus* method, is in essence a generalization of the diagram of Figure 2.1. It shows the movement of poles as a parameter on which they depend is varied.

2.1.2 Higher-Order Equations and Linear System Models

There is no reason to limit ourselves to equations of second order. The general nth order homogeneous linear differential equation is

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0. \tag{2.4}$$

When the functions a_i are constant, we talk of *constant-coefficient linear differential equations*; this is the class of equations that will be of concern in this course.

In physical systems, the homogeneous equation describes the **unforced system**, roughly when external forces are zero. When we have a non-zero right-hand side, the *forced system* that is obtained corresponds to our notion of a **controlled system**.

The main point of the analysis we are about to present is that, in a natural sense to be made precise soon, **no new behaviour is observed** beyond that of first and second-order systems.[2] The key is a generalization of the auxilliary equation method that reveals the **modal structure** of linear systems. It is based on the Laplace transform.

In order to motivate the introduction of the Laplace transform, let us look at a **linear system** which, for now, will mean a system with input u and output y, the two being related through a linear, constant-coefficient ordinary differential equation (see Figure 8.1 of Chapter 1.) This implies that there is a *linear relation* involving the derivatives of u and y—up to a certain maximal order, say n. *Linear relation* here means the same as a *linear combination* of vectors in vector algebra. Thus we assume that the linear system is described by the equation

$$\sum_{i=0}^{n} a_{n-i}\frac{d^i y}{dt^i} = \sum_{j=0}^{m} b_{m-j}\frac{d^j y}{dt^j}, \tag{2.5}$$

where the coefficients $\{a_i\}$ and $\{b_j\}$ are constant and $a_0 \neq 0$. We assume that $n \geq m$ (for reasons that will become clear later.)

[2]This is not quite true, since we could have, for example, multiple complex pairs; these are quite rare in practice, though—non-generic in the modern terminology. The reader is advised to think of how a real polynomial can always be factorized into linear and quadratic factors, corresponding to real roots or pairs of complex roots, respectively.

Now suppose the input is given by the complex exponential function

$$u(t) = e^{st}$$

(remember how the auxilliary equation was derived.) The parameter s is called a *complex frequency*. Since taking the derivative of this waveform is the same as multiplying it by s, let us check that an output of the form

$$y(t) = g(s)e^{st},$$

for $g(s)$ some (in general, complex) number, actually *solves* the differential equation (more precisely is a particular integral of the d.e.) Substituting the given u and y into equation 2.5, we get

$$g(s)(\sum_{i=0}^{n} a_{n-i}s^i)e^{st} = (\sum_{j=0}^{m} b_{m-j}s^j)e^{st}. \tag{2.6}$$

Now, **provided s is not a zero of the algebraic equation**

$$\sum_{i=0}^{n} a_{n-i}s^i = 0$$

we can satisfy the above differential equation with

$$g(s) = \frac{\sum_{j=0}^{m} b_{m-j}s^j}{\sum_{i=0}^{n} a_{n-i}s^i}.$$

The rational function expression on the right-hand side is called the *transfer function* of the linear system. This is similar to the **eigenvalue/eigenvector** concept for linear transformations (matrices.) We say that the complex exponential e^{st} is an **eigenvector** of the linear system, with **eigenvalue** the complex number $g(s)$. The intervention of complex numbers in a *real* number context should not worry us more that the use of complex numbers in solving the eigenvalue problem in matrix analysis (necessitated by the fact that a polynomial with real coefficients may have complex roots.)

In the next section, we start by defining the Laplace transform and giving some of its properties. We then give the method for finding the general solution of equation 2.4.

Exercise 2.2. Find the complete solution $y = y_{CF} + y_{PI}$ of the second-order equation

$$\ddot{y} + a\dot{y} + by = u,$$

where $u(t) = e^{st}$ and s is not a root of the auxilliary equation. Under what conditions does the complementary function y_{CF} go to zero as $t \to \infty$?

Hence verify that the transfer function of the linear system with input u and output y described by above the linear differential equation is

$$g(s) = \frac{1}{s^2 + as + b}$$

and justify the statement that the output $g(s)u(t)$ is the steady-state response of the linear system to the input $u = e^{st}$.

2.2 The Laplace Transform Method

The reader may have already met **Fourier series** expansions of periodic signals $x(t + T) = x(t)$

$$x(t) = \sum_n c_i e^{2\pi int/T},$$

where the coefficients are given by

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-2\pi int/T}dt$$

or the **Fourier transform** of finite energy signals,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

The Laplace transform is a more powerful technique and applies to a much larger class of signals than either of the above methods. The limitation is that we consider time functions on the half-infinite interval $[0, \infty)$ only, instead of the whole real axis.

Definition 2.1. If $f(t)$, $t \geq 0$ is a signal, the **Laplace transform** $\mathcal{L}(f)$ of f is defined by

$$\mathcal{L}(f)(s) = \int_0^{\infty} f(t)e^{-st}dt, \tag{2.7}$$

where s is a complex number for which the integral converges.

Roughly, the Laplace transform is a function of the complex argument s, well-defined at least in the region of the complex plane where the above integral converges. Since most of us are used to thinking of integrals in the *real* domain, this intrusion of complex numbers may seem confusing. If you are happier with real integrals, use the polar form

$$e^{st} = e^{(\sigma+i\omega)t} = e^{\sigma t}(\cos\omega t + i\sin\omega t) \tag{2.8}$$

and write the defining integral in the form

$$\mathcal{L}(f)(\sigma,\omega) = \int_0^\infty f(t)e^{-\sigma t}\cos(\omega t)dt + i\int_0^\infty f(t)e^{-\sigma t}\sin(\omega t)dt, \tag{2.9}$$

involving only real integrals. In practice this is unnecessary and one considers s as a *parameter* and integrates as if it were real. The only concern is then with whether the upper limit at ∞ is defined. Let us look at the simplest example.

Example 2.1. The Laplace transform of the real exponential function e^{at} (a real, but arbitrary) is

$$\mathcal{L}(e^{at})(s) = \int_0^\infty e^{-(s-a)t}dt = \lim_{T\to\infty}\left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^T. \tag{2.10}$$

Here the upper limit is

$$\lim_{t\to\infty} e^{-(\sigma-a)t}e^{-i\omega t}$$

which is zero provided $\sigma > a$.

The condition $\sigma > a$ defines a *half-plane* in $\mathbb{C}$, to the right of the line

$$\{s \in \mathbb{C}\,; \quad \Re(s) > a\}.$$

Note that, for a negative (a decaying exponential), this half-plane contains the imaginary axis. For a positive (an *'unstable'* exponential in positive time), the imaginary axis is not in the region of convergence of the integral. This is related to the fact that the Fourier transform is defined for a stable exponential, but not for an unstable one.

Now we can write down the Laplace transform of e^{at} by only evaluating the lower limit $t \to 0$:

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}} \tag{2.11}$$

Remark 2.1. Note that the right-hand side of equation 2.11 is in fact *defined for all complex numbers* $s \neq a$ (the whole $\mathbb{C}$ plane except for a single point on the real axis.) This arises, of course, from the notion of *analytic continuation*, but we leave these considerations to a course in complex analysis. In this course, we shall quickly forget the convergence issue and work with the analytically continued transform, implicitly assuming that a suitable region can be found to make the evaluation at the upper limit vanish.

Exercise 2.3. By writing

$$\cos \omega_0 t = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$$

or otherwise check that the Laplace transform of $\cos \omega_0 t$ is

$$\boxed{\mathcal{L}(\cos(\omega_0 t)) = \frac{s}{s^2 + \omega_0^2}} \tag{2.12}$$

(What is the region of convergence?)

Similarly, check that

$$\boxed{\mathcal{L}(\sin(\omega_0 t)) = \frac{\omega_0}{s^2 + \omega_0^2}} \tag{2.13}$$

2.2.1 Properties of the Laplace Transform

We have started building up a list of **Laplace transform pairs** which are going to be of use later on in solving differential equations and in handling control problems of interest. In order to add to the list, the best way forward is to first develop some basic properties of the Laplace transform and use them to derive transforms of larger and larger classes of functions. From now on, we shall use the notation

$$X(s) = \mathcal{L}(x(t))(s).$$

Since the Laplace transform is defined by the integral of x multiplied by the complex exponential function e^{-st}, it is clearly **linear** in x; in other words

$$\mathcal{L}(ax_1(t) + bx_2(t)) = a\mathcal{L}(x_1(t)) + b\mathcal{L}(x_2(t)) = aX_1(s) + bX_2(s) \tag{2.14}$$

Property 1 (Frequency-shift). *If $X(s)$ is the Laplace transform of $x(t)$, then, for a a real number,*

$$\mathcal{L}(e^{-at}x(t)) = X(s+a)$$

The proof is straightforward:

$$\mathcal{L}(e^{-at}x(t)) = \int_0^\infty x(t)e^{-(s+a)t}dt.$$

(Note that this changes the region of convergence.)

As an application, derive the Laplace transform pairs

$$\boxed{\mathcal{L}(e^{-at}\cos(\omega_0 t)) = \frac{s+a}{(s+a)^2+\omega_0^2}} \tag{2.15}$$

and

$$\boxed{\mathcal{L}(e^{-at}\sin(\omega_0 t)) = \frac{\omega_0}{(s+a)^2+\omega_0^2}} \tag{2.16}$$

Property 2 (Differentiation of Laplace Transform). *If $X(s)$ is the Laplace transform of $x(t)$, then*

$$\frac{dX}{ds} = \mathcal{L}(-tx(t)).$$

More generally,

$$\frac{d^nX(s)}{ds^n} = \mathcal{L}((-t)^n x(t)).$$

The proof is again direct:

$$\frac{d}{ds}\int_0^\infty x(t)e^{-st}dt = \int_0^\infty x(t)\frac{de^{-st}}{ds}dt = \int_0^\infty (-t)x(t)e^{-st}dt.$$

The $(-1)^n$ factor can be taken to the other side, if desired. Thus, for example,

$$\boxed{\mathcal{L}(te^{-at}) = \frac{1}{(s+a)^2}} \tag{2.17}$$

and, more generally,

$$\boxed{\mathcal{L}(t^n e^{-at}) = \frac{n!}{(s+a)^{n+1}}} \tag{2.18}$$

(Note how the minus signs conveniently disappear from the above formuæ.)

The case $a = 0$ is traditionally listed separately:

$$\boxed{\mathcal{L}(1) = \frac{1}{s}} \qquad \boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}} \tag{2.19}$$

Let us remember that what we mean by the constant function 1 is, in fact, only the part on the closed half-infinite line $[0, \infty)$. In the Engineering literature, this is called the **Heaviside unit step function** and is written

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Some texts have the highly recommended habit of multiplying any function whose Laplace transform is to be taken by the Heaviside step function. Thus $\mathcal{L}(x(t))$ is really

$$\mathcal{L}(h(t)x(t)).$$

Since the lower limit of the integral defining the transform is zero and since h is identically equal to one on the positive real axis, this is harmless and prevents confusion between, for example, the cos function in $(-\infty, \infty)$ and the 'cos' function in the interval $[0, \infty)$ whose Laplace transform was found to be $s/(s^2 + \omega_0^2)$.

The above may seem a bit pedantic, until we get to the point where we address the problem of using the Laplace transform to solve differential equations *with given initial conditions*. For this, we use the next property of $\mathcal{L}$. Let $x(t)$ be a smooth real function, considered on the whole of the real axis; if we only consider the positive real axis, we require smoothness on $t > 0$, but we may have a jump at 0. Using the familiar right and left limits

$$x(0^+) = \lim_{\epsilon \to 0, \epsilon > 0} x(t + \epsilon)$$

and

$$x(0^-) = \lim_{\epsilon \to 0, \epsilon > 0} x(t - \epsilon),$$

we require

$$x(0) = x(0^+).$$

This condition is called **right-continuity**. The derivative of x is assumed to exist also for all $t \geq 0$.

Property 3 (Transform of Derivatives). *The Laplace transform of the derivative of a function $x(t)$ satisfying the conditions above is*

$$\mathcal{L}(\frac{dx}{dt}) = sX(s) - x(0). \tag{2.20}$$

More generally, if all the derivatives and limits exist and are right-continuous,

$$\mathcal{L}(\frac{d^n x}{dt^n}) = s^n X(s) - x(0)s^{n-1} - \dot{x}(0)s^{n-2} - \ldots - \frac{d^{n-2}x}{dt^{n-2}}(0)s - \frac{d^{n-1}x}{dt^{n-1}}(0). \tag{2.21}$$

The proof of 3 uses integration by parts.

$$\int_0^\infty \frac{dx}{dt} e^{-st} dt = \left[x(t)e^{-st}\right]_0^\infty - \int_0^\infty x(t) \frac{de^{-st}}{dt} dt$$

and the result follows since we have assumed (see remark 2.1) that s is chosen so that the upper limit of the fisrt term vanishes. (Also notice that the derivative is with respect to t this time.)

Exercise 2.4. Rederive the transform of $\sin \omega_0 t$ from the transform of $\cos \omega_0 t$ and the identity $d \cos x/dt = -\sin x$. Do not forget to use the initial value of cos at $t = 0$.

Property 4 (Positive Time Shift). *Suppose*

$$X(s) = \mathcal{L}(h(t)x(t)).$$

For any positive time τ, the transform of the function x shifted by time τ is

$$\mathcal{L}(h(t-\tau)x(t-\tau)) = e^{-\tau s} X(s). \tag{2.22}$$

Since τ is positive,

$$\mathcal{L}(h(t-\tau)x(t-\tau)) = \int_\tau^\infty x(t-\tau)e^{-st} dt$$

and the change of variable $u = t - \tau$ gives

$$\mathcal{L}(h(t-\tau)x(t-\tau)) = \int_0^\infty x(u)e^{-s(u+\tau)} du = e^{-s\tau} X(s),$$

as claimed.

2.2.2 Solving Linear Differential Equations

We now have all the tools we need to give a complete solution method for linear, constant-coefficient ordinary differential equations of arbitrary degree. The steps of the method will be made clear through an example.

Example 2.2. Consider the second-order differential equation

$$\ddot{y} + 2\dot{y} + 5y = u(t),$$

and the initial conditions

$$y_0 = 1, \quad \dot{y}_0 = -1.$$

Taking the Laplace transform of both sides and using the properties of the transform, we have

$$(s^2 Y(s) - s y_0 - \dot{y}_0) + 2(sY(s) - y_0) + 5Y(s) = U(s). \tag{2.23}$$

This algebraic (polynomial) relation between the transforms of y and u, $Y(s)$ and $U(s)$ is then solved for $Y(s)$ to give

$$Y(s) = \frac{U(s) + (s+2)y_0 + \dot{y}_0}{(s^2 + 2s + 5)} = \frac{U(s) + (s+1)}{(s^2 + 2s + 5)}. \tag{2.24}$$

If we are considering the response to a specific input, say $u(t) = 2t - 1$, then

$$U(s) = 2\frac{1}{s^2} - \frac{1}{s} = \frac{2-s}{s^2}$$

and

$$Y(s) = \frac{2 - s + s^2(s+1)}{s^2(s^2 + 2s + 5)} = \frac{s^3 + s^2 - s + 2}{s^2(s^2 + 2s + 5)}. \tag{2.25}$$

Finally, we invert the Laplace transform of Y to get the time signal $y(t)$, for $t > 0$. This involves breaking the denominator polynomial into its simple (irreducible) factors and then writing the ratio of polynomials as a sum of simpler terms, each involving just one of the simple factors of the denominator.

We shall look into this procedure in detail in the next section. For now, let us formulate the most general problem that can be treated using this method.

Definition 2.2. A rational function $\frac{n(s)}{d(s)}$ is called **proper** if the denominator degree is at least equal to the numerator degree, $\deg d(s) \geq \deg n(s)$.

A rational function is **strictly proper** if $\deg d(s) > \deg n(s)$.

Definition 2.3. A **linear differential relation** between the variables

$$x_1, \ldots, x_n$$

is the equating to zero of a finite linear sum involving the variables and their derivatives. We denote such a relation by

$$\mathcal{R}(x_1, \ldots, x_n) = 0.$$

This means an expression involving the variables and their derivatives, each term involving a single variable with a **constant coefficient**. Thus, a linear differential relation cannot involve *products* of the variables or their derivatives.

Since we assume that the sum is finite, there is a maximal derivative defined for each variable and hence a maximal degree of the relation. Thus, in the linear differential relation

$$\ddot{x}_2 + 3x_1^{(iv)} - x_2 + 10\dot{x}_1 = 0$$

the maximal degree of the relation is four. We write $\deg x_i$ for the maximal degree of the variable x_i in the relation.

It is easy to write down a linear differential relation in the Laplace transform domain, making use of the properties of the Laplace transform, in particular Property 3. Note that we do not assume zero initial conditions. Thus, $\mathcal{R}$ becomes

$$a_1(s)X_1(s) + a_2(s)X_2(s) + \ldots + a_n(s)X_n(s) = b(s).$$

Here $X_i(s)$ are the transforms of the variables $x_i(t)$, $a_i(s)$ are polynomials of degree at most equal to the maximal degree of the relation $\mathcal{R}$ and $b(s)$ is a polynomial arising from the initial conditions.

If we are given, say, N linear relations $\mathcal{R}_1, \ldots, \mathcal{R}_N$, the equivalent description in the s-domain is therefore nothing but a **system of linear equations** in the variables $X_1, \ldots, X_n$, the difference being that the coefficients are **polynomials**, rather than *scalars*. We can use the notation

$$A(s)X = \mathbf{b}(s), \tag{2.26}$$

where $A(s)$ is a matrix of dimension $N \times n$ with polynomial elements and $\mathbf{b}$ is an n-vector with polynomial entries.

For a *square system* (one where $N = n$), it is reasonable to assume that $\det A(s)$ is not the zero polynomial and that the solution

$$X(s) = A^{-1}(s)\mathbf{b}(s)$$

gives a rational function for each X_i that is **strictly proper**.
* Note that Property 3 shows that the initial condition terms in $\mathcal{L}(\frac{d^m x}{dt^m})$ are of degree $m-1$ at most. However, the strict properness we assumed above does not follow directly from this. The reader should try to find a counter-example.

Definition 2.4. A **linear control system** is a system of linear differential relations between a number m of input variables $u_1, \ldots, u_m$ and a number p of output variables $y_1, \ldots, y_p$.

A **scalar linear system**, or **single-input, single-output** (SISO) system, is a linear system involving a single input u and a single output y.

In terms of Laplace transforms, a linear control system is thus, for the moment, given by the following matrix relations:

$$D(s)Y(s) + N(s)U(s) = \mathbf{b}(s), \tag{2.27}$$

where we simply separated the terms involving the outputs from those involving the inputs in equation 2.26. It is commonly assumed that the matrix $D(s)$ is invertible.

If the initial conditions are zero, we get the so-called **transfer function matrix** description of the linear control system. This is

$$Y(s) = -D^{-1}(s)N(s)U(s),$$

which is a generalization to a multiple input and output system of the *transfer function* of a SISO system that is defined in the following Chapter. Staying with this level of generality clearly raises many interesting questions and leads to a rich theory; for the purposes of this introductory course, however, we mostly stick to SISO systems (see the Exercise section for some multi-variable problems.)

A SISO control system is described by the relation

$$Y(s) = \frac{n(s)U(s) + b(s)}{d(s)}. \tag{2.28}$$

We shall always assume that the linear system we consider is strictly proper and take n to be its degree (the degree of the output y.)

Let $\mathcal{R}(u, y)$ be a strictly proper scalar linear system. Suppose we want to find the output $y(t)$, $t \geq 0$ to the input[3] $u(t)$, $t \geq 0$, satisfying a set of initial conditions $y_0, \dot{y}_0, \ldots, y_0^{(n-1)}$.

1. Take the Laplace transform of the linear differential relation defining the linear system. Recall that the relation is a constant-coefficient ode. Also note that the initial conditions are taken into account because of Property 3.

2. Solve the algebraic equation resulting from the first step for the transform $Y(s)$ of $y(t)$.

3. Apply the partial fractions method to write the expression for $Y(s)$ as a sum of terms whose inverse Laplace transform is known.

4. Finally, invert these simpler Laplace transforms to get the output $y(t)$ for $t \geq 0$.

Suitably generalized, the above method can also be applied to arbitrary linear control systems (involving more than one input and output variable.)

* **A note on (strict) properness** We have been rather remiss, so far, in not clearly justifying the assumption of properness made at various stages in the above exposition. The essential reason for these assumptions is that **real systems have dynamics**, in the sense that their state cannot be changed instantaneously. Thus, if I push a cart on a level floor, it will take time to change from the rest condition to a moving one. Similarly for other systems of physical origin. In addition, we have Property 3 that says that the initial condition terms in a linear equation in **one** variable will lead to a strictly proper expression for the Laplace transform of the variable, $Y(s) = \frac{b(s)}{d(s)}$.

We similarly expect that the effect of control cannot be instantaneous on the system it is controlling: a pilot cannot throttle up the engines with zero delay. At a deeper level, properness has to do with **causality**. Roughly, causality means that a system cannot at time t respond to the input values prior to t (see the advanced textbook [6].)

[3] We assume implicitly that this input has a known Laplace transform; often, we even assume this transform is a rational function.

2.3 Partial Fractions Expansions

2.3.1 Polynomials and Factorizations

The method we have just given for solving differential equations produces an algebraic expression for the output $Y(s)$ which is, in fact, a **rational function**[4] in s (provided the input transform $U(s)$ is a rational function.) As before, we shall make the assumption that this rational function is **proper**.

Thus, we have that

$$Y(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = \frac{n(s)}{d(s)}, \tag{2.29}$$

with $m \leq n$ and we write n and d for the numerator and denominator polynomials.[5] In the case when $m = n$, polynomial division permits us to write $Y(s)$ as *a constant plus a strictly proper rational function*. For simplicity, we assume from now on that the ratio is actually *strictly proper*, $n > m$.

Let us look at the denominator polynomial

$$d(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n.$$

By assumption, it is a real polynomial of degree n and therefore has n roots, which we shall call **poles** in this context; they are either *real* or appear as *complex conjugate pairs*. Hence the factorized form of $p(s)$ is

$$d(s) = \prod_{\Im p_i = 0} (s - p_i) \prod_{\Im p_j \neq 0} (s - p_j)(s - \bar{p_j})$$

where we have grouped together the real roots and complex-pair roots. If we write

$$p_j = \alpha_j + i\beta_j$$

the factorization takes the form

$$d(s) = \prod_{\Im p_i = 0} (s - p_i) \prod_{\Im p_j \neq 0} ((s - \alpha_j)^2 + \beta_j^2).$$

[4] A *rational function* is a ratio of polynomials.

[5] Question: Is there a loss in generality in taking the leading coefficient of p to be equal to one?

We conclude that the real polynomial has two type of factors: **linear factors** corresponding to **real roots** and **quadratic factors** corresponding to a **complex conjugate pair of roots**.

Let us also note that each root p_i may appear more than once in the list of roots of $p(s)$; the **multiplicity** of p_i, k_i, counts how many times it appears as a root. Naturally,

$$\sum k_i = n.$$

2.3.2 The Partial Fractions Expansion Theorem

Proposition 2.1. *If the strictly proper rational function $n(s)/d(s)$ of equation 2.29 has more than one distinct roots, then it can be written as a sum of rational functions of denominator degree less than* $\deg d(s)$*, as follows:*

1. *To every* simple real root p *of the denominator* $d(s)$*, so that* $d(s) = (s-p)q(s)$*, with* $q(p) \neq 0$*, corresponds a single term of the form*

$$\frac{a}{s-p},$$

where

$$\boxed{a = \frac{n(p)}{q(p)} = \lim_{s \to p}(s-p)G(p)} \tag{2.30}$$

The coefficient a is known as the **residue** *of the simple pole at p.*

2. *To every* multiple real root p *of multiplicity k, ($k \geq 2$) of the denominator $d(s)$, so that $d(s) = (s-p)^k q(s)$, with $q(p) \neq 0$, correspond exactly k terms in the partial fractions expansion,*

$$\frac{c_1}{(s-p)} + \frac{c_2}{(s-p)^2} + \ldots + \frac{c_k}{(s-p)^k} \tag{2.31}$$

where, letting

$$h(s) = \frac{n(s)}{q(s)},$$

$$\boxed{\begin{array}{rcl} c_k & = & h(p) = \frac{n(p)}{q(p)} = \lim_{s \to p}(s-p)G(p) \\ c_{k-1} & = & h'(p) \\ & \cdots & \\ c_1 & = & \frac{1}{(k-1)!}\frac{d^{k-1}h}{ds^{k-1}}(p) \end{array}} \tag{2.32}$$

The coefficient c_k is known as the **residue** *of the pole at p.*

3. *To every* simple pair of complex conjugate roots $\alpha \pm i\beta$ *of the denominator* $d(s)$*, so that*

$$d(s) = ((s-\alpha)^2 + \beta^2)q(s),$$

with $q(\alpha \pm i\beta) \neq 0$*, correspond exactly two terms of the form*

$$\frac{A(s-\alpha)}{(s-\alpha)^2 + \beta^2} + \frac{B\beta}{(s-\alpha)^2 + \beta^2}, \tag{2.33}$$

where the coefficients A *and* B *are real and are found from the* **complex residue**

$$R = Res_{\ \alpha+i\beta} Y(s) = \frac{n(s)}{(s-\alpha+i\beta)q(s)}|_{s=\alpha+i\beta} \tag{2.34}$$

and are given by

$$\boxed{\begin{array}{rcl} A & = & 2\Re(R) \\ B & = & -2\Im(R) \end{array}} \tag{2.35}$$

4. *To every* multiple pair of complex poles $\alpha \pm i\beta$ *of multiplicity* k*,* $k \geq 2$*, correspond exactly* k *terms of the form*

$$\frac{A_1(s-\alpha) + B_1\beta}{(s-\alpha)^2 + \beta^2} + \frac{A_2(s-\alpha) + B_2\beta}{((s-\alpha)^2 + \beta^2)^2} + \ldots + \frac{A_k(s-\alpha) + B_k\beta}{((s-\alpha)^2 + \beta^2)^k} \tag{2.36}$$

(We omit the formulæ for the coefficients (A_i, B_i) *as this case is rarely met in practice; they are derived by combining the above two cases.)*

The partial fractions expansion of $Y(s)$ *is exactly equal to the sum of the above terms.*

Remark 2.2. The above statement makes possible the *'inversion'* of the Laplace transform $Y(s)$ and casts it in a very convenient form, since the answers can be read directly off the Laplace transform table (see the final section.)

In particular, note that the complex pair yields PFE terms that have the obvious Laplace transform inverses

$$e^{-\alpha t}\cos\beta t = \mathcal{L}^{-1}\left(\frac{(s-\alpha)}{(s-\alpha)^2 + \beta^2}\right)$$

and

$$e^{-\alpha t}\sin\beta t = \mathcal{L}^{-1}\left(\frac{\beta}{(s-\alpha)^2 + \beta^2}\right)$$

Proof. The formula for the real residue a is familiar.

For a multiple real root p, break up the PFE into terms involving p and terms not involving it, so that we get

$$Y(s) = \frac{c_1(s-p)^{k-1} + c_2(s-p)^{k-2} + \cdots + c_{k-1}(s-p) + c_k}{(s-p)^k} + \frac{n'(s)}{q(s)}$$

for some polynomial $n'(s)$. Multiplying both sides by $(s-p)^k$, we see that, clearly, $c_k = (s-p)^k Y(s)|_{s=p} = h(p)$ and the other coefficients are found by taking successive derivatives.

The case of a simple complex pair is derived by writing the PFE terms for the two simple (but *complex*) poles $\alpha \pm i\beta$ and making use of the property of residues that states that the two residues corresponding to conjugate roots are themselves complex conjugates:[6]

$$\frac{R}{s-\alpha-i\beta} + \frac{\bar{R}}{s-\alpha+i\beta} = \frac{(R+\bar{R})(s-\alpha) + (R-\bar{R})i\beta}{(s-\alpha)^2 + \beta^2}$$

We now note that

$$R + \bar{R} = 2\Re(R), \quad \text{and} \quad R - \bar{R} = 2i\Im(R).$$

□

It is time to give examples (they are taken from Kreyszig [12].)

Example 2.3. Find the output $y(t)$, $t \geq 0$, of the linear system described by the differential equation

$$\ddot{y} - 3\dot{y} + 2y = u(t)$$

when $u(t) = (4t)h(t)$ and the initial conditions are: $y_0 = 1, \dot{y}_0 = -1$.

Taking the Laplace transform of both sides and since

$$\mathcal{L}(4t) = \frac{4}{s^2},$$

we obtain

$$(s^2 - 3s + 2)Y(s) = \frac{4}{s^2} + s - 4 = \frac{4 + s^3 - 4s^2}{s^2}$$

[6] This relies on the original polynomials having **real** coefficients.

and finally

$$Y(s) = \frac{s^3 - 4s^2 + 4}{s^2(s-2)(s-1)}.$$

Using the PFE result, we have that

$$Y(s) = \frac{a}{s-2} + \frac{b}{s-1} + \frac{c_1}{s} + \frac{c_2}{s^2},$$

where

$$a = \frac{s^3 - 4s^2 + 4}{s^2(s-1)}|_{s=2} = -1$$

$$b = \frac{s^2 - 4s^2 + 4}{s^2(s-2)}|_{s=1} = -1$$

$$c_2 = \frac{s^3 - 4s^2 + 4}{s^2 - 3s + 2}|_{s=0} = 2$$

and

$$c_1 = \frac{d}{ds}\frac{s^3 - 4s^2 + 4}{s^2 - 3s + 2}|_{s=0} = 3.$$

The output is then

$$\boxed{y(t) = (3 + 2t) - e^{2t} - e^t} \tag{2.37}$$

(As a check, verify that $y_0 = 1$ and $\dot{y}_0 = -1$.)

Example 2.4 (Periodically Forced Harmonic Oscillator). Consider the linear system representing a harmonic oscillator with input that is sinusoidal, but of a frequency different from the natural frequency of the harmonic oscillator.

$$m\ddot{y} + ky = K_0 \sin \omega t,$$

where $\omega_0 = \sqrt{k/m} \neq \omega$. Divide by m and take the Laplace transform to get

$$(s^2 + \omega_0^2)Y(s) = \frac{(K_0/m)\omega}{s^2 + \omega^2}.$$

Letting $K = K_0/m$, we write

$$Y(s) = \frac{K\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)} = \frac{A_1 s + B_1 \omega_0}{s^2 + \omega_0^2} + \frac{A_2 s + B_2 \omega}{s^2 + \omega^2},$$

the PFE of Y. In order to find the coefficients A_i, B_i, we first compute the complex residues

$$R_0 = \frac{K\omega}{(s+i\omega_0)(s^2+\omega^2)}\Big|_{s=i\omega_0} = \frac{K\omega}{(2i\omega_0)(-\omega_0^2+\omega^2)}$$

and

$$R = \frac{K\omega}{(s^2+\omega_0^2)(s+i\omega)}\Big|_{s=i\omega} = \frac{K}{(\omega_0^2-\omega^2)(2i)}.$$

These are both *pure imaginary* and, since $1/i = -i$, we find that

$$A_1 = 0, \quad B_1 = \frac{K\omega/\omega_0}{\omega^2-\omega_0^2}$$

and

$$A_2 = 0, \quad B_2 = \frac{K}{\omega_0^2-\omega^2}$$

and finally

$$\boxed{y(t) = \frac{K}{\omega^2-\omega_0^2}\left(\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t\right).} \tag{2.38}$$

(Valid only for $\omega \neq \omega_0$.)

2.4 Summary

- The **Laplace transform** converts *linear differential equations* into *algebraic equations*. These are **linear** equations with polynomial coefficients. The solution of these linear equations therefore leads to **rational function** expressions for the variables involved.

- Under the assumption of properness, the resulting rational functions are written as a sum of simpler terms, using the method of Partial Fractions Expansion. This is a far-reaching formalization of the concept of *system decomposition into* **modes**.

- *Simple real roots* of the denominator correspond to exponential time functions. *A simple complex pair* corresponds to a modulated trigonometric function, $e^{at}\cos(\omega t + \phi)$.

- *Multiple roots* imply the multiplication of the time function corresponding to a simple root by a polynomial in t.

- A **linear control system** in the s-domain is simply a set of linear equations involving the sets of inputs and outputs.

- This course focuses mainly on *single-input, single-output* systems, described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{n(s)}{d(s)}.$$

2.5 Appendix: Table of Laplace Transform Pairs

Time Function	Laplace Transform
$h(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\frac{t^n}{n!}e^{-at}$	$\frac{1}{(s+a)^{n+1}}$
$\cos \omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
$\sin \omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

2.6 Exercises

1. In this exercise, and in reference to the discussion in Section 2.1.2, we avoid using the complex exponential e^{st} and insist that the input be real.

   For the second-order linear system of Exercise 2.2, with $a \neq 0$, use the form

   $$\cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-\omega t})$$

   to show that the output $y(t)$ to the *real* input

   $$u(t) = \cos\omega t$$

   is

   $$y(t) = |g(i\omega)|\cos(\omega t + \angle g(i\omega)),$$

   where $|g(i\omega)|$ and $\angle g(i\omega)$ are the *amplitude* and *phase* of the transfer function $g(s)$ evaluated at $s = i\omega$. Thus, the output waveform is also sinusoidal of the same frequency; its amplitude is scaled by $|g(i\omega)|$ and there is a phase shift by $\angle g(i\omega)$.

2. Find the Laplace transform $X(s)$ of the following functions $x(t)$

   (a) The *square pulse* $x(t) = h(t) - h(t-1)$, where

   $$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

   is the unit step.

   (b) The *ramp function* $x(t) = t \cdot h(t)$

   (c) $x(t) = (1 - e^{-3t})h(t)$

   (d) $x(t) = t^2 e^{-4t} h(t)$

3. Assuming zero initial conditions, solve the following differential equations

   (a) $\frac{dx}{dt} + x = 2$

   (b) $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 1$

   (c) $\frac{dx}{dt} - 15x = \sin 4t$

(d) $\frac{dx}{dt} + 2y = 1, \quad \frac{dy}{dt} + 2x = 0$

4. A thermometer is thrust into a hot liquid of constant temperature ϕ^oC. At time $t = 0$, the temperature of the thermometer $\theta = 0^o$C. Newton's law of heating gives the temperature θ satisfying

$$\frac{d\theta}{dt} = \frac{1}{T}(\phi - \theta),$$

where T is the time constant. The thermometer takes one minute to reach 98% of the value ϕ.

(a) Find the time constant using the Laplace transform approach.

(b) If ϕ is time-varying and satisfies the equation

$$\frac{d\phi}{dt} = 10^o\text{C per minute},$$

how much error does the reading θ indicate as $t \to \infty$?

5. A model of water pollution in a river is given by the differential equation

$$\begin{aligned} \dot{x}_1 &= -1.7x_1 + 0.3x_2 \\ \dot{x}_2 &= -1.8x_2 + 1.5u, \end{aligned}$$

where x_1 =dissolved oxygen deficit (DOD) and x_2 =biochemical oxygen deficit (BOD) and the control u is the BOD content of effluent discharged into the river from the effluent treatment plant. The time scale is in days.

Calculate the response of DOD and BOD to a unit step input of effluent, $u(t) = h(t)$, assuming initial conditions

$$x_1(0) = A, \quad x_2(0) = B.$$

6. A Galitzin seismograph has dynamic behaviour modelled by

$$\begin{aligned} \ddot{x} + 2K_1\dot{x} + n_1^2 x &= \lambda\dddot{\xi} \\ \ddot{y} + 2K_2\dot{y} + n_2^2 y &= \mu\dot{x}, \end{aligned}$$

where y is the displacement of the mirror, x is the displacement of the pendulum and ξ is the ground displacement.

When $K_1 = K_2 = n_1 = n_2 = n > 0$,

(a) Determine the response $y(t)$ to a unit ground velocity shift, $\dot{\xi}(t) = h(t)$, assuming

$$\xi(0) = \dot{\xi}(0) = x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$$

(b) Show that the maximum value of $y(t)$ occurs at time $t = (3 - \sqrt{3})/n$ and the minimum value of $y(t)$ at time $t = (3 + \sqrt{3})/n$.

Chapter 3

Transfer Functions and Feedback Systems in the Complex s-Domain

3.1 Linear Systems as Transfer Functions

So far in this course we have used the notion of *linear system* as synonymous with the notion of a *system of linear differential relations* between a set of inputs and a set of outputs; in turn, this is equivalent to a system of *linear, constant-coefficient ordinary differential equations* relating the two sets of variables. Now, with the Laplace transform in hand, we can give a more algebraic definition of linear systems. Since we only deal with *single-input, single-output (SISO) control systems*, we shall only give the definiton for this simple case.

Definition 3.1. A **linear control system** with input u and output y is given by specifying that the ratio of the output to the input in the s-domain is a proper rational function $G(s)$, called the **transfer function** of the linear system. This means

$$\frac{Y(s)}{U(s)} = G(s)$$

If we write

$$G(s) = \frac{n(s)}{d(s)}$$

where $n(s)$ and $d(s)$ are the numerator and denominator polynomials which we assume have **no common factors**, we require

$$\deg d(s) \geq \deg n(s).$$

Suppose concretely that

$$d(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

and

$$n(s) = b_0 s^m + \ldots + b_{m-1} s + b_m.$$

Please do not confuse the use of n as the degree of $d(s)$ and $n(s)$ as the numerator polynomial of $G(s)$.

The equivalence of this definition with the one using differential equations is established by writing the above in the form

$$d(s)Y(s) = n(s)U(s)$$

and translating it into the differential equation in the by now familiar way

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_{n-1} \frac{dy}{dt} + a_n y = b_0 \frac{d^m u}{dt^m} + \ldots + b_{m-1} \frac{du}{dt} + b_m u. \quad (3.1)$$

Note that to get from one object to the other, we have assumed **zero initial conditions**.

For simplicity, it will be assumed from now on that the transfer function is **strictly proper**. (If it is not, use polynomial division to take out a *constant term* relating input and output.)

Definition 3.2. The roots (or zeros) of the numerator polynomial $n(s)$ of $G(s)$ are called the (open-loop) **zeros** of the linear system.

The roots of the denominator polynomial $d(s)$ of $G(s)$ are called the (open-loop) **poles** of the system.

(The qualifier *'open-loop'* will be useful later when we wish to distinguish the open-loop poles and zeros from the *closed-loop* poles and zeros.) We shall refer to simple and multiple roots of $d(s)$ as *'simple' and 'multiple' poles.*

Definition 3.3. The linear system is **stable** if all its poles are in the open left-half plane

$$\mathbb{C}_- = \{s \in \mathbb{C}\,;\ \Re(s) < 0\}.$$

The box diagram representing a linear system in the Laplace transform domain is shown in Figure 3.1.

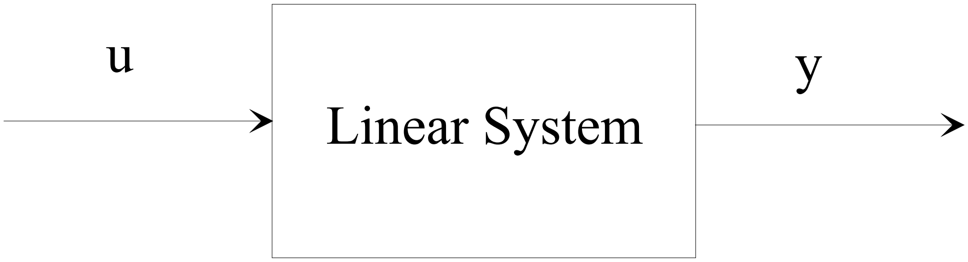


Figure 3.1: A Single-Input Single-Output Linear System

3.2 Impulse and Step Responses

We propose to study linear systems given as strictly proper transfer functions $G(s)$. The main aspect of a linear system that interests us is its *input-output behaviour*: what is the *output* $y(t)$ of the linear system to the various possible *inputs* $u(t)$ of interest (considered mainly in the interval $t \geq 0$.)

Definition 3.4. The **step response** $y_s(t)$ of the linear system $G(s)$ is the output to the input $u(t) = h(t)$ (a unit step.)

Note that the step response is obtained by inverting (using PFE) the rational function

$$\frac{G(s)}{s}.$$

The *impulse or delta function* $\delta(t)$ is obtained informally by a limiting process from the unit step:

$$\delta(t) = \lim_{c \to \infty} c(h(t) - h(t - 1/c)).$$

Definition 3.5. The **impulse response** $g(t)$ of the linear system $G(s)$ is the output to the unit-impulse input $u(t) = \delta(t)$.

Remark 3.1. We can avoid the use of the delta function by defining $g(t)$ (as the notation suggests) as the inverse Laplace transform of the system transfer function $G(s)$. This is because one can show that the Laplace transform of the delta function is the function equal to one identically, so that

$$g(t) = \mathcal{L}^{-1}(1 \cdot G(s)).$$

Exercise 3.1. Prove this statement.

Proposition 3.1. *For a* **stable** *linear system, the impulse response decays to zero as $t \to \infty$ and the step response asymptotically approaches a constant value.*

You should be able to give a quick proof of this by considering the partial fractions expansion of $Y_s(s) = G(s)/s$ and $G(s)$, given the stability assumption. In particular, since $G(s)$ is stable, $s = 0$ is a simple pole of $Y_s(s)$ and hence its PFE has a unique term

$$Y_s(s) = \frac{A}{s} + \text{other terms corresponding to the stable poles,}$$

where, by the formula given in the previous chapter,

$$A = \lim_{s \to 0} s(G(s)/s) = G(0).$$

Inverting term-by-term, we see that all the stable terms decay to zero and what is left is the term $Ah(t)$. Hence the asymptotic value of the step response is exactly A.

For example, the asymptotic value $\lim_{t\to\infty} y_s(t)$ of the step response of the linear system

$$G(s) = \frac{10(s+2)}{s^2 + 2s + 6}$$

is 10/3 (Note that we did not have to compute the PFE for this.)

Before leaving this subject, we mention that there is in fact a general property of the Laplace transform (and thus applicable to transforms more general than rational functions) from which the final value of $y_s(t)$ can be computed.

Property 5 (Final-Value Theorem). *Let $X(s)$ be the Laplace transform of $x(t)$. Then*

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

(provided these exist.)

For the proof, recall that

$$\mathcal{L}(\frac{dx}{dt}) = sX(s) - x(0).$$

Hence

$$\lim_{s\to 0}(sX(s) - x_0) = \lim_{s \to 0} \int_0^\infty \dot{x}(t) e^{-st} dt =$$
$$= \int_0^\infty \dot{x}(t) (\lim_{s\to 0} e^{-st}) dt = \lim_{t\to\infty} x(t) - x(0).$$

3.3 Feedback Control (Closed-Loop) Systems

We saw in the Introduction the importance of *feedback* in compensating for deviations from the desired path of systems. Let us now use our *Laplace transform tool* to discuss feedback arrangements and the ways in which they improve the system characteristics.

Let $G(s) = \frac{n(s)}{d(s)}$ be a linear system representing the real-world system that we wish to control. A typical feedback arrangement is to pass the output y through a **controller** C, itself taken to be a linear system, and then subtract this from a *reference signal* r, forming an error signal that is fed back to the system (see Figure 3.2.) Since the controller is a linear system, its output is

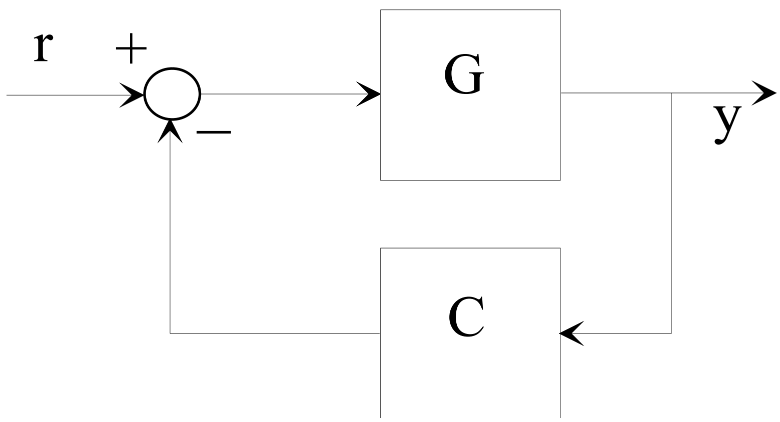


Figure 3.2: A SISO Linear Control System

related to its input through a transfer function, so that the error signal in the s-domain is, in fact

$$E(s) = R(s) - C(s)Y(s),$$

where $E(s), R(s)$ and $Y(s)$ are the Laplace transforms of $e(t), r(t)$ and $y(t)$, respectively, and $C(s)$ is the *controller transfer function*, a rational function that we usually take to be *proper*.

Definition 3.6. The *closed-loop transfer function* $H(s)$ is the ratio of the system output $Y(s)$ to the reference input $R(s)$ in the s-domain.

The closed-loop transfer function (CLTF) is the fundamental object of study in classical (i.e. frequency-domain) control theory. Since it depends on the controller C, we shall make explicit how control can be used to improve system characteristics.

Proposition 3.2. *The CLTF of this single-input/single-output feebdack system is*

$$\boxed{H(s) = \frac{G(s)}{1 + G(s)C(s)}}$$

Indeed,

$$Y(s) = G(s)E(s) = G(s)(R(s) - C(s)Y(s)),$$

so that

$$(1 + G(s)C(s))Y(s) = G(s)R(s).$$

It is useful to have an alternative expression for the CLTF H in terms of the polynomials appearing in the transfer functions G and C. Suppose $C(s) = \frac{n_c(s)}{d_c(s)}$, with $\deg d_c(s) \geq \deg n_c(s)$,[1] so that, since $G(s) = \frac{n(s)}{d(s)}$,

$$H(s) = \frac{n/d}{1 + (nn_c)/(dd_c)} = \frac{n(s)d_c(s)}{d(s)d_c(s) + n(s)n_c(s)}.$$

We can easily check that, under our assumptions, $H(s)$ is *strictly proper*. In general (though not always, think of an example), the rational function H will be *reduced* if G and H are reduced, in other words there will be no *'cancellable factors'*.

Definition 3.7. The roots of the numerator polynomial of $H(s)$, $n(s)d_c(s)$ are called the **closed-loop zeros** of the feedback system.

The roots of the denominator polynomial of $H(s)$,

$$\boxed{d(s)d_c(s) + n(s)n_c(s)}$$

are called the **closed-loop poles** of the system.

The above equation is also known as the **characteristic equation** of the closed-loop system.

[1] As for $G(s)$, we again assume that $n_c(s)$ and $d_c(s)$ are polynomials with **no common factors**.

3.3.1 Constant Feedback

Arguably the most important (certainly the most studied) case of feedback controller is the case of **constant-gain feedback**

$$C(s) = k, \quad k \geq 0.$$

See Figure 3.3. (Only non-negative k is considered, since it is the *difference*, or error, between (scaled) output and (reference) input that needs to be formed, as we saw in the introductory discussion on feedback. This is not to say that negative k is never used–this is known as **positive feedback**.) The

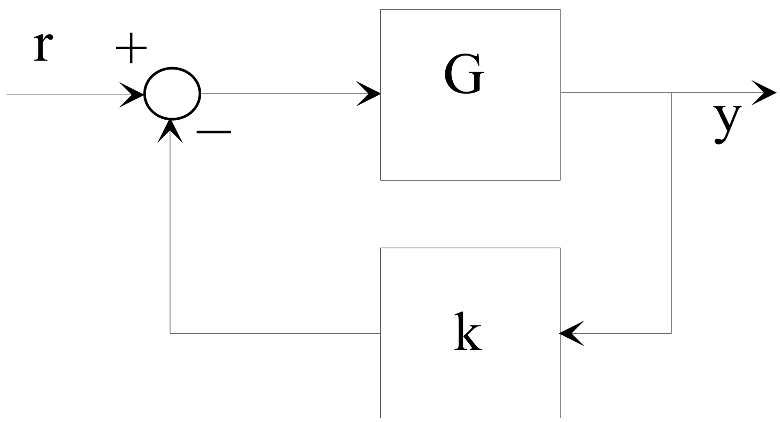


Figure 3.3: A Constant-Gain Control System

CLTF now simplifies to

$$\boxed{H(s) = \frac{G(s)}{1 + kG(s)} = \frac{n(s)}{d(s) + kn(s)}}$$

Remark 3.2. We conclude from the above that

- The **closed-loop zeros** coincide with the **open-loop zeros**.
- The **closed-loop poles** are a function of k, in general, and are the solutions of the k-dependent nth order characteristic equation

$$\boxed{d(s) + kn(s) = 0}$$

(where $n = \deg d(s)$.) This equation is at the core of the present course: we shall study criteria for stability (the Routh-Hurwitz Stability Criterion of the next Chapter) and the way the roots move as k varies (the *Root-Locus method* that follows it.)

Let us look at the simplest possible example to see how the closed-loop poles can vary with k.

Example 3.1. Let

$$G(s) = \frac{1}{s+a}$$

be a first-order system that is (open-loop) **stable** for $a > 0$ and **unstable** for $a < 0$.

The closed-loop transfer function is

$$H(s) = \frac{1}{s + (a+k)}$$

and we can see that if the OL system is unstable, the single CL pole at $s = -(a+k)$ eventually becomes stable for large enough feedback gain k (for $k > -a$.)

This is the first instance we meet where control is used to stabilize an unstable system.

Example 3.2. Now consider the stable second-order OL system

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}.$$

How do the CL poles depend on k?

The CLTF is

$$H(s) = \frac{(s+1)}{(s^2+5s+6)+k(s+1)} = \frac{(s+1)}{(s^2+(5+k)s+(6+k)}$$

and it is less immediate, but true, that the closed-loop system is stable for all $k \geq 0$.

In Chapter 5, we shall give a powerful graphical method for the analysis of such problems. For now, try this as an exercise.

Exercise 3.2. Show that the CLTF is stable for all non-negative gains k.

Hint: Use the formula for the roots of a quadratic polynomial and look at the discriminant $\Delta(k)$, itself a quadratic polynomial in k.

3.3.2 Other Control Configurations

The reader may have questioned the use of the *'error signal'* $e(t) = \mathcal{L}^{-1}(E(s)$, where

$$E(s) = R(s) - C(s)Y(s),$$

rather than the more obvious *'output-minus-input'*

$$e(t) = r(t) - y(t).$$

One can, in fact, consider the alternative feedback configuration where the error signal $y - r$ is fed back to a controller C, whose output goes into the system G (see Figure 3.4.)

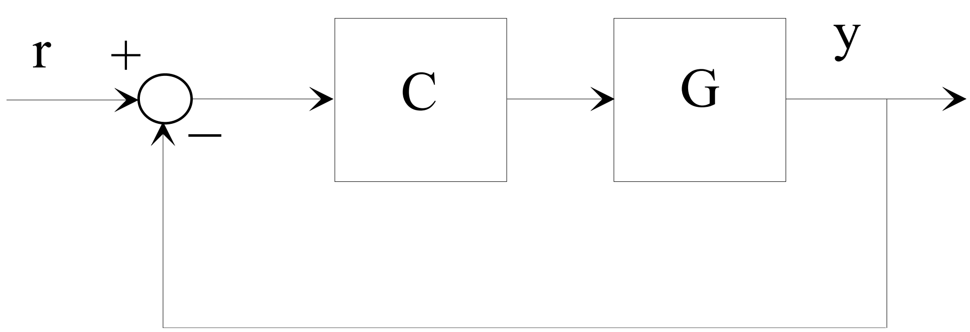


Figure 3.4: An Alternative Control Configuration

Exercise 3.3. Show that the CLTF is now

$$H(s) = \frac{G(s)C(s)}{1 + G(s)C(s)},$$

or, in the case of simple gain feedback,

$$H(s) = \frac{kn(s)}{d(s) + kn(s)},$$

so that **the equation for the closed-loop poles is identical to the previous case**.

The *block diagram technique* that we have been using so far, in the sense that we relate the input and output of linear systems via the linear system transfer function of each system component, can be generalized to the more complex situations that one often meets in practise. For an example, see Figure 1.2 of Chapter 1. The basic building block is then a single 'black box' representing a linear system, with a single arrow entering it, the *input*, and a single arrow leaving it, the *output*. We shall also allow summation and difference nodes, again with directed arrows distinguishing between inputs and outputs.

As in the simple feedback configurations that we have just examined, one is interested in the collection of **transfer functions**, each defined as the ratio of a selected output arrow to a selected input arrow.

Disturbances: In many cases, the input to the system or plant G is not the output of our controller but a corrupted signal that has a disturbance $d(t)$ added to it. This leads to the revised block diagram shown in Figure 3.5. By way of practising your skills of manipulating block diagrams, show that

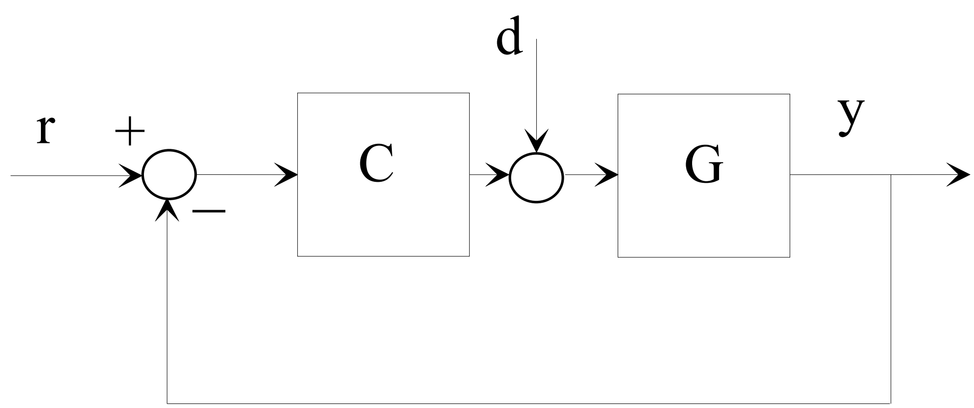


Figure 3.5: Control System with Additive Disturbance

the output Y is related to the reference input R and the disturbance D by

$$Y = \frac{G}{1+GC}D + \frac{GC}{1+GC}R.$$

3.4 The Sensitivity Function

The linear system model that we are using to represent our real-world system or plant is always an approximation: we may have had to ignore certain aspects of its dynamical behaviour so as to keep the system order low. Of course, assuming **linearity** is also itself an approximation, as in practice all system are **nonlinear**. In this section we address the issue of **modelling uncertainties** *in the context of linear system models.*

We thus assume that the real system model $G(s)$ differs from the *nominal* system model, say $G_0(s)$, by an additive factor $\Delta G(s)$, so that

$$G(s) = G_0(s) + \Delta G(s),$$

where $\Delta G(s)$ is itself a rational transfer function and is in some sense **small**; we denote this, rather informally, by

$$|\Delta G(s)| < \epsilon, \quad \epsilon \text{ small.}$$

A natural question to ask is then: *what is the effect of the modelling uncertainty $\Delta G(s)$ on the output y?* Being careful to distinguish relative from absolute errors and assuming that the real output (always in the s-domain) differs from the nominal by a factor as follows

$$Y(s) = Y_0(s) + \Delta Y(s),$$

we make the following definition:

Definition 3.8. The **sensitivity function** of the closed-loop system with nominal plant $G_0(s)$ and controller $C(s)$ is the function

$$S(s) = \frac{\Delta Y(s)/Y(s)}{\Delta G(s)/G_0(s)} = \frac{\Delta Y}{\Delta G} \cdot \frac{G_0}{Y}$$

Roughly, the percentage error in the output Y is the product of the percentage error in the system model G scaled by the sensitivity function. It is then desirable to have a *'small'* sensitivity function (since this is a function of the *'complex frequency* s, this may only hold for a range of s of most interest—compare the frequency response of your HiFi; more on this frequency response aspect of linear systems later.)

Proposition 3.3. *The sensitivity function is given by*

$$\boxed{S(s) = \frac{1}{1 + G_0(s)C(s)}}$$

The conclusion from the above is that to make the sensitivity **small**, we need to make the product $G_0 C$ **large**, in other words

$$|G_0(s)C(s)| \gg 1,$$

for all s where the uncertainty is concentrated (often at *'high frequencies'*.)

We shall give *two* informal derivations of this result.

Derivation 1. The first way uses the very informal Δ-calculus: write all quantities of interest as sums of nominal+error and ignore terms of order Δ^2 or higher.

Now we assumed

$$Y = Y_0 + \Delta Y$$

and we know that the nominal output is

$$Y_0 = H_0 R = \frac{G_0}{1 + G_0 C} R,$$

while the true output is

$$\begin{aligned} Y &= \frac{G}{1+GC} R = \frac{G_0 + \Delta G}{1 + (G_0 + \Delta G)C} R = \\ &= \frac{G_0 + \Delta G}{(1 + G_0 C) + \Delta G C} R = \frac{G_0 + \Delta G}{(1 + G_0 C)(1 + \frac{\Delta G C}{1 + G_0 C})} R \end{aligned} \tag{3.2}$$

Using the formula

$$\frac{1}{1+\epsilon} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \cdots \simeq 1 - \epsilon,$$

for small ϵ, it follows that

$$\begin{aligned} Y &\simeq \frac{G_0 R + \Delta G R}{1 + G_0 C}\left(1 - \frac{\Delta G C}{1 + G_0 C}\right) = \\ &= \frac{G_0 R}{1 + G_0 C} + \frac{\Delta G R}{1 + G_0 C} - \frac{\Delta G G_0 R C}{(1 + G_0 C)^2} + \mathcal{O}(\Delta G^2) \simeq \\ &\simeq Y_0 + \frac{\Delta G R}{1 + G_0 C}\left(1 - \frac{G_0 C}{1 + G_0 C}\right) = Y_0 + \Delta G \frac{R}{(1 + G_0 C)^2} \end{aligned} \tag{3.3}$$

Comparing with the expression $Y = Y_0 + \Delta Y$, we find that

$$\Delta Y = \Delta G \frac{R}{(1 + G_0 C)^2}$$

and so, finally,

$$\begin{aligned} S = \frac{\Delta Y}{\Delta G} \cdot \frac{G}{Y} = \frac{R}{(1+G_0C)^2} \cdot \frac{G}{GR}(1+GC) = \\ = \frac{1+GC}{(1+G_0C)^2} = \frac{1}{1+G_0C} + \mathcal{O}(\Delta G), \quad (3.4) \end{aligned}$$

and we are done. □

Derivation 2. The second derivation is just as informal as the first, but is quicker, in that it treats all rational functions as *'variables'* and computes a 'derivative.'

The expression

$$Y = \frac{G}{1+GC} R$$

is then taken to represent the variable Y as a 'function' of the variable G. The sensitivity function redefined to be

$$S = \frac{dY}{dG} \frac{G}{Y}.$$

Differentiating the above expression for Y, we obtain

$$\frac{dY}{dG} = \left(\frac{(1+GC) - GC}{(1+GC)^2} \right) R = \frac{R}{(1+GC)^2}$$

and so

$$S = \frac{R}{(1+GC)^2} \frac{G}{GR}(1+GC) = \frac{1}{1+GC},$$

as before. □

Note that, informal though the derivations were, the resulting sensitivity function is a well-defined rational function (since G and C are rational functions.)

Example 3.3. The sensitivity function for the first-order linear system

$$G(s) = \frac{1}{s+a}$$

with control $u = -ky$ is

$$S(s) = \frac{1}{1 + k/(s+a)} = \frac{s+a}{s+(a+k)}.$$

(We wrote G rather than G_0, taking the second derivation's viewpoint that G is the plant that is pertubed by an 'infinitesimal' dG.)

For the second-order system

$$G(s) = \frac{s+1}{(s+2)(s+3)},$$

with the same constant-gain control, we get

$$S(s) = \frac{s^2+5s+6}{s^2+(5+k)s+(6+k)}.$$

3.5 Summary

- In system theory, we use block diagrams to represent interconnections of linear systems. Each system is represented by its *transfer function*, a description that is equivalent to the linear differential equation description (but recall the zero initial condition assumption.)

- The *impulse response* of a linear system is the response to a unit impulse—equivalently, it is the inverse Laplace transform of its transfer function. It essentially gives the *'dynamics'* inherent in the system.

- The *step response* is the response of the output to a unit step (assuming zero initial conditions, again.) The steady-state value of the output is given by $G(0)$, if this is finite.

- **Feedback control** is achieved by *'closing the loop'* and using a controller, either in the feedback path or in the forward path.

- The **closed-loop transfer function** is the main object of study of linear, SISO control system theory. Control action is used to give the CL transfer function better properties than the open-loop transfer function. Its numerator zeros are the **closed-loop zeros** and its denominator zeros are the **closed-loop poles**.

- The closed-loop system is **stable** if all the CL poles are stable.

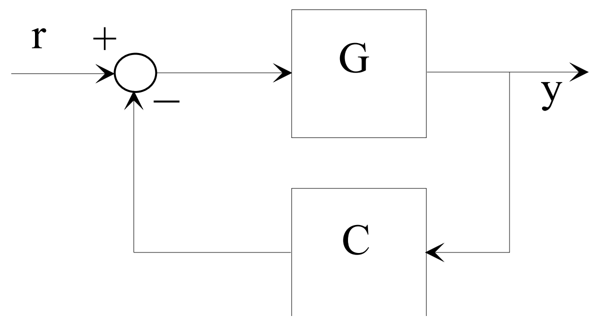
- The sensitivity function is a measure of the extent to which the system output can vary from its nominal value if the system model is inaccurate. It should be small for a well-designed system. In practice, it is enough that it be small only over the range of frequencies of unmodelled dynamics.

3.6 Exercises

1. For the standard feedback control configuration shown, find the transfer functions $Y(s)/R(s)$ and $U(s)/R(s)$, where $U(s) = R(s) - C(s)Y(s)$, for

   (a) $G(s) = \frac{1}{s^2+15s+6}$ and $C(s) = (s+1)$.

   (b) $G(s) = \frac{s+3}{s^3}$ and $C(s) = s^2 + s + 1$.



   Also find the impulse responses of the above systems $G(s)$.

2. The open-loop transfer function of a *unity feedback* control system (constant gain equal to one) is

$$G(s) = \frac{1}{s(As+B)},$$

   for A, B some positive constants. Find the steady-state closed-loop errors for unit step, ramp and *parabolic* ($r(t) = t^2$) inputs. (The error is defined to be $e(t) = y(t) - r(t)$.) What do your results say about the ability of the system to follow each type of input?

   Another OL pole is introduced at $s = 0$, so that the OL transfer function becomes

$$G(s) = \frac{(s+1)}{s^2(As+B)}.$$

   How does this change the errors due to ramp and parabolic inputs?

   You may assume that the quadratic and cubic (respectively) polynomials that are the denominators of the transfer function $E(s)/R(s)$ are stable in both cases.

Chapter 4

Stability and the Routh-Hurwitz Criterion

4.1 Introduction

We would like to be able to determine the stability of the closed-loop system

$$H(s) = \frac{G(s)}{1 + G(s)C(s)},$$

especially in the case of constant-gain feedback, $u = -ky$. Recall from the previous Chapter that the equation for the closed-loop poles is

$$d(s)d_c(s) + n(s)n_c(s) = 0,$$

or, for $C(s) = k$,

$$d(s) + kn(s) = 0.$$

Clearly, unless the order of the system G is no more than two, we cannot hope to find the roots of the nth-order polynomials above explicitly, unless we appeal to a computer package such as **Matlab**.

It turns out that there is an effective test for determining stability that **does not** require an explicit solution of the algebraic equation—rather, it only involves computing the values of some two-by-two determinants. This is the **Routh-Hurwitz Stability Criterion** that is the subject of this Chapter.

If all the roots of a polynomial $p(s)$ have negative real parts, we shall say that the polynomial is **stable** (sometimes the term *Hurwitz* is used.)

4.2 The Routh-Hurwitz Table and Stability Criterion

Let

$$p(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

be a polynomial with real coefficients, with $a_0 \neq 0$. Starting with the leading coefficient a_0, fill two rows of a table, as follows:

$$\begin{array}{r|llll} s^n & a_0 & a_2 & a_4 & \cdots \\ s^{n-1} & a_1 & a_3 & a_5 & \cdots \end{array} \tag{4.1}$$

Thus, the first row contains all coefficients with *even* subscript (but not necessarily even powers of s) and the second those with *odd* subscripts. The labels s^n and s^{n-1} are used to indicate the highest power of s, whose coefficient is in that row.

Now a **Routh-Hurwitz Table** (RHT) with $(n+1)$ rows is formed by an algorithmic process, described below:

$$\begin{array}{r|llll} s^n & a_0 & a_2 & a_4 & \cdots \\ s^{n-1} & a_1 & a_3 & a_5 & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s^1 & \ell_1 & 0 & 0 & \cdots \\ s^0 & a_n & 0 & 0 & \cdots \end{array} \tag{4.2}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1},\ b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \ldots,$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1},\ c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, \dots,$$

and so on. Zeros are placed whenever a row is too short. That the last two rows have the properties stated, in other words that in the s^1 row only the first element is (in general) non-zero and that in the s^0 row we have a_n and a row of zeros to its right, is easy to show, but we omit the proof.

The elements of the *first column* will be called **pivots**. It should be clear that the numerators in the expressions for the coefficients $\{b_i\}$, $\{c_i\}$ etc. are *the negatives of two-by-two determinants containing the elements from the two rows above and from two columns: the pivot column and the column to the right of the element to be computed.* For example, we highlight below the computation of the element b_2:

$$\begin{array}{r|cccc} s^n & \mathbf{a_0} & a_2 & \mathbf{a_4} & \cdots \\ s^{n-1} & \boxed{\mathbf{a_1}} & a_3 & \mathbf{a_5} & \cdots \\ s^{n-2} & b_1 & \mathbf{b_2} & b_3 & \cdots \end{array}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} \tag{4.3}$$

Proposition 4.1. *The polynomial $p(s)$ is (Hurwitz) stable if and only if all the pivot elements (the elements of the first column) of its Routh-Hurwitz Table are non-zero and of the same sign.*

The proof will not be given, as it would take us too far afield. It is nevertheless instructive for the reader to try and see whether the obvious approach using the familiar identities relating the coefficients $\{a_i\}$ to the elementary symmetric polynomials in the roots $\{\lambda_i\}$

$$\sum_{i=1}^{n} \lambda_i = -\frac{a_1}{a_0}, \quad \sum_{1 \le i < j \le n} a_i a_j = \frac{a_2}{a_0}, \quad \prod_{i=1}^{n} \lambda_i = (-1)^n \frac{a_n}{a_0}$$

has any chance of working (it does not, by itself!)

Exercise 4.1. Show that a **necessary** condition for stability of $p(s)$ is that all the coefficients are non-zero and of the same sign (all positive or all negative). (Only do the simpler case of real roots, if you want.)

This result, together with its refinement below, Proposition 4.2, is of tremendous value in understanding how the coefficients of a polynomial determine the stability of its roots. Let us see how it applies in the simplest cases:

Example 4.1 (Quadratic Polynomials). The RHT of the quadratic polynomial

$$p(s) = a_0 s^2 + a_1 s^1 + a_2$$

is

$$\begin{array}{c|cc} s^2 & a_0 & a_2 \\ s^1 & a_1 & 0 \\ s^0 & a_2 & 0 \end{array} \tag{4.4}$$

and we conclude that **a quadratic polynomial is stable if and only if all its coefficients are of the same sign** (and in particular are non-zero).

Example 4.2 (Cubic Polynomials). For the **cubic** polynomial

$$p(s) = a_0 s^3 + a_1 s^2 + a_2 s^1 + a_3$$

the RHT is

$$\begin{array}{c|cc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & b_1 & 0 \\ s^0 & a_3 & 0 \end{array} \tag{4.5}$$

with

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}.$$

We conclude that the cubic polynomial is stable provided either

$$a_0 > 0,\ a_1 > 0,\ a_3 > 0 \text{ and } a_1 a_2 > a_0 a_3,$$

or

$$a_0 < 0,\ a_1 < 0,\ a_3 < 0 \text{ and } a_1 a_2 > a_0 a_3.$$

Note that the last inequality is the same in both cases.

We claim that in either case, it is necessary for **all** coefficients to be of the same sign: if $a_0 > 0$, the nonlinear inequality says that

$$a_2 > \frac{a_0 a_3}{a_1} > 0$$

and, if $a_0 < 0$, that the product $a_1 a_2$ must be positive (sine it is $> a_0 a_3 > 0$), so, since $a_1 < 0$, a_2 must also be negative (or simply use the fact a polynomial and its negative have the same roots.)

4.3 The General Routh-Hurwitz Criterion

The information contained in the Routh-Hurwitz Table of the polynomial $p(s)$ is in fact greater than simply a test for stability: *it is possible to determine the exact number of stable and unstable roots of $p(s)$, in general.* (The qualifier *'in general'* is necessary here, as we shall see.)

For any non-zero number x, define

$$\operatorname{sign}(x) = \begin{cases} +1, & x > 0 \\ -1 & x < 0 \end{cases} \tag{4.6}$$

Proposition 4.2. *Suppose that all the elements of the pivot (first) column of the RHT are non-zero. Consider the sequence of signs, a sequence of length $(n+1)$:*

$$(\operatorname{sign}(a_0), \operatorname{sign}(a_1), \operatorname{sign}(b_1), \ldots, \operatorname{sign}(a_n)) \tag{4.7}$$

This sequence can have at most n changes of sign.

Then the number of unstable roots of $p(s)$ is equal to the number of changes of sign of the above sequence.

By *unstable* root, we mean a root with a *positive real part.*

Example 4.3. The coefficients of the polynomial

$$p(s) = s^5 + s^4 + 6s^3 + 5s^2 + 12s + 20$$

are all positive. However, this polynomial is **not Hurwitz**! The RHT gives

$$\begin{array}{c|cccc}
s^5 & 1 & 6 & 12 & 0 \\
s^4 & 1 & 5 & 20 & 0 \\
s^3 & 1 & -8 & 0 & 0 \\
s^2 & 13 & 20 & 0 & 0 \\
s^1 & -124/13 & 0 & 0 & 0 \\
s^0 & 20 & 0 & 0 & 0
\end{array} \tag{4.8}$$

Gives the sign sequence

$$(+1, +1, +1, +1, -1, +1)$$

which has **two sign changes**. The polynomial thus has **two unstable roots**.

This example demonstrates that the condition on the signs of the coefficients of a polynomial is **not sufficient** for stability. **Matlab** can be used to confirm this: indeed, the three stable roots of p are $-0.5986 \pm 2.1011i$ and -1.2649 and the two unstable roots are $0.7311 \pm 1.6667i$.

4.3.1 Degenerate or Singular Cases

Consider the polynomial

$$p(s) = s^5 + s^4 + 5s^3 + 5s^2 + 12s + 10.$$

It, too, satisfies the positivity condition on its coefficients. However, when we try to complete the entries of the RHT, we come up against a zero in the first column:

$$\begin{array}{c|cccc} s^5 & 1 & 5 & 12 & 0 \\ s^4 & 1 & 5 & 10 & 0 \\ s^3 & 0 & 2 & 0 & 0 \\ s^2 & ? & \cdots & & \end{array} \tag{4.9}$$

and it is not clear how we are to continue from this point onwards, as the pivot element is zero. The roots of this polynomial are well away from zero: **Matlab** gives them as $0.6519 \pm 1.7486i$, $-0.7126 \pm 1.6617i$ and -0.8784. What went wrong is that, by a pure coincidence, it works out that the symmetric functions of the roots that give the coefficients happened to be identical in pairs: $a_1 = a_0$ and $a_2 = a_3$, so that a 2×2 determinant vanishes.

The key is to consider an **arbitrary** perturbation of this configuration of the roots; then the coefficients will move by a small amount as well (by continuity of the symmetric functions), ridding us of the degeneracy, in general.

It turns out that this can be done without an explicit perturbation of the roots, as follows: replace the zero by a small ϵ (of arbitrary sign) and continue filling in the entries of the RHT. In our case, we get:

$$\begin{array}{c|cccc} s^5 & 1 & 5 & 12 & 0 \\ s^4 & 1 & 5 & 10 & 0 \\ s^3 & \epsilon & 2 & 0 & 0 \\ s^2 & c_1 = 5 - 2/\epsilon & 10 & 0 & 0 \\ s^1 & d_1 = 2 - (10\epsilon/c_1) & 0 & 0 & 0 \\ s^0 & 10 & 0 & 0 & 0 \end{array} \tag{4.10}$$

Sine it was already quite a rare occurence to have the single zero, it is very unlikely that we get stuck again (an informal statement that can, however, be rigorously justified.) In the RHT above, since ϵ is small, $c_1 \ll 0$, so that it is easy to determine the sequence of signs: it is

$$(+1, +1, +1, -1, +1, +1)$$

if $\epsilon > 0$ and

$$(+1, +1, -1, +1, +1, +1)$$

if $\epsilon < 0$. In either case, we have *two sign changes*, confirming that the polynomial has two unstable roots.

Pure imaginary pair of roots: So far, all our results excluded the possibility that the polynomial $p(s)$ has pure imaginary roots; we examined conditions for roots to lie in the **open** left-half plane (stable) and right-half plane (unstable) in $\mathbb{C}$.

It turns out that *if* $p(s)$ *has a simple pair of imaginary roots at* $\pm i\omega$ *and no other imaginary roots, then this can be detected in the RHT, in that it will have the whole of the* s^1 *row equal to zero and only two non-zero elements in the* s^2 *row, of the same sign. If* α *and* β *are these numbers, then*

$$\alpha(s^2 + \omega^2) = \alpha s^2 + \beta.$$

Example 4.4. Let

$$p(s) = s^4 + 5s^3 + 10s^2 + 20s + 24.$$

The Routh-Hurwitz Table is:

$$\begin{array}{c|ccc} s^4 & 1 & 10 & 24 \\ s^3 & 5 & 20 & 0 \\ s^2 & 6 & 24 & 0 \\ s^1 & 0 = \frac{120-120}{6} & 0 & 0 \\ s^0 & 24 & 0 & 0 \end{array} \tag{4.11}$$

where we used the ϵ method to allow us to bring the last coefficient, $a_4 = 24$, to its place at the bottom of the pivot column.

By the above result,

$$6s^2 + 24 = 6(s^2 + 4)$$

is a factor of the polynomial. Indeed, one can check that

$$p(s) = s^4 + 5s^3 + 10s^2 + 20s + 24 = (s^2 + 4)(s + 2)(s + 3).$$

Remark 4.1. Notice how the sign of ϵ affects stability in this case: If we choose a positive ϵ, we get a stable $p(s)$; if a negative one, then we get *two unstable roots*. This is a consequence of the fact that a pair of pure imaginary roots can be made stable or unstable using infinitesimally small perturbations (moving them to either of the two sides of the imaginary axis and since they have to move as a pair, by the properties of the roots of a *real* polynomial.)

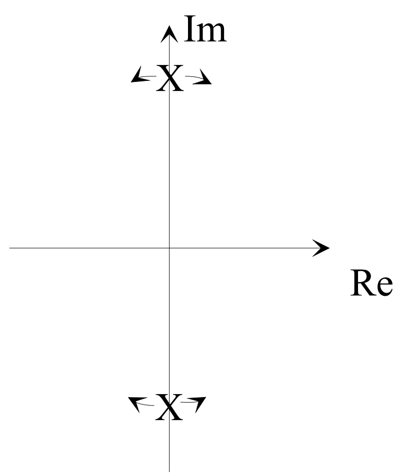


Figure 4.1: Perturbations of a pair of imaginary roots

***Parameter-Dependent Roots** We have discussed in this section the idea that when the RH Table is degenerate, it is useful to examine polynomials that are *close by*, in the sense that their roots are infinitesimally close to those of the given $p(s)$.

Such a situation often arises when the coefficients of the polynomial depend on a number of **parameters**:

$$p(s, k_1, \dots, k_r) = \sum_{i=1}^{n} a_i(k_1, \dots, k_r)s^i.$$

Since the roots of the polynomial depend continuously on the parameters, we get in the complex plane a *movement* of the roots—in the case of a **single** parameter, for each root, we get a branch of a curve in $\mathbb{C}$.

In the next Chapter and in the context of constant-gain feedback, we shall study precisely such kinds of *'curves' or 'loci'*; they will be called **root-locus diagrams**. We have seen in Chapter 2 an example of this: we studied the dependence of the roots of the quadratic

$$s^2 + 2\zeta\omega_0 s + \omega^2$$

on the damping ratio ζ. This discussion should be bourne in mind in going through the section that follows.

4.4 Application to the Stability of Feedback Systems

The above situation, where two roots pass through the imaginary axis, arises in the context of constant-gain feedback. Recalling the equation for the closed-loop poles

$$d(s) + kn(s) = 0,$$

we see that, as k varies, it is possible for a pair of stable roots to become unstable, or the other way round.

Let us look at an example:

Example 4.5. Consider the feedback system with the marginally stable open-loop system

$$G(s) = \frac{1}{s(s^2 + 3s + 2)}.$$

The closed-loop poles satisfy

$$s^3 + 3s^2 + 2s + k = 0.$$

The parameter k (the feedback gain) enters the Routh-Hurwitz Table:

$$\begin{array}{c|ccc} s^3 & 1 & 2 & 0 \\ s^2 & 3 & k & 0 \\ s^1 & \frac{6-k}{3} & 0 & 0 \\ s^0 & k & 0 & 0 \end{array} \tag{4.12}$$

For $0 < k < 6$, the first column consists of positive elements and the closed-loop system is therefore **stable**. For $k > 6$, the sign sequence is $(+1, +1, -1, +1)$ and we conclude from the Generalized RH Criterion that there are **two unstable CL poles**. The *'critical case'* $k = 6$, by the theory presented above, yields two CL poles satisfying

$$3s^2 + 6 = 3(s^2 + 2) = 0,$$

i.e. $s = \pm 2i$.

Remark 4.2. The first column of the Routh-Hurwitz Table for the k-dependent polynomial for the closed-loop poles of a constant-gain feedback system

$$d(s) + kn(s)$$

is, in general, k-dependent, starting with the second element:

$$(a_0, a_1(k), b_1(k), \ldots, a_n(k)).$$

The expressions for $b_1(k), c_1(k)$ and so on are **ratios of polynomials** in k. Thus, for k not coinciding with one of the zeros of these polynomials, it will be possible to determine the number of stable and unstable CL poles, by the generalized RH Criteron.

This information will be combined with the graphical root-locus method in the next Chapter, to give a more complete understanding of the movement of the CL poles.

Example 4.6. For the closed-loop system with

$$G(s) = \frac{s+2}{s^4 + 7s^3 + 15s^2 + 25s},$$

the RH Table is

$$
\begin{array}{c|cccc}
s^4 & 1 & 15 & 2k & 0 \\
s^3 & 7 & 25+k & 0 & 0 \\
s^2 & \frac{80-k}{7} & 2k & 0 & 0 \\
s^1 & c_1(k) & 0 & 0 & 0 \\
s^0 & 14k & 0 & 0 & 0
\end{array}
\tag{4.13}
$$

where

$$c_1(k) = \frac{(80-k)(25+k) - 98k}{(80-k)}.$$

The elements of the first column of the RHT are

$$(1, 7, \frac{80-k}{7}, \frac{(80-k)(25+k) - 98k}{(80-k)}, 14k).$$

Critical values of k are at $k = 80$ and at the roots of the quadratic equation

$$q(k) = -k^2 - 43k + 2000 = 0.$$

Using **Matlab**, these are approximately $k = 28.12, -71.12$ and clearly, since the leading coefficient of $q(k)$ is negative, $q(k) > 0$ for $-71.12 < k < 28.12$. Only non-negative gains need concern us, though, so the interval of positivity is $0 < k < 28.12$.

Putting all this together, we conclude that the closed-loop system will be **stable** for $0 < k < 28.12$ and it will have two unstable CL poles for $k > 28.12$. Note that for $k > 80$ the number of sign changes is still two, so there are *still* only two unstable CL poles (this is a degenerate case, as in Section 4.3.1.)

At the critical $k_c = 28.12$, the pure imaginary roots are found from the s^2-row:

$$\frac{(80 - 28.12)}{7} s^2 + 2(28.12) = 0.$$

(Approximately, $s = \pm 2.75i$.)

4.5 A Refinement of the Stability Criterion

There is a simple way of modifying the Routh-Hurwitz criterion so that it yields extra information about the location of the zeros of $p(s)$. Instead of asking whether the roots are stable or not, we ask: *can we determine whether the roots lie in the half-plane*

$$\mathcal{H}_\alpha = \{s \in \mathbb{C};\ \Re(s) < \alpha\},$$

or in its complement? (We mostly think of negative α.) In other words, since the real part of a root is related to the speed of response of the corresponding time function—think of the Laplace transform pair

$$\frac{(s-\alpha)}{\left((s-\alpha)^2+\omega^2\right)} \longrightarrow e^{\alpha t}\cos(\omega t),$$

we are asking whether the roots are *at least as fast as* the exponential $e^{\alpha t}$ or not.

The modification of the RH Test is very direct: We let

$$\tilde{s} = s + \alpha$$

and consider the polynomial

$$\tilde{p}(\tilde{s}) = p(\tilde{s} - \alpha).$$

Then, if $\tilde{s} = \beta + \omega i$ is a root of $\tilde{p}(\tilde{s})$, $s = (\beta - \alpha) + \omega i$ is a root of $p(s)$. For example, a pure imaginary root of $\tilde{p}(\tilde{s})$ corresponds to a root of $p(s)$ with real part exactly equal to α.

Testing the stability of the new polynomial $\tilde{p}(\tilde{s})$ is thus equivalent to testing whether the roots of $p(s)$ lie in $\mathcal{H}_\alpha$ or not.

Example 4.7. Suppose we want to know how big the gain must be so that both CL poles of the system with $G(s) = \frac{1}{s(s+10)}$ have real part less than -2. We let $s = \tilde{s} - 2$ in the characteristic polynomial

$$s^2 + 10s + k = 0,$$

to obtain the equation

$$(\tilde{s}^2 - 4\tilde{s} + 4) + 10(\tilde{s} - 2) + k = \tilde{s}^2 + 6\tilde{s} + (k - 16) = 0.$$

The RH Table is

$$\begin{array}{c|ccc} s^2 & 1 & (k-16) & 0 \\ s^1 & 6 & 0 & 0 \\ s^0 & (k-16) & & \end{array} \tag{4.14}$$

and we conclude that, for $k > 16$, the modified system is stable or, equivalently, the roots of the original characteristic equation are to the left of the line $\{\Re(s) = -2\}$.

Further examples are found in the Exercises. The shortcoming of this method is of course that a separate RH Table must be computed for every choice of α.

4.6 Kharitonov's Theorem and Robust Stability

The issue of **robustness** is an important one in control: will a desirable property (such as stability) continue to hold if the system is only approximately known? We gave in Chapter 3 the definition of the *sensitivity function* which measured the variation in the system output as the plant varied from its nominal model.

In the context of the topic of this Chapter, namely the *stability of polynomials*, suppose that the coefficients of the polynomial

$$p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$$

are not known exactly, but there are intervals given for their variation:

$$\alpha_i^- \leq a_i \leq \alpha_i^+,$$

where $\alpha_i^- < \alpha_i^+$ for all $i = 1, \ldots, n$.

It would be desirable to have a test for the stability of the polynomial $p(s)$ *for all possible values of its coefficients in the above intervals.* It is an amazing fact that not only such a test exists, but also that it involves the stability test of just **four** polynomials.

* One sees the convenience of this test better if we consider that the intervals above define a 'cuboid' in $\mathbb{R}^n$,

$$(a_1, \ldots, a_n) \in [\alpha_1^-, \alpha_1^+] \times [\alpha_2^-, \alpha_2^+] \times \ldots \times [\alpha_n^-, \alpha_n^+] \subset \mathbb{R}^n$$

which has 2^n vertices. It may seem reasonable to expect that it is sufficient to check stability of the resulting 2^n polynomials (one for each choice of an n-tuple of pluses and minuses.) Yet Kharitonov's theorem says that it is enough to check just four of these vertices!

This result, due to Kharitonov ([11]) was published in the Soviet literature in 1978. It took a few years before it was known in the West, where it quickly led to the development of a very active area of research in the robust stability of linear systems (see the recent book by Barmish [3].)

The theory behind Kharitonov's test is quite classical and is related to the Routh-Hurwitz test as we discussed it in the present Chapter (the book by Gantmacher, [10], is the classic reference on the theory of matrices and has a nice chapter of the RH criterion, containing the results needed for Kharitonov's theorem, on pp.225–232.) More recently, generalizations have appeared where the coefficients of $p(s)$ are allowed to vary in an arbitrary convex polytope.

Theorem 4.1 (Kharitonov's Theorem). *The polynomial $p(s)$ is stable for all choices of coefficients in the ranges indicated above if and only if the following four polynomials are stable:*

$$\begin{aligned}
q_1(s) &= s^n + \alpha_1^- s^{n-1} + \alpha_2^- s^{n-2} + \alpha_3^+ s^{n-3} + \alpha_4^+ s^{n-4} + \alpha_5^- s^{n-5} + \alpha_6^- s^{n-6} + \ldots \\
q_2(s) &= s^n + \alpha_1^+ s^{n-1} + \alpha_2^+ s^{n-2} + \alpha_3^- s^{n-3} + \alpha_4^- s^{n-4} + \alpha_5^+ s^{n-5} + \alpha_6^+ s^{n-6} + \ldots \\
q_3(s) &= s^n + \alpha_1^- s^{n-1} + \alpha_2^+ s^{n-2} + \alpha_3^- s^{n-3} + \alpha_4^+ s^{n-4} + \alpha_5^- s^{n-5} + \alpha_6^+ s^{n-6} + \ldots \\
q_4(s) &= s^n + \alpha_1^+ s^{n-1} + \alpha_2^- s^{n-2} + \alpha_3^+ s^{n-3} + \alpha_4^- s^{n-4} + \alpha_5^+ s^{n-5} + \alpha_6^- s^{n-6} + \ldots
\end{aligned} \tag{4.15}$$

Note the pattern of upper and lower bounds on the coefficients that are used to form the four Kharitonov polynomials; in terms of signs they are:

$$(-,-,+,+,-,-,+,+,\ldots),(+,+,-,-,+,+,-,-,\ldots),$$

$$(-,+,-,+,-,+,-,+,\ldots),(+,-,+,-,+,-,+,\ldots).$$

In our context, it is common practice to write the polynomial with **interval coefficients**,

$$p(s) = s^n + [\alpha_1^-, \alpha_1^+]s^{n-1} + [\alpha_2^-, \alpha_2^+]s^{n-2} + \ldots + [\alpha_{n-1}^-, \alpha_{n-1}^+]s + [\alpha_n^-, \alpha_n^+].$$

Stability of the interval polynomial is then understood to mean the stability of *all* polynomials with coefficients in the given intervals.

Example 4.8. The cubic interval polynomial

$$p(s) = s^3 + [1,2]s^2 + [1,2]s + [1,3]$$

is **not** stable, since

$$q_1(s) = s^3 + s^2 + s + 3$$

is not stable.

4.7 Summary

- The **Routh-Hurwitz table** of a polynomial is computed easily using two-by-two determinants. It contains important stability information.
- A polynomial is (Hurwitz) stable if all the elements of the first column of the RHT are of the same sign.
- If the elements of the first column are all non-zero and there are exactly k sign changes, then there are exactly k unstable poles, the remainder being stable.
- The presence of *zeros* in the first column of the RHT is not generic. It reflects unusual configurations of poles, such as two poles on the imaginary axis.
- The stability information can still be extracted from the RHT even in the presence of zeros, usually by replacing the zero by a small quantity ϵ.
- The presence of poles on the imaginary axis can be detected and their exact values computed from the RHT.
- *Kharitonov's theorem* is a powerful generalization of the RH test to polynomials whose coefficients vary within known intervals. It is simple to apply, as it involves the RH test of just *four* polynomials.

4.8 Exercises

1. Use the Routh-Hurwitz Criterion to determine the stability of the polynomials below. In each case, determine the number of stable and unstable roots.

   (a) $s^4 + 2s^3 + 8s^2 + 4s + 3$
   (b) $s^4 + 2s^3 + s^2 + 4s + 2$
   (c) $s^4 + s^3 + 2s^2 + 2s + 5$
   (d) $s^5 + s^4 + 3s^3 + 9s^2 + 16s + 10$
   (e) $s^6 + 3s^5 + 2s^4 + 9s^3 + 5s^2 + 12s + 20$

2. The characteristic equation for the CL poles of two constant-gain feedback control systems are given below. Determine the range of values of k for the CL systems to be stable.

   (a) $s^4 + 4s^3 + 13s^2 + 36s + k = 0$
   (b) $s^3 + (4 + k)s^2 + ks + 12 = 0$

   Give the OL transfer function in each case.

3. For what values of k, $k > 0$, do the equations

   (a) $s^4 + 8s^3 + 24s^2 + 32s + k = 0$
   (b) $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + k = 0$

   have roots with zero real part? Determine these roots.

4. The characteristic equation of a feedback system is

   $$s^3 + 3(k + 1)s^2 + (7k + 5)s + (4k + 7) = 0.$$

   By setting $\tilde{s} = s + 1$ and applying the RH Criterion to the resulting polynomial in $\tilde{s}$, determine the positive values of k such that the roots of the above equation are more negative than $s = -1$, in other words

   $$\Re(s) < -1.$$

5. A constant-gain feedback control system has the OL transfer function

   $$G(s) = \frac{(s + 13)}{s(s + 3)(s + 7)}.$$

(a) Calculate the range of values of k for the system to be stable.

(b) Check if, for $k = 1$, all the CL poles have real parts more negative than $-1/2$.

6. The OL transfer function of a constant-gain feedback control system is given by

$$G(s) = \frac{(s+1)}{s^3 + as^2 + 2s + 1},$$

where a is a constant parameter and $k \geq 0$.

Use the Routh-Hurwitz criterion to determine the values of k and of a such that the CL transfer function has poles at $\pm 2i$.

Chapter 5

The Root Locus Plot

5.1 Introduction

In this chapter we shall be working with the constant-gain control system of Figure 5.1. If $G(s) = \frac{n(s)}{d(s)}$, recall that the closed-loop transfer function is

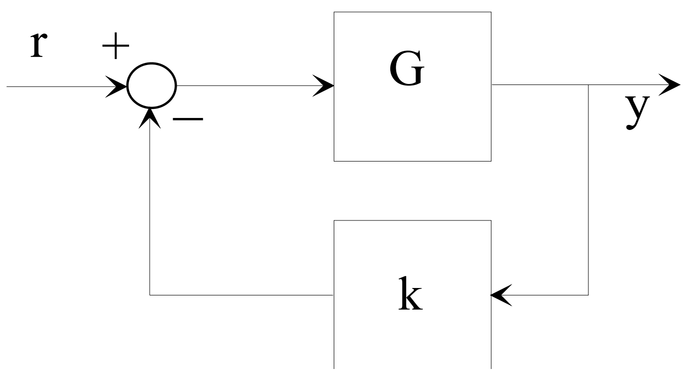


Figure 5.1: Constant-Gain Control System

$$H(s) = \frac{G(s)}{1 + kG(s)} = \frac{n(s)}{d(s) + kn(s)}$$

and we have assumed that $n(s)$ and $d(s)$ have no common factors and that $\deg d(s) > \deg n(s)$.

The closed-loop poles then depend on the gain k and satisfy

$$d(s) + kn(s) = 0.$$

We shall give a *graphical method* for sketching how the CL poles move in the complex plane as k increases from zero to infinity. These will be called **root locus plots** or diagrams (plots of the loci of the CL poles as a function of the scalar parameter k.) This will help us understand:

1. How the gain can be used to improve system performance (by making the CL poles stable, or *'more stable'*, in the sense of making the real part of the poles move further away from the imaginary axis.)

2. The presence of *'trade-offs'* between performance and, for example, stability: it can happen that if we push the gain too high, the system will become unstable.

It must be stressed that quick sketches of root locus plots are easy to obtain, by following a number of rules that we shall present shortly. However, a computer package such as **Matlab** will produce more accurate plots, if needed.

5.2 Definition and Basic Properties

Let $d(s)$ and $n(s)$ be as in the previous section. Suppose that $d(s)$ is of degree n.

Definition 5.1. The **root locus diagram** of the constant-gain feedback system with OL system transfer function $G(s) = n(s)/d(s)$ is a plot in the complex plane $\mathbb{C}$ consisting of n branches, one for each CL pole, each branch following the variation of the pole as the gain k varies in the range $0 \leq k < \infty$.

Let us analyze the characteristic equation for the closed-loop poles in more detail. Write

$$d(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

and

$$n(s) = b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m$$

with $m < n$ and where we normalized the non-zero coefficient of s^n to one in the ratio $n(s)/d(s)$. We also assume $b_0 \neq 0$. If $p_1, \ldots, p_n$ are the OL poles and $z_1, \ldots, z_m$ are the OL zeros, the characteristic equation takes the form

$$\boxed{(s-p_1)(s-p_2)\cdots(s-p_n) + kb_0(s-z_1)\cdots(s-z_m) = 0} \tag{5.1}$$

An alternative form of this equation, for $k > 0$, is

$$G(s) = -\frac{1}{k} \Longleftrightarrow \frac{n(s)}{d(s)} = -\frac{1}{k} \tag{5.2}$$

or, in factorized form,

$$\boxed{\frac{b_0(s-z_1)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = -\frac{1}{k}} \tag{5.3}$$

This is the right starting point for the derivation of the root locus plot.

Let $\lambda_1(k), \lambda_2(k), \ldots, \lambda_n(k)$ be the closed-loop poles, as a function of the gain k. First note that, from equation 5.1, $\lambda_i(0) = p_i$, i.e. rather obviously the CL poles for gain equal to zero coincide with the OL poles. As k increases from zero, the CL poles move in such a way that, if $\lambda = \lambda(k)$ is a closed-loop pole,

$$\frac{b_0(\lambda-z_1)\cdots(\lambda-z_m)}{(\lambda-p_1)(\lambda-p_2)\cdots(\lambda-p_n)} = -\frac{1}{k}.$$

Thus, a point in the complex plane, $s = \lambda$, belongs to the root locus plot if it satisfies the above equation.

Now we use the **polar form** (modulus-argument) for complex numbers, $z = |z|e^{i\angle(z)}$, and the elementary properties:

1. $|z_1 z_2| = |z_1||z_2|$ and $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$,
2. $\angle(z_1 z_2) = \angle(z_1) + \angle(z_2)$, $\angle(\frac{z_1}{z_2}) = \angle(z_1) - \angle(z_2)$,

so that the polar form of the number minus one is

$$-1 = 1e^{i\pi(2\ell+1)},\ \ell \text{ any integer},$$

to derive the basic condition for s to lie on the root locus plot, for $k > 0$:

$$\frac{|n(s)|}{|d(s)|} = \frac{1}{k}$$

and

$$\angle(n(s)) - \angle(d(s)) = (2\ell+1)\pi.$$

These are known as the **Modulus or Magnitude Condition** and **Argument or Phase Condition**, respectively.

In terms of the factorized forms of $n(s)$ and $d(s)$, the modulus and argument conditions are, respectively,

$$\boxed{|b_0| \cdot \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{j=1}^{n} |s - p_j|} = \frac{1}{k}} \tag{5.4}$$

and

$$\boxed{\sum_i \angle(s - z_i) - \sum_j \angle(s - p_j) = (2\ell + 1)\pi, \quad \ell = 0, \pm 1, \pm 2, \ldots} \tag{5.5}$$

if $b_0 > 0$ and

$$\boxed{\sum_i \angle(s - z_i) - \sum_j \angle(s - p_j) = 2\ell\pi, \quad \ell = 0, \pm 1, \pm 2, \ldots} \tag{5.6}$$

if $b_0 < 0$.

In the complex plane, each term of the form $s - z_i$ or $s - p_j$ represents a vector from the point s on the root locus to one of the zeros, z_i, or poles, p_j, resp. Therefore, the *modulus condition* says that the ratio of the products of the lengths of such vectors corresponding to all the zeros, divided by the product of the lengths of the vectors for the poles, is equal to $1/k$.

The *argument condition* states that the sum of the angle contributions of vectors corresponding to the zeros minus the sum of the contributions from the poles is equal to an odd multiple of π (for $b_0 > 0$.)

Remark 5.1. Note that the argument condition **does not depend on** k! This means that *the argument condition alone can be used to find the root locus branches, the modulus condition only coming in later to fix the corresponding gain* k.

Example 5.1. Let

$$G(s) = \frac{1}{(s+1)(s+3)}.$$

The argument condition is

$$-\angle(s+1) - \angle(s+3) = (2\ell + 1)\pi.$$

As Figure 5.2 shows, this condition is satisfied on the bisector of the interval from -3 to -1 and also along the interval from -3 to -1. This is because,

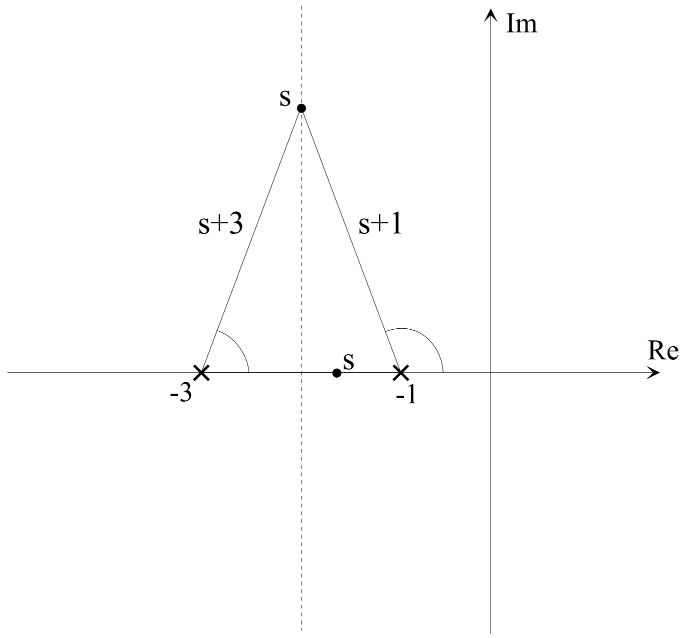


Figure 5.2: Condition for s to be in the Root Locus plot of $G(s) = \frac{1}{(s+1)(s+3)}$

for the isosceles triangle of Figure 5.2, the interior angle plus the exterior angle of the opposite vertex sum up to pi; also, for s between -3 and -1, the angle contribution of $(s+1)$ is equal to π, while that of $(s+3)$ is zero. However, outside this interval, the sum of the angles is either 0 or 2π. It can be checked that only the points of the bisector and of the interval $[-3, -1]$ belong to the root locus.

The gain dependence is as follows: the CL poles start at the positions of the OL poles at $-3, -1$ and move towards each other on the real axis with increasing gain, until a double pole occurs. Subsequently, for still higher gains, the two CL poles have constant real part and form a complex conjugate pair going to infinity as $k \to \infty$.

This can be checked directly by looking at the discriminant of the characteristic equation

$$s^2 + 4s + (3 + k) = 0,$$

$$\Delta = 4^2 - 4(3 + k) = 4(1 - k),$$

which is positive for $0 < k < 1$ and negative for $k > 1$, and using the formula for the poles

$$s = \frac{-4 \pm \sqrt{\Delta}}{2} = -2 \pm \frac{\sqrt{\Delta}}{2}.$$

Even though we managed to find the root locus plot of a simple system directly from the two conditions, the root loci of more complex systems cannot be found so easily without a more systematic approach. This is what we turn our attention to in the next Section.

5.3 Graphical Construction of the Root Locus

The root locus can be *sketched* using the following steps; of course, for greater accuracy, a computer package such as `matlab` will be used in practice. Note that the root locus is **symmetric** about the real axis, since roots of polynomial equations with *real coefficients* occur in complex conjugate pairs.

1. **OL Poles and Zeros:** In the complex plane, mark the n **open-loop poles** with an $\times$ and the m **open-loop zeros** with a $\circ$ (we assume $n > m$.)

2. **Branches of the root locus:** The root locus has exactly n **branches**, each branch starting at an open-loop pole. Of these, m branches will approach the m open-loop zeros as $k \to \infty$. The remaining $n - m$ branches will go to infinity (in modulus.)

   * The root locus has n branches because it is the locus of the roots of a polynomial of degree $\deg d(s) = n$ as a parameter, k, varies.

   Roughly, for $k >> 1$, the polynomial $n(s)$ dominates $d(s)$ in $d(s) + kn(s)$ and, asymptotically, m of the roots tend to the roots of $n(s)$, in other words the OL zeros. The excess $n - m$ roots go to infinity. The full justification for this involves the consideration of the polynomial $d + kn$ in the projective space $\mathbb{R}P^n$.

3. **Asymptotic rays:** The $n - m$ branches going to infinity asymptotically approach $n - m$ rays originating at a single point on the real axis. These rays are the **asymptotes** and the common origin is the **centre of the asymptotes**.

The **angles of the asymptotes** are evenly spaced around the unit circle and they are given by

$$\alpha = \frac{2\ell + 1}{n - m}\pi,$$

for $\ell = 0, \pm 1, \pm 2, \ldots$

More explicitly, we tabulate the first few cases:

$n - m$	α
1	π
2	$\pm\pi/2$
3	$\pm\pi/3, \pi$
4	$\pm\pi/4, \pm 3\pi/4$

The **centre of the asymptotes** is at the real point

$$\sigma_c = \frac{\sum_j p_j - \sum_i z_i}{(n - m)}$$

* The asymptotic angles are found by considering a point on the root locus *'far away'*, in the sense that $|s| >> |p_j|, |z_i|$, for all OL poles and zeros, see Figure 5.3. Then the angles of the vectors $s - p_j$, $s - z_i$ are all approximately equal and the angle condition gives:

$$m\alpha - n\alpha = (2\ell + 1)\pi,$$

from which the asymptotic ray angle condition follows.

4. **Real Part of the Root Locus:** The parts of the real axis to the left of an odd number of poles and zeros belong to the root locus, by a simple application of the angle condition, see Figure 5.4.

   * Indeed, as a consequence of the angle condition, for a point on the real axis with, say, k_p real OL poles and k_z real OL zeros to its right—and any number of poles and zeros to its left—and such that $k_p + k_z$ is **odd** (so

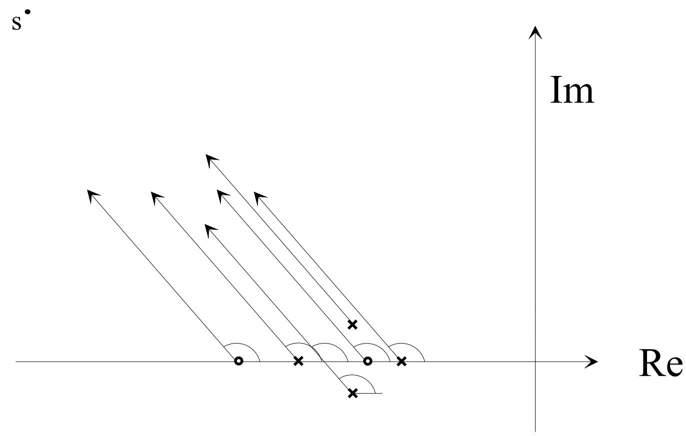


Figure 5.3: All angles to a faraway point are approximately equal

in particular non-zero), the angle contributions are: zero for the poles and zeros to the left of s and π for each pole or zero to the right. The complex conjugate poles and zeros give **zero** net contribution, by symmetry. Thus, we have total angle

$$(k_z - k_p)\pi = (k_z + k_p - 2k_p)\pi$$

which is an **odd** multiple of π.

5. **Breakaway Points:** These are points at which two (or more) branches of the root locus leave the real axis. Similarly, points where two (or more) branches coalesce on the real axis and move, as $k \to \infty$, towards two real OL zeros. We shall describe only the case of an interval bounded by two OL poles, the other case being similar.

   Suppose we have found that an interval in the real axis between two poles is part of the root locus (see Figure 5.5.) Since a branch of the RL originates at each pole, the two loci move towards each other until a double pole occurs; subsequently, the branches move off in opposite vertical directions in the complex plane. A method to find the breakaway point is to notice that the gain k, as a function of the *real* roots s, must have a local maximum at the breakaway points, so that, with

$$k(s) = -d(s)/n(s)$$

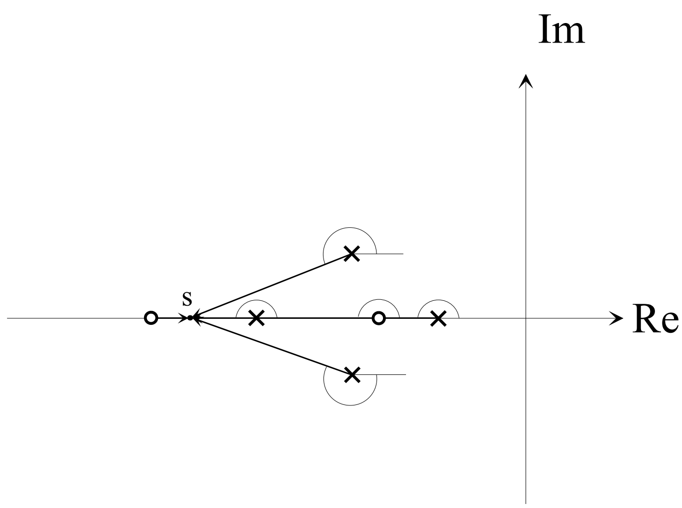


Figure 5.4: The point s in in the root locus

and s considered a **real** variable, we require

$$\frac{dk(s)}{ds} = 0.$$

This can often be quite a lengthy computation and the resulting stationarity condition impossible to solve by hand. As a result, one often assumes in practice that the break point is at the mid-point of the interval bounded by the two poles. Alternativley, one can attempt to plot the function $-d(s)/n(s)$.

* Taking the case of two real OL poles, notice that $k = 0$ at the endpoints of the interval, namely at the two OL poles. As k increases from zero, two real CL poles move towards each other until they form a double CL pole for some value of k. It is clear that, on the interval between the poles k, as a function of the real variable s, achieves a maximum at the break point.

Similarly, if the interval is bounded by two OL zeros, we know that $k \to \infty$ as s approaches the OL zeros, so k achieves a *minimum* at the point of arrival of the two branches on the real axis.

The convention of taking the break point at the mid-point of the interval can now be seen to involve approximating the function $k(s)$ by a parabola—since the values at the end-points are zero, the maximum of the quadratic approximation occurs at the mid-point of the interval. Naturally, if the

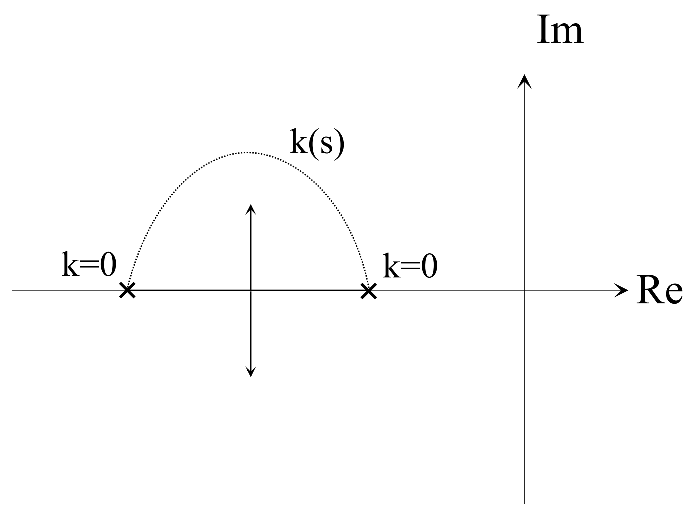


Figure 5.5: The gain k achieves a maximum at the breakaway point

interval is long enough for the quadratic approximation to be inaccurate, this will fail.

6. **Crossover Points:** Find the points where the branches of the root locus cross the imaginary axis.

   This is most easily done using the **Routh-Hurwitz criterion**. In general, we expect that exactly two branches cross the imaginary axis *cleanly* (or *transversely*), in other words at an angle other than $\pm\pi/2$. The RHT can then be used to detect the values of the gain k where two poles change their *stability type* (they either pass from being stable to being unstable or the other way round, depending on the direction of the crossing of the imaginary axis.) Recall that the RHT also gives us *precise information on the location of the two pure imaginary poles at the point of crossover* (extracting the quadratic factor from the s^2-row of the RHT.)

7. **Angles of Departure:** For a pair of complex conjugate poles $p, \bar{p}$ (such that $Im(p) \neq 0$) determine the **angles of departure** of the corresponding branches. (By symmetry, only one need be computed.) Similarly, one computes the **angles of arrival** at a pair of complex zeros. If ϕ_p is the angle of departure for the pole p, then (see Figure 5.6)

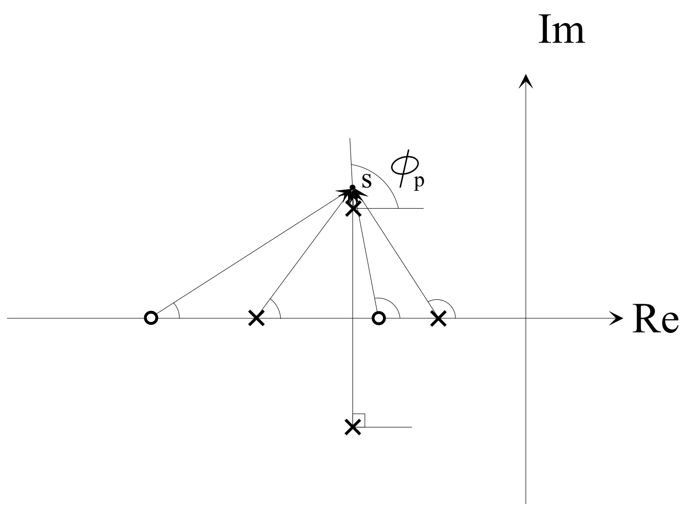


Figure 5.6: Angle of departure ϕ from a pole

$$\phi_p = \sum_i \angle(s - z_i) - \sum_{j,\, p_j \neq p} \angle(s - p_j) - (2\ell + 1)\pi,$$

where the summation over the pole contributions to the angle excludes the pole at p (but not the pole at $\bar{p}$.)

* The above follows directly from the argument or angle condition: take a point s on the root locus and arbitrarily close to the pole p, so that the slope of the segment $s - p$ is approximately equal to the angle of departure ϕ. Then

$$\sum_i \angle(s - z_i) - \sum_j \angle(s - p_j) - \phi_p = (2\ell + 1)\pi.$$

The above procedure is followed in most cases where a Root Locus plot is to be sketched (see Example 5.3 for an exception.) Even though the steps we gave seem to concentrate on the *asymptotic* and *local* behaviour of the locus (near breakawy or crossover points), the resulting sketch contains considerable information on the closed-loop poles, as a function of the gain k.

5.4 Examples

In all the examples to follow, $G(s)$ is the transfer function of a system to be controlled using constant-gain feedback, with $k \geq 0$.

Example 5.2. Let

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}.$$

The CL pole equation is

$$s^3 + 6s^2 + 11s + (6+k) = 0.$$

The OL poles are $p = -1, -2, -3$ and there are no OL zeros. The RL has three branches, all going to infinity along the asymptotic rays centred at

$$\sigma_c = \frac{-1-2-3}{3} = -2$$

and at angles $\pm\pi/3, \pi$. The half-line $(-\infty, -3]$ and the interval $[-2, -1]$ are the real part of the RL.

The breakpoint is computed using

$$k(s) = -\frac{d(s)}{n(s)} = -(s^3 + 6s^2 + 11s + 6)$$

by setting

$$\frac{dk(s)}{ds} = -3s^2 - 12s - 11 = 0.$$

This has roots

$$s = -2 \pm \frac{\sqrt{3}}{3},$$

but only the root $s = -2 + \frac{\sqrt{3}}{3}$ lies in the interval $(-2, -1)$ and is hence the **breakaway point** for the two branches originating at -2 and -1.

The **crossover points** are found from the Routh-Hurwitz Table:

$$\begin{array}{c|cc} s^3 & 1 & 11 \\ s^2 & 6 & 6+k \\ s^1 & 10-\frac{k}{6} & 0 \\ s^0 & 6+k & 0 \end{array} \tag{5.7}$$

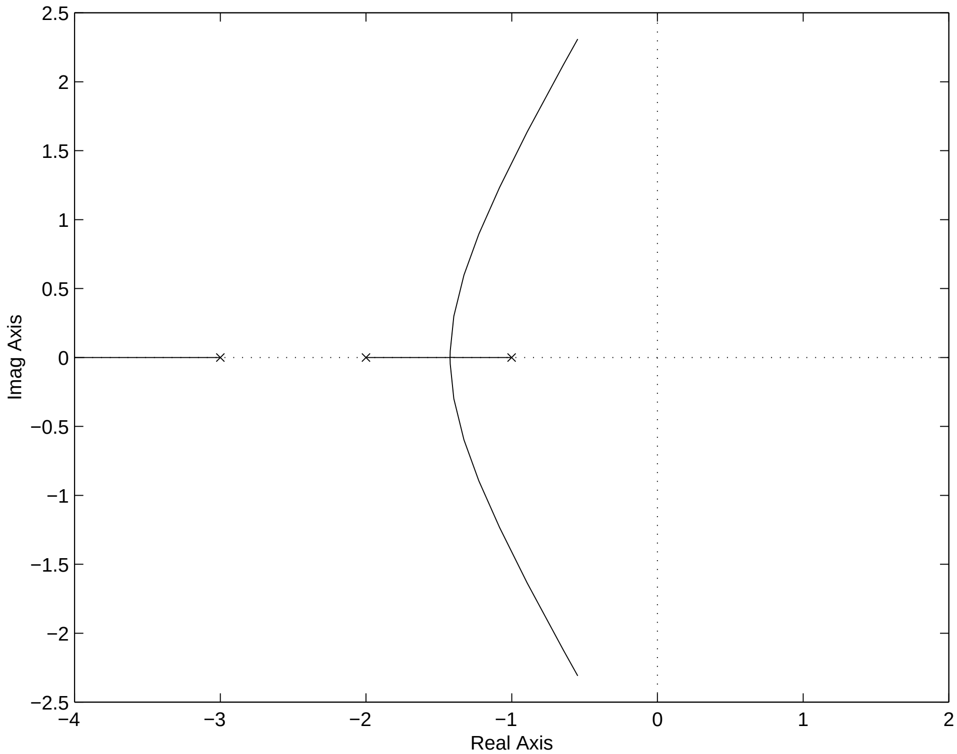


Figure 5.7: The root locus plot for $G(s) = \frac{1}{(s+1)(s+2)(s+3)}$

The critical k is $k_c = 60$ and the quadratic factor is then

$$6s^2 + 66 = 6(s^2 + 11),$$

and we conclude that the crossover points are at $s = \pm i\sqrt{11} \simeq i3.317$. Note that $\tan(\pi/3)2 = \sqrt{3}2 \simeq 3.464$, so that the asympotic ray stays above the crossover point at $s = i\sqrt{11}$.

Example 5.3. Let

$$G(s) = \frac{(s+4)}{s(s+2)}.$$

The CL pole equation is

$$s^2 + (k+2)s + 4k = 0.$$

The OL poles are $p = 0, -2$ and the OL zero at $z = -4$. Just one branch goes to infinity at an angle of π. The half-line$(-\infty, -4]$ and the interval $[-2, 0]$ form the real part of the RL.

From the asymptotic directions on the branches we conclude that there must be both a breakaway point (between -2 and 0) and a point of arrival to the left of the zero at -4.

In fact, we claim that *the RL away from the real axis lies on a circle centred at* -4 *and of radius* $2\sqrt{2}$.

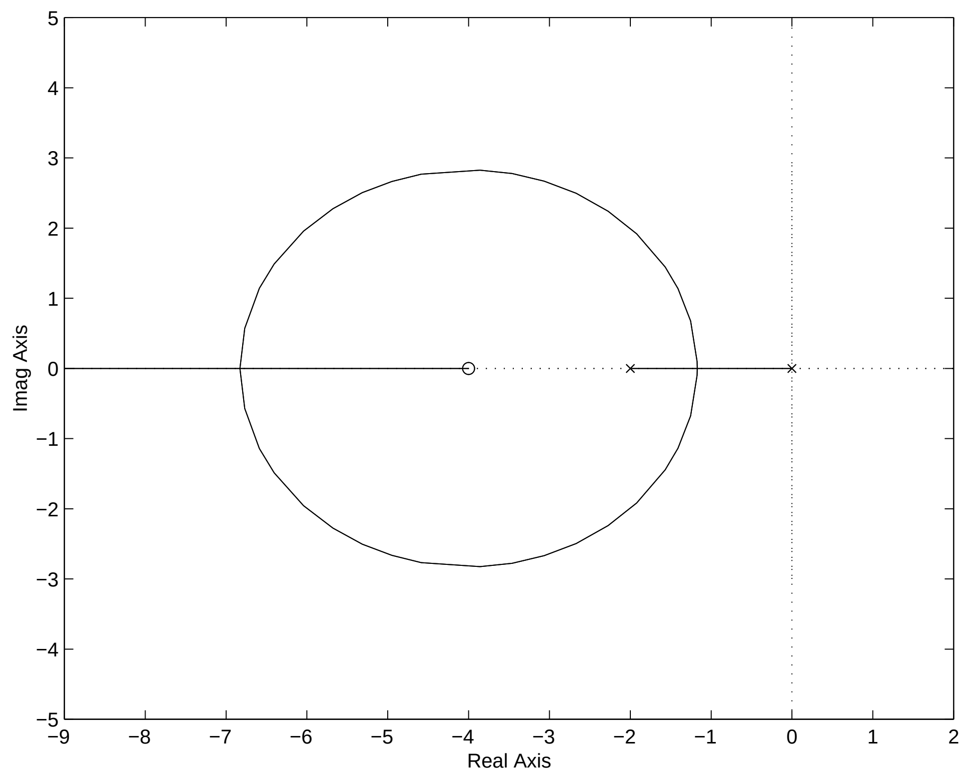


Figure 5.8: The root locus plot for $G(s) = \frac{(s+4)}{s(s+2)}$

Write $s = \sigma + i\omega$; substituting into the characteristic equation, we have

$$(\sigma^2 - \omega^2) + 2i\sigma\omega + (k+2)(\sigma + i\omega) + 4k = 0.$$

Setting the real and imaginary parts of this to zero, we get

$$\begin{aligned}(\sigma^2 - \omega^2) + (k+2)\sigma + 4k &= 0 \\ 2\sigma\omega + (k+2)\omega &= 0\end{aligned} \tag{5.8}$$

The second equation gives

$$k = -2(\sigma + 1)$$

which, substituted into the first equation, gives

$$\sigma^2 + \omega^2 + 8\sigma + 8 = 0.$$

Completing the square, we get the equation of the circle, as claimed:

$$(\sigma + 4)^2 + \omega^2 = 8.$$

Clearly, the desired break points are at $-4 \pm 2\sqrt{2}$.

5.5 Dominant Poles

Suppose that, for some gain k, the closed-loop system is **stable** and the configuration of the closed-loop poles is such that there is a complex pair $(p, \bar{p})$ such that

$$\Re(p_i) << \Re(p),$$

for all other CL poles p_i ($p_i \neq p, \bar{p}$), see Figure 5.9. In practice, we may take this separation in the real parts to mean that, say,

$$\Re(p_i) < 2\Re(p).$$

We would like to argue that the pole pair $(p, \bar{p})$ is **dominant** in the sense that the closed-loop response of the system is composed primarily of the *modes* corresponding to the poles $p, \bar{p}$. Here, by *'mode'*, we mean a term in the partial fractions expansion of $Y(s)$ corresponding to a unique pole.

Indeed, we have that the *impulse response*, for example, can be written

$$g(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left(\frac{A(s+\alpha) + B\omega}{(s+\alpha)^2 + \omega^2} + \sum_{p_i \neq p, \bar{p}} \sum_j \frac{c_{ij}}{(s - p_i)^j}\right),$$

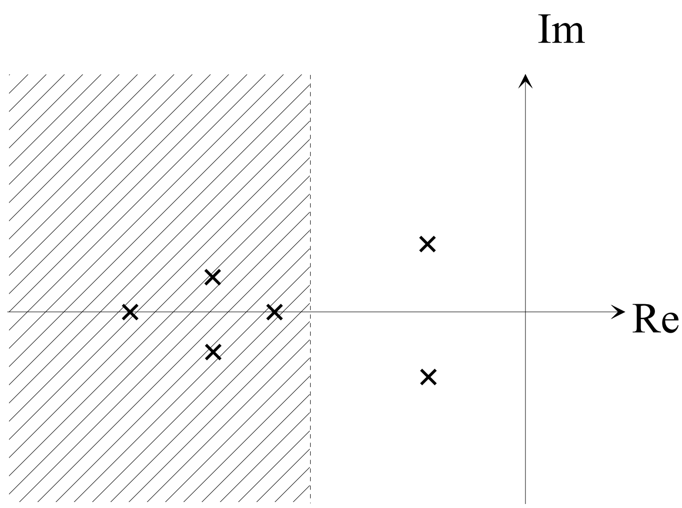


Figure 5.9: A dominant pair of CL poles

where $p = -\alpha + i\omega$ and the real parts, say α_i, of all the other poles are more than twice as negative as α.

Since this gives

$$g(t) = e^{-\alpha t}(A\cos(\omega t) + B\sin(\omega t)) + \sum e^{-\alpha_i t} \sum_j t^j (\text{sinusoidal terms}),$$

we see that the *much faster terms* from poles $p_i \neq p, \bar{p}$ will decay first, leaving the main part of the response to be the much **slower** response due to the pair $(p, \bar{p})$.

Now this separation of time scales will continue to hold for other inputs as well, for example for the *step response* of the CL system. This is seen from the partial fractions expansion of the output, $Y(s) = G(s)U(s)$ for other $u(t)$.

If a dominant pair exists for a fixed closed-loop system, the **second-order approximation** of the CL system is obtained by retaining only the terms corresponding to the dominant pair of CL poles.

Let us look at this approximation in more detail as it is often the case that in concrete design cases, desirable performance requirements are imposed on the second-order system, in the hope that then the full system will satisfy these requirements.

5.5.1 Performance Requirements for the Second-order Approximation

Consider the stable second-order system

$$G(s) = \frac{cs + d}{s^2 + 2\zeta\omega_0 s + \omega_0^2},$$

with $0 < \zeta < 1$. The correspondence with $p = -\alpha + i\omega$ is clearly that

$$\alpha = -\zeta\omega_0 \text{ and } \omega = \omega_0\sqrt{1-\zeta^2}.$$

Recall that

$$|p| = \omega_0\sqrt{\zeta^2 + 1 - \zeta^2} = \omega_0.$$

Hence note (see Figure 5.10) that the angle ϕ that p makes with the negative real axis satisfies

$$\cos\phi = \frac{\zeta\omega_0}{\omega_0} = \zeta.$$

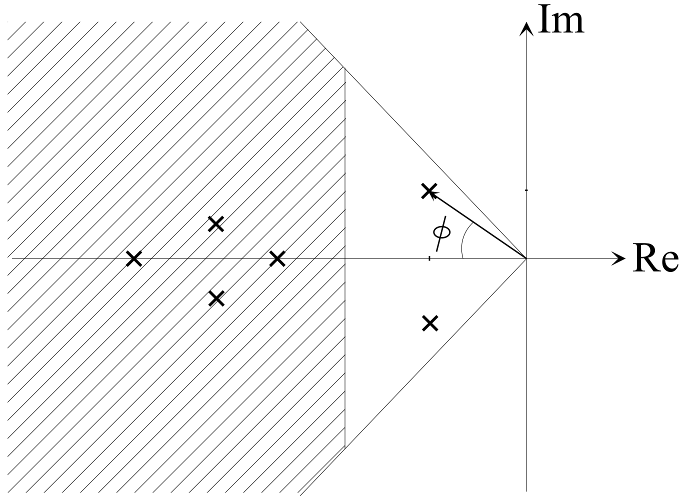


Figure 5.10: A dominant pair of CL poles

Performance Requirements: We would like the closed-loop response (say to a unit step) to be

1. **Fast**, in the sense that it reaches its steady-state value quickly. In practice, since the convergence to the steady-state value is asymptotic, we would like to be within, say, 2% of the final value within a fixed time, called the **settling time**, T_s. The magnitude of T_s clearly depends on the application we have in mind.

2. **Well-damped** in the sense of having a high enough damping ratio ζ; since $\cos\phi = \zeta$, this implies that the oscillatory (sinusoidal) component $i\omega_0\sqrt{1-\zeta^2}$ (the *imaginary part* of the pole) is small relative to the real part. In turn, this means that the final values is reached without excessive **overshoot** (going above the final value, see below.)

We shall only consider these two specifications. They will be referred to as the **settling time** requirement and the **overshoot** requirement. Let us look at them in turn.

1. **Settling Time:** Since the response of the second-order system lies inside the *exponential envelope* $\pm e^{-\zeta\omega_0 t}$, if we want to be within two percent of the final value in T_s seconds, this implies that

$$e^{-\zeta\omega_0 T_s} = 0.02.$$

Hence

$$\boxed{T_s = \frac{\ln(0.02)}{-\zeta\omega_0} \simeq \frac{4}{\zeta\omega_0}} \tag{5.9}$$

since $\ln(0.02) \simeq -3.91 \simeq 4$.

2. **Overshoot Requirement:** Consider the PFE expansion of the *step response* of the second-order system above. Suppose that the initial condition is zero. We have

$$Y_s(s) = \frac{G(0)}{s} + \frac{A(s+\alpha) + B\omega}{s^2 + 2\zeta\omega_0 s + \omega^2}$$

so that

$$y_s(t) = G(0) + e^{-\alpha t}(A\cos(\omega t) + B\sin(\omega t))$$

and, since we assumed $y_s(0) = 0$, it is checked that

$$A = -G(0)$$

and so the maximal overshoot from the final value $G(0)$ is due to the **cosine** term in the response $y_s(t)$ and in particular *after half a period.* Since $\omega = \omega_0\sqrt{1-\zeta^2}$, the period T is

$$T = \frac{2\pi}{\omega}$$

and we conclude that the maximal possible overshoot occurs at

$$t_m = \frac{T}{2} = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}}.$$

At that time, the exponential envelope has decayed to the value

$$e^{-\alpha T/2} = e^{-\zeta\omega_0(\pi/\omega_0\sqrt{1-\zeta^2})} = e^{-\zeta\pi/\sqrt{1-\zeta^2}}.$$

The **percent overshoot** is then defined as

$$\boxed{POV = 100 \cdot e^{-\zeta\pi/\sqrt{1-\zeta^2}}} \tag{5.10}$$

Exercise 5.1. Show that the settling time requirement constrains the pole pair to lie to the left of a vertical line in the complex plane and that the overshoot requirement constrains the pole pair to lie inside a cone

$$\{\cos\phi \le \zeta_c\}.$$

Find the line and the critical ζ and show that the resulting region in the complex plane looks as in Figure 5.10.

5.6 * Closed-loop stability and Kharitonov's theorem

Kharitonov's theorem, 4.15, asserts the stability of an interval polynomial based on the stability of just four 'vertex' polynomials. In the control context, we have considered polynomials of the form

$$p(s) = d(s) + kn(s),$$

(the characteristic polynomial of a constant-gain feedback system.) Fixing a gain interval $k \in [k^-, k^+]$ gives intervals of variation for the coefficients of $p(s)$ (find them explicitly if $d(s)$ is of degree n and $n(s)$ is of degree $m < n$.)

It is conceivable that the following may happen: at the two endpoints k^- and k^+ the closed-loop system is stable, yet for some intermediate value (or interval) the CL system is unstable. This would appear to contradict Kharitonov's theorem, since only the vertices are relevant to the test.

The following exercise is meant to convince the reader of the truth of Kharitonov's theorem even in this case.

Exercise 5.2. Consider the open-loop transfer function

$$G(s) = \frac{(s^2 + 0.3s + 4)}{(s^2 + 3s + 2)(s - 0.5)}.$$

The root locus plot of this system is shown in Figure 5.11. The characteristic

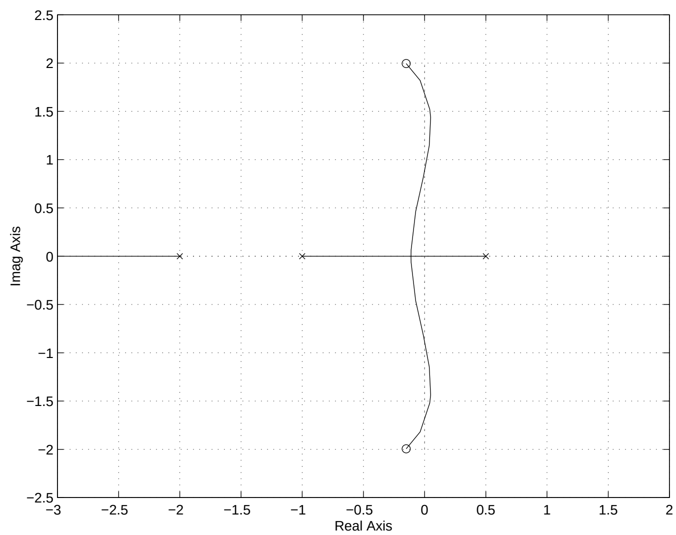


Figure 5.11: This CL system is unstable for $0.91 < k < 8.26$ (approx.)

equation is

$$s^3 + (k + 2.5)s^2 + (0.3k + 0.5)s + (4k - 1) = 0$$

and a quick application of the Routh-Hurwitz table gives **CL instability** for $0 < k < 0.25$ and for $0.908 < k < 8.258$, approximately. In particular, the CL system is stable for $k = 0.5$ and for $k = 10$. Using the gain interval $[0.5, 10]$, we obtain the *interval polynomial*

$$p(s) = s^3 + [3, 12.5]s^2 + [0.65, 3.5]s + [1, 39].$$

Let us see whether the Kharitonov test can detect the instability for values in the interval we gave above, lying between $k = 0.5$ and $k = 10$. Note that, since the lower and upper bounds of the intervals correspond exactly to $k = 0.5$ and $k = 10$, respectively, we know that both the $(+, +, +)$ and the $(-, -, -)$ vertex give stable systems.

However, for the $(-, -, +)$ vertex, we get the Kharitonov polynomial

$$q_1(s) = s^3 + 3s^2 + 0.65s + 39,$$

which is easily seen to be unstable. Thus the Kharitonov test **failed** and the CL system is not stable for all intermediate values, as had to be the case.

5.7 Summary

- The *Root Locus plot* of the OL transfer function $G(s)$ gives the locus of the closed-loop poles as the feedback gain varies from zero to $+\infty$.
- The RL plot is easy to sketch using a simple, step-by-step graphical procedure (see the main body of the text) and can be of great help in control design.
- A stable pair of complex poles is *dominant* when it is considerably *slower* than all the other CL poles. Thus, the fast terms contribute transients that disappear, leaving the bulk of the response to be due to the dominant pole pair. Control design based on a dominant pair is often easier and useful.

5.8 Appendix: Using the Control Toolbox of Matlab

It is extremely easy to use `matlab` as an aid in the analysis of control systems. Most commands one uses are part of the comprehensive `control toolbox` that is frequently updated and improved on. Remember that `matlab` has a very good help facility, accessed by typing `help` *command*, for any command whose name you know, or just `help` to get the help menu.

The specification of a system is straightforward: Suppose we want to study the transfer function

$$G(s) = \frac{(s+2)}{s(s+3)(s+4)} = \frac{(s+2)}{s^3 + 7s^2 + 12s}.$$

We input the numerator and denominator polynomials by typing the coefficients of the descending powers of s (*caution*: if, as above, there is no constant term, we must remember to input a zero.)

```
>> num = [1 2];
>> den = [1 7 12 0];
```

If we want `matlab` to multiply the factors, we can use

```
>> den1 = [1 3 0];
>> den2 = [1 4];
>> den = conv(den1, den2);
```

The `conv` command performs a discrete convolution which, in this case, is the same as multiplying the two factors We can check that we have typed the correct system with the 'print system' command

```
>> printsys(num,den);
```

If we want to find the roots of a polynomial that is not given in factorized form, use the `roots` command

```
>> roots(den);
```

As a first step in the analysis of a system, we can obtain its response to either a *step input* (the input is the unit step function $h(t) = 1$ for $t \geq 0$ and zero otherwise) or a *unit impulse* (the input is the Dirac delta $\delta(t)$ and so its transform is identically equal to one):

```
>> step(num,den);
>> impulse(num,den);
```

In each case, a plot of output versus time is given. (Recall the `hold on`, `hold off` facility of `matlab` for getting more than one plot on the same figure window.)

Now suppose we close the loop using a gain k, so that we have the **closed-loop transfer function**

$$H(s) = \frac{G(s)}{1 + kG(s)}.$$

The *root locus* of this CLTF is obtained by typing

```
>> rlocus(num, den);
```

This plots the curve of complex closed-loop poles parametrized by the gain k, $k \geq 0$ (upper limit selected automatically by `matlab`.) Again, the plot comes without the need to use a separate plot command.

If you need more detail, you can use the `axis` command

```
>> axis([x0 x1 y0 y1]);
```

where the real axis interval of the plot is $[x0, x1]$ and the imaginary axis interval is $[y0, y1]$.

Another useful command for control design is the root locus *find* command

```
>> [k, poles] = rlocfind(num,den);
```

This sets up crosshairs for picking a point on the root locus (such as the imaginary axis crossing) and reading off the k values together with the corresponging poles; you must type `k` and `poles` at the `matlab` prompt to get these values.

It is just as easy to use *state space* descriptions of systems (as we shall see later in the course) and also to convert from state space to transfer function and vice versa.

Finally, remember the up and down arrows that can be used to scroll through previous commands: they are very handy for modifying your system, for example.

5.9 Exercises

1. Draw the root locus plots for the closed-loop constant-gain systems with OL transfer functions

   (a) $G(s) = \frac{1}{s(s+3)(s+6)}$

   (b) $G(s) = \frac{(s+2)}{s^2+2s+3}$

   (c) $G(s) = \frac{1}{s(s+3)(s^2+2s+2)}$

2. Draw the root locus plot for the closed-loop constant-gain system with OL transfer function
$$G(s) = \frac{(s+1)}{s(s+2)(s+3)}.$$
   To evaluate the breakaway point, plot the graph of
$$p(s) = -\frac{s(s+2)(s+3)}{(s+1)}$$
   for s between -2 and -3 on the real axis.

3. Draw the root locus plot for the closed-loop constant-gain system with OL transfer function
$$G(s) = \frac{1}{s(s+4)(s+5)}$$

   (a) Determine the value of the gain k which gives a closed-loop pole at -0.6 using the magnitude criterion.

   (b) Determine graphically the value of $k = k_c$ giving two dominant complex closed-loop poles corresponding to a damping ratio $\zeta = 0.5$ of a second-order system.

   (c) Determine graphically the approximate position of the third pole when $k = k_c$. Comment on the validity of the control design in parts (ii) and (iii).

4. Draw the root locus plot for the closed-loop constant-gain system with OL transfer function
$$G(s) = \frac{1}{s(s+3)(s^2+2s+2)}.$$
   You can take the breakaway point from the real axis to be at $s = -2.3$.

(a) Determine graphically the value of $k = k_c$ giving two dominant closed-loop poles corresponding to a damping ratio $\zeta = 0.5$ of a second-order system.

(b) For $k = k_c$, determine all the closed-loop poles of the system. Are the two complex poles you found in part (i) dominant?

(c) Determine k_{max} such that the CL system is stable for $0 < k < k_{max}$.

(d) For this k_{max}, describe the CL impulse response of the system as $t \to \infty$.

Chapter 6

The Nyquist Plot and Stability Criterion

6.1 Introduction

The **Nyquist plot** of a linear system $G(s)$ is the locus in the complex plane of the complex function $G(i\omega)$, for all ω in $(-\infty, +\infty)$. Roughly, it gives information on the response of the linear system to **sinusoidal inputs** of frequencies in the range $\omega \in [0, \infty)$. But it also contains important **stability** information for the closed-loop system with gain k in the feedback (or forward) path (roughly because it is the map of $G(s)$ restricted to the dividing line between stable and unstable systems.)

Like the *Root locus plot* (of the CL poles satisfying $d(s) + kn(s)$), the *Nyquist plot* is easy to sketch. We shall give the steps for sketching the Nyquist plot in the next Section. The stability criterion is given in Section 6.4. The Nyquist plot is very useful in control design and has been used since the early fifties. *Stability margins*, called **phase and gain margins** are defined next, and their use in control design explained.

Here, by way of explaining why it is useful to consider the transfer function for $s = i\omega$, let us again consider the response of a linear system to a *sinusoidal input*.

Recall that $Y(s) = G(s)U(s)$ and suppose that the input $u(t) = \cos(\omega t)$. Then

$$U(s) = \frac{s}{s^2 + \omega^2}$$

and, in the partial fractions expansion of $Y(s)$,

$$Y(s) = \frac{As + B\omega}{s^2 + \omega^2} + \text{terms corr. to poles of G(s)},$$

we compute the coefficients A and B from the residue

$$R = \frac{sG(s)}{(s + i\omega)}|_{s=i\omega} = \frac{i\omega G(i\omega)}{2i\omega} = \frac{G(i\omega)}{2}.$$

Hence, writing

$$G(i\omega) = G_R(i\omega) + iG_I(i\omega) = |G(i\omega)|e^{i\angle(G(i\omega))},$$

$$A = 2\Re(R) = G_R(i\omega) \text{ and } B = -2\Im(R) = -G_I(i\omega)$$

and so, *assuming for the moment that* $G(s)$ *is stable,*

$$y(t) = G_R(i\omega)\cos(\omega t) - G_I(i\omega)\sin(\omega t) + \ldots =$$

$$= |G(i\omega)|\cos(\omega t + \angle G(i\omega)) + \ldots .$$

(where the dots represent *transient terms*, going to zero as $t \to \infty$.) Thus the output is (asymptotically) a sinusoid of the same frequency, scaled by $|G(i\omega)|$ and with a phase shift given by $\angle G(i\omega)$. The above justifies calling $G(i\omega)$ the **frequency response** of the open-loop system $G(s)$.

Definition 6.1. The **Nyquist plot** of the linear system $G(s)$ is a plot of $G(i\omega)$ for all frequencies $\omega \in (-\infty, \infty)$.

Note that we allow negative frequencies, even though their physical meaning is not too clear. The reason for this will become apparent in Section 6.4.

6.2 Nyquist Plots

Let us look at the Nyquist plot as the complex function from the *imaginary axis* to $\mathbb{C}$

$$\begin{aligned} \mathbb{C} \supseteq \{\Re(s) = 0\} &\rightarrow \mathbb{C} \\ i\omega &\mapsto G(i\omega). \end{aligned} \tag{6.1}$$

As in the previous Chapter, write $G(i\omega)$ in factorized form

$$G(i\omega) = \frac{(i\omega - z_1)\cdots(i\omega - z_m)}{(i\omega - p_1)\cdots(i\omega - p_n)},$$

where we assume a strictly proper G ($n > m$) and we took $b_0 = 1$, for simplicity (see Figure 6.1.)

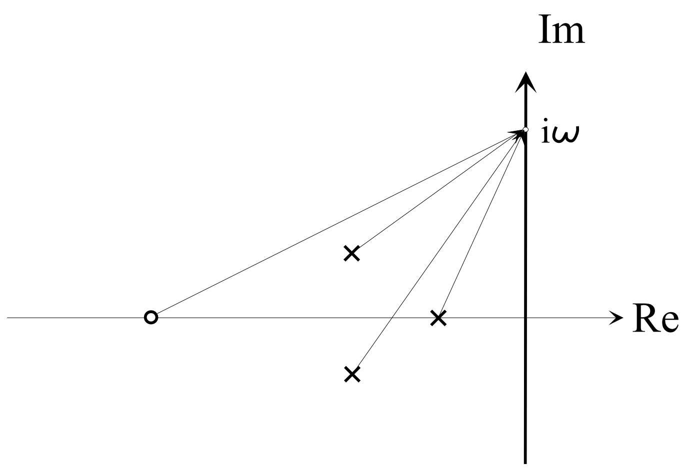


Figure 6.1: Setting $s = i\omega$ in $G(s)$—the Nyquist map

1. **Asymptotic behaviour:** As $\omega \to \infty$, $G(i\omega) \to 0$; more precisely, the approach to zero is asymptotic to the ray $\{\rho e^{-i(n-m)\pi/2};\ \rho > 0\}$. This is since

$$\lim_{\omega\to\infty} \frac{(i\omega - z_1)\cdots(i\omega - z_m)}{(i\omega - p_1)\cdots(i\omega - p_n)} = \lim_{\omega\to\infty} \frac{(i\omega)^m}{(i\omega)^n}$$
$$\simeq i^{m-n} \lim_{\omega\to\infty} \frac{1}{\omega^{(n-m)}} = e^{-i(n-m)\pi/2} \lim_{\omega\to\infty} \frac{1}{\omega^{(n-m)}}.$$

   Concretely, for $n - m = 1$, the Nyquist plot approaches the origin at an asymptotic angle of $-90°$; for $n - m = 2$ at an angle of $180°$ and so on.

2. **Zero frequency behaviour:** There are two cases:

   (a) If there are no OL poles at the origin, then the value at $\omega = 0$ is simply $G(0)$.

(b) If there are k OL poles at $s = 0$, so that

$$G(s) = \frac{\tilde{G}(s)}{s^k}, \text{ with } \tilde{G}(0) \neq 0,$$

then

$$\lim_{\omega \to 0} G(i\omega) = \tilde{G}(0) \lim_{\omega \to 0} \frac{1}{(i\omega)^k} = \tilde{G}(0) e^{-ik\pi/2} \lim_{\omega \to 0} \frac{1}{\omega^k}.$$

Thus $|G(i\omega)| \to \infty$ as $\omega \to 0$ and the Nyquist plot asymptotically, for small frequencies, approaches the ray

$$\{Re^{-ik\pi/2};\ R >> 0\}$$

Thus, for a single pole at zero, the Nyquist plot comes in from 'infinity' at an angle of $-90°$; for a double pole at zero, at an angle of $180°$ and so on.

3. **Monotonicity:** In a system with **no OL zeros**, the plot of $G(i\omega)$ will decrease monotonically as ω rises above the level of the largest imaginary part of the poles; this should be clear from Figure 6.1. This will also be true for large enough ω even in the presence of zeros, even though we do not show it.

4. **Crossings of the Real and Imaginary Axes:** These are found from the conditions:

$$\Re(G(i\omega)) = 0$$

for crossings of the **imaginary** axis and

$$\Im(G(i\omega)) = 0$$

for crossings of the **real** axis. Alternatively, when we insist on crossings of the **negative real axis**, the condition

$$\angle G(i\omega) = \pi$$

is used.

The above steps are sufficient for quick sketches of simple transfer functions. A more refined will be presented after we go through a number of examples.

6.3 Examples

Example 6.1. The first-order OL transfer function

$$G(s) = \frac{1}{s+2}$$

has the Nyquist plot shown in Figure 6.2. Note that $G(i0) = \frac{1}{2}$ and that the approach to zero is along the 'negative' imaginary axis (asymptotic angle $-90°$.) Also, it should be clear from Figure 6.1 that the magnitude of $G(i\omega)$ decreases monotonically to zero, as $\omega \to \infty$.

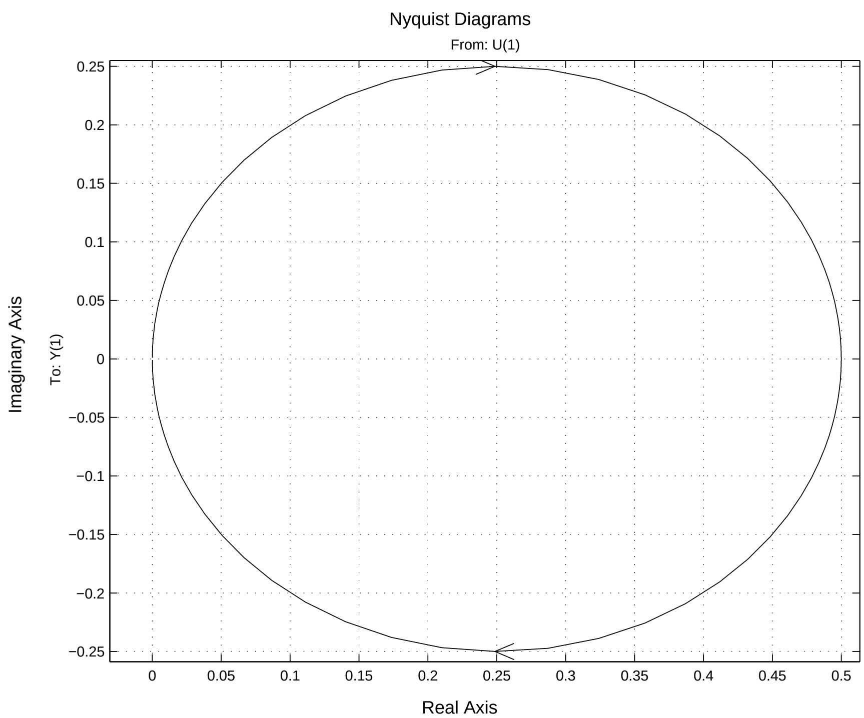


Figure 6.2: Nyquist plot of $\frac{1}{(s+2)}$

It is not hard to show that the root locus is a **circle** *centred at* $1/4$ *and of radius* $1/4$.

Exercise 6.1. Show this. *Hint:* show that, with $G(i\omega) = G_R(i\omega) + iG_I(i\omega)$,

$$(G_R - \frac{1}{4})^2 + G_I^2 = \frac{1}{4}^2.$$

Example 6.2. The second-order system

$$G(s) = \frac{1}{(s+1)^2 + 1}$$

has the Nyquist plot shown in Figure 6.3. The zero-frequency behaviour is:

$$G(i0) = \frac{1}{2}$$

and the asymptotic angle is 180°.

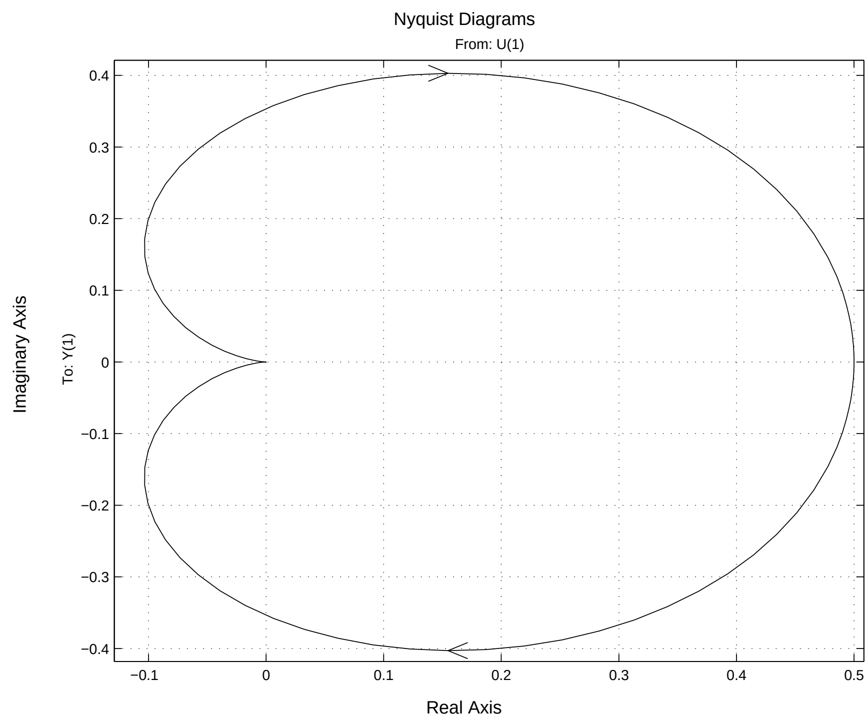


Figure 6.3: Nyquist plot of $\frac{1}{s^2+2s+2}$

Exercise 6.2. Find the points of crossing of the imaginary axis.

Is the magnitude $|G(i\omega)|$ a strictly decreasing function of omega or not?

In the two examples that follow, the asymptotic behaviour is identical, since in both cases $n - m = 1$. The $\omega = 0$ behaviour is also similar (finite positive real values for $G(0)$.) There are important differences, however.

Example 6.3. Consider the OL transfer function

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}.$$

Notice that the single OL *zero* at -1 is smaller, in magnitude, than the two OL poles at -2 and -3. As a result, the positive phase contribution of the term $(i\omega + 1)$ in

$$G(i\omega) = \frac{(i\omega+1)}{(i\omega+2)i\omega+3)}$$

is more important for small frequencies ω; in fact, the positive phase dominates the negative phase contributions from the two poles, and the Nyquist plot is as shown in Figure 6.4.

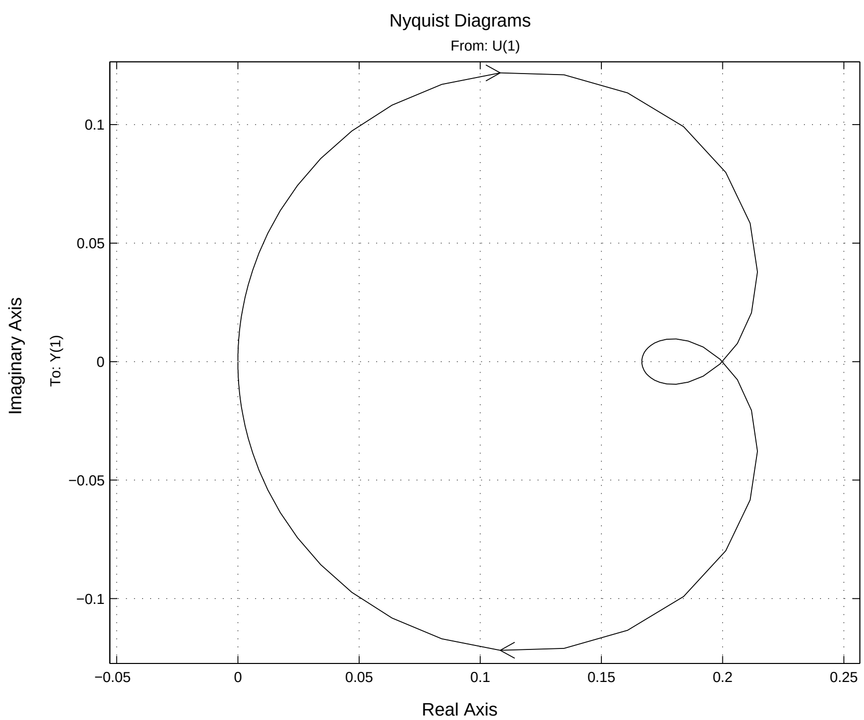


Figure 6.4: Nyquist plot of $\frac{(s+1)}{(s+2)(s+3)}$

Example 6.4. Now consider the OL transfer function

$$G(s) = \frac{(s+5)}{(s+1)(s+2)}.$$

Here the zero at -5 is far to the left of the two OL poles at -1 and -2. Hence *for small frequencies*, the positive phase contribution of the term $(i\omega+5)$ will be overhwelmed by the negative phase due to the two poles. For large enough ω, this term will, still, contribute the $+90°$ that will cancel the contribution of one of the poles to give the expected asymptotic angle of $-90°$.

The Nyquist plot is shown in Figure 6.5. Note that, if the zero term $(s+5)$ were absent, the Nyquist plot would look like Figure 6.3; thus, the term $(s+5)$ *'bends'* the Nyquist plot at high frequency so that the asymptotic angle is correct. In the previous example, this bending came much earlier.

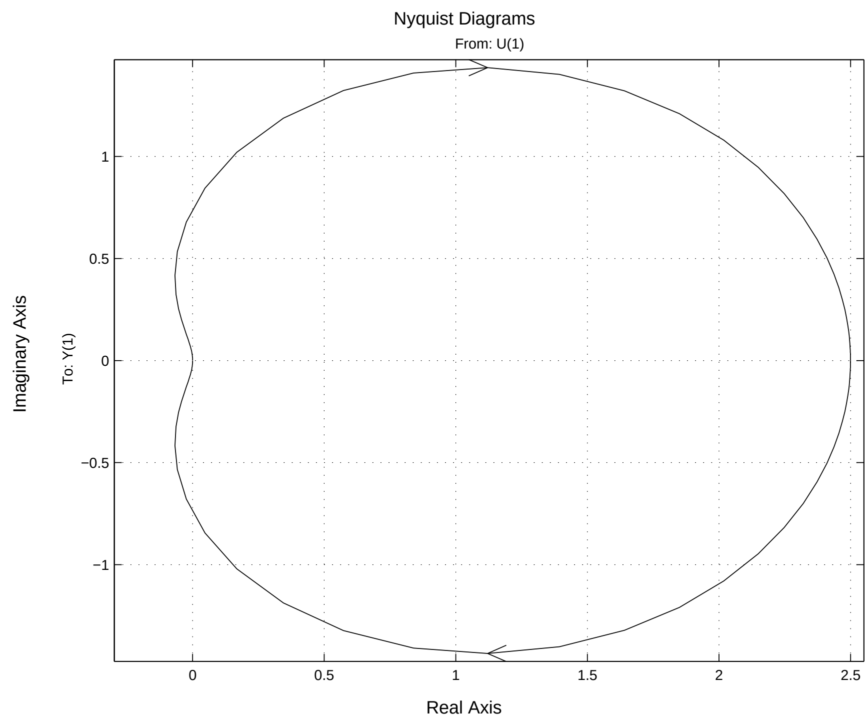


Figure 6.5: Nyquist plot of $\frac{(s+5)}{(s+1)(s+2)}$

Example 6.5. Finally, an example with a zero on the imaginary axis. Here

$$G(s) = \frac{s(s+1)}{s^3+5s^2+3s+4}$$

so that $G(i0) = 0$. As in the two previous examples, $n - m = 1$, so that the asymptotic ray is at $-90°$. It is checked that the real pole of G is to the left of the zero at -1 (there is also a *complex pair* at $-0.23 \pm i0.9$, whose overall phase contribution is very nearly **zero**, *for small frequencies*, by symmetry), so that the Nyquist plot resembles that of Figure 6.4 above (with the $G(0)$ point moved to the origin.)

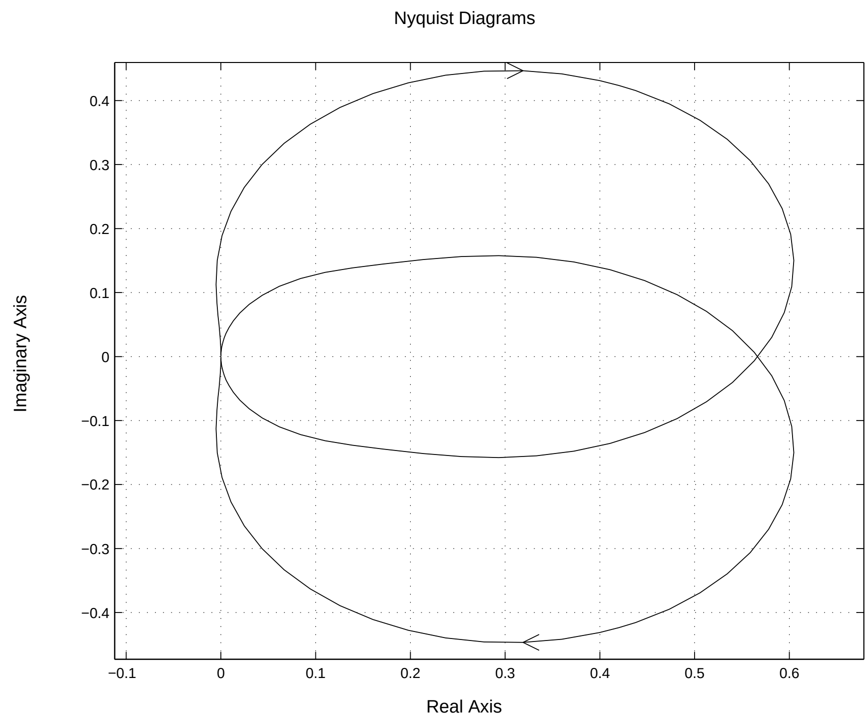


Figure 6.6: Nyquist plot of $\frac{s(s+1)}{s^3+5s^2+3s+4}$

The examples above taught us that we must in some cases look in more detail at the relative phase contributions of the zeros and poles and how their importance is a function of the frequencies considered.

We treat the case where there are OL *poles* on the imaginary axis in Section 6.5.

6.4 The Nyquist Stability Criterion

For the next two Sections, we make the simplifying assumption that $G(s)$ **has no poles on the imaginary axis**. Hence the Nyquist plots are assumed bounded. We relax this assumption in Section 6.5.

6.4.1 Nyquist map of the D-Contour

Even though we saw that, having assumed a strictly proper OL transfer function $G(s)$, the Nyquist plot $\{G(i\omega);\ -\infty < \omega < \infty\} \subset \mathbb{C}$ goes to zero as $\omega \to \pm\infty$, we would now like to insist that the Nyquist plot is a **closed curve** in the complex plane.

In anticipation of the stability result that follows, we make the following

Definition 6.2. The D-**contour**, γ_D, in the complex plane is the closed curve defined as the union of the part of the imaginary axis between $-iR$ and $+iR$, for a large R, and the semi-circle of large radius

$$\{Re^{-i\theta};\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$$

in the right-half plane. It is to be considered oriented in the *clockwise* sense.

Now the image of the D-contour under G *is a closed and bounded curve* and we claim that it is almost the same thing as the Nyquist plot that we have previously defined. This is because the image of the large-radius semi-circle above is part of a circle of *very small radius*, $\frac{1}{R^{n-m}}$. From now on, we shall consider these closed versions of the Nyquist plot and, to distinguish them, we shall refer to them as **Nyquist contours**. Evidently, the Nyquist contour of $G(s)$ is precisely

$$\Gamma_D := G(\gamma_D).$$

6.4.2 Encirclements and Winding Number

We shall be needing the notion of encirclements of a point by a closed curve in the plane. This is an intuitively obvious concept and we make no appeal to a precise mathematical definition.

Consider a closed, bounded curve γ in the plane, oriented counter-clockwise, and a point p such that γ does not pass through it. Then the **winding number** of p by γ is the net number of counter-clockwise encirclements of the point p by γ. Roughly, we place a vertical stick on the point p and we consider the curve γ as a rubber band, with orientation. We are allowed to stretch, untwist etc. the curve, so long as we do not go through the vertical pole at p. We then count the net number of times that γ goes around p. This number can be *negative*, if after untwisting the wrapping is in the *clockwise sense*. Of course, the winding number is an integer.

6.4.3 Statement of the Nyquist Stability Criterion

Consider an OL system $G(s)$ that is strictly proper and has exactly ℓ unstable poles and $n-\ell$ stable ones (recall that we are assuming, for the moment, that there are no OL poles on the imaginary axis.)

Theorem 6.1. *In the constant-gain feedback configuration, with OL transfer function $G(s)$, the closed-loop system is stable for the gain value k if and only if the net number of counter-clockwise encirclements of the point $-1/k$ by the Nyquist contour is exactly equal to the number ℓ of unstable OL poles.*

In particular, for **OL stable** systems, we get

Corollary 6.1. *If the OL system is stable, then the closed-loop system with gain k is stable if and only if the Nyquist contour does not encircle the point $-1/k$.*

Let us look at a couple of examples. It will be very useful to combine the viewpoints afforded by all the methods we have presented so far; thus, the root locus (and, in the background, the Routh-Hurwitz criterion) will provide *complementary* views of the CL stability problem. This is an approach that can be profitably pursued in many instances.

Example 6.6. The OL system

$$G(s) = \frac{(s+2)(s+3)}{(s+1)[(s-1)^2+1]}$$

is OL unstable, with poles $1 \pm i$ and -1. The **root locus plot** for this system (Figure 6.7) shows that large gain will stabilize the CL system.

Indeed, since $n-m=3-2=1$, only one branch goes to infinity, at an asymptotic angle of π; the other two branches tend towards the two OL zeros, which are in the stable half of $\mathbb{C}$. The exact value of k at crossover, say k_c, can be found from the Routh-Hurwitz table.

Now let us look at the **Nyquist contour** of this G. Because of the two unstable poles, it turns out that the net angle $\angle G(i\omega)$, for small $\omega>0$, is **positive**. Since the asymptotic ray is at $-\pi/2$, the Nyquist contour will bend around, as shown in Figure 6.8, crossing both the positive imaginary and the negative real axis (the exact points can be computed by the method given.) Now, as k increases from zero, the point $-1/k$ will come in from negative infinity and tend towards zero, for large values of k. For $k>k_c$,

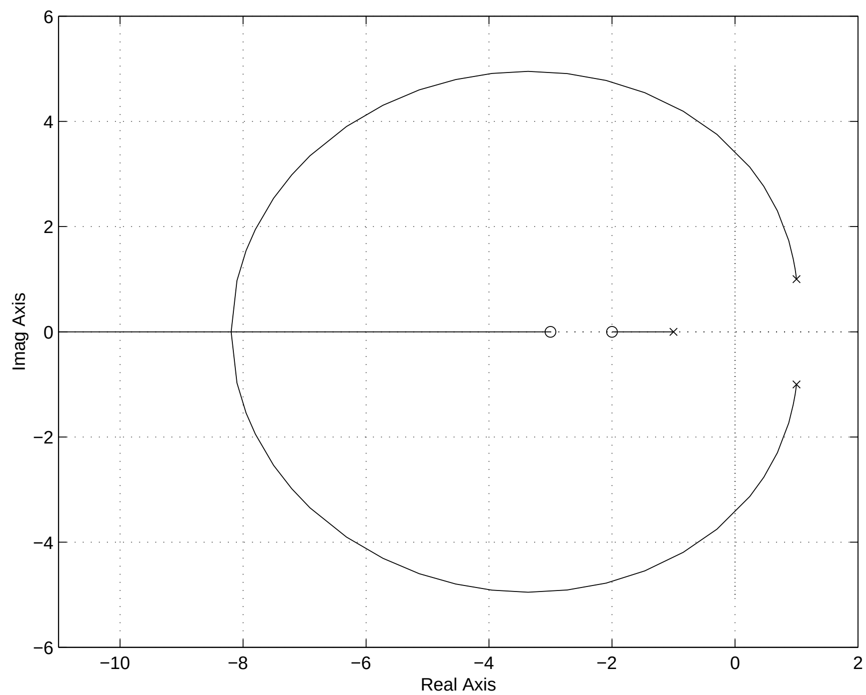


Figure 6.7: Root locus plot of $G(s) = \frac{(s+2)(s+3)}{(s+1)[(s-1)^2+1]}$

the point $-1/k$ is inside the smaller of the two loops of the Nyquist contour. Counting the number of encirclements, we get **two net, counter-clockwise encirclements**. By the Nyquist Stability Criterion, the CL system is stable for $k > k_c$, since G had two unstable poles. For $0 < k < k_c$ we get zero encirclements, hence the CL system is *unstable*. This agrees with our conclusion from the root locus plot.

6.4.4 Proof of the Nyquist Stability Criterion

Theorem 6.2 (The Argument Principle). *Let $f(s)$ be a rational function. Let γ be a simple, closed curve in the complex plane not passing through any of the poles and zeros of $f(s)$ and oriented in the clockwise sense.*

Let Z_γ and P_γ be the numbers of zeros and poles of $f(s)$ inside γ. Let $\Gamma = f(\gamma)$ be the image of the curve γ under the rational map $f(s)$.

Then, the net number of clockwise encirclements of the origin by Γ is equal to $Z_\gamma - P_\gamma$.

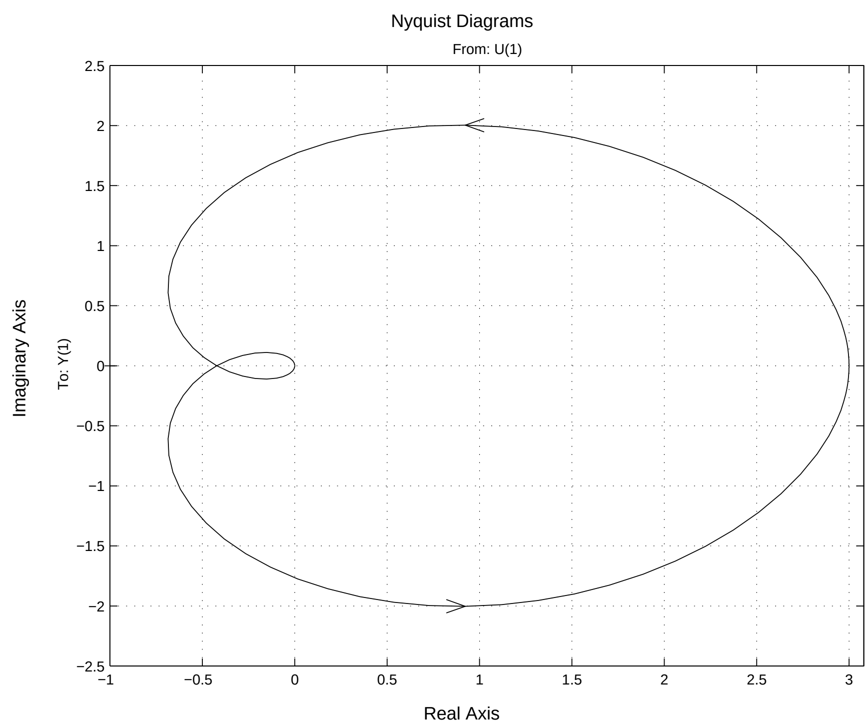


Figure 6.8: Nyquist plot of $G(s) = \frac{(s+2)(s+3)}{(s+1)[(s-1)^2+1]}$

(A **simple curve** is one without self-intersections; its interior is then well defined.) The proof of this result assumes a knowledge of complex analysis; it is given in the Appendix.

Application to the Nyquist contour: Let

$$\boxed{f(s) = 1 + kG(s)} \tag{6.2}$$

Note that

$$f(\gamma_D) = 1 + kG(\gamma_D) \quad \text{and so, for } k > 0, \qquad \frac{f(\gamma_D)}{k} = \frac{1}{k} + G(\gamma_D).$$

Hence, except for a scaling factor, the image of the D-contour by f is simply the Nyquist contour of G shifted to the left by $1/k$. The conclusion is that **the encirclement of the origin by $f(s)$ is the same as the encirclement of the point $-1/k$ by the Nyquist contour.**

Now we only need to observe that *the zeros of $f(s)$ coincide with the closed-loop poles for the gain k and the poles of $f(s)$ coincide with the OL poles of $G(s)$.* The D-contour was chosen so as to include almost the whole of the right-half complex plane, and therefore will contain, for large enough R, any unstable CL poles.

Sine we want a **stable** CL system for the gain k, the former number is zero and hence the Argument Principle gives that the net number of *clockwise* encirclements of $-1/k$ by the Nyquist contour of G must be equal to minus the number of unstable OL poles. The minus sign implies that the encirclements must be in the reverse direction, in other words **counter-clockwise**, and the Nyquist Stability Criterion is proved.

6.5 OL Poles on the imaginary axis

We have left out the case where $G(s)$ has pure imaginary poles so as not to obscure the main ideas of the Nyquist stability criterion. Let us now present the modifications necessary when there are pure imaginary OL poles.

We have already seen that if the system $G(s)$ has k poles at the origin, the Nyquist plot comes in from an asymptotic angle of $-k\pi/2$. Clearly, this is unsatisfactory from the application of the stability criterion since we want a bounded Nyquist contour. On the other hand, if the D-contour is to contain as much of the right-half complex plane, we cannot throw away too much.

The compromise adopted is to draw semi-circles of small radius ρ, centred at each of the imaginary poles $i\omega_p$:

$$\{i\omega_p + \rho e^{i\theta}\ ;\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$$

The image of these semi-circles will be circular arcs of large, but finite, radius. Since these arcs are important for the application of the Nyquist criterion, care must be taken to sketch them correctly in each case.

Example 6.7. As an example, consider the transfer function

$$G(s) = \frac{(s+1)}{(s^2+1)}$$

which has the OL poles $\pm i$. Letting

$$s = i + \rho e^{i\theta}$$

gives

$$G(i+\rho e^{-i\theta}) \simeq \frac{1+i}{2i\rho e^{i\theta}} = \frac{1}{\sqrt{2}\rho} e^{i(\pi/4-\pi/2-\theta)} = \frac{1}{\sqrt{2}\rho} e^{i(-\pi/4-\theta)}$$

and, as θ goes from $-\pi/2$ to $\pi/2$, we get a circular arc of a circle of radius $\frac{1}{\sqrt{2}\rho}$, centred at the origin, and angle ranging from $\pi/4$ to $-3\pi/4$, clockwise (see Figure 6.9 for a sketch.)

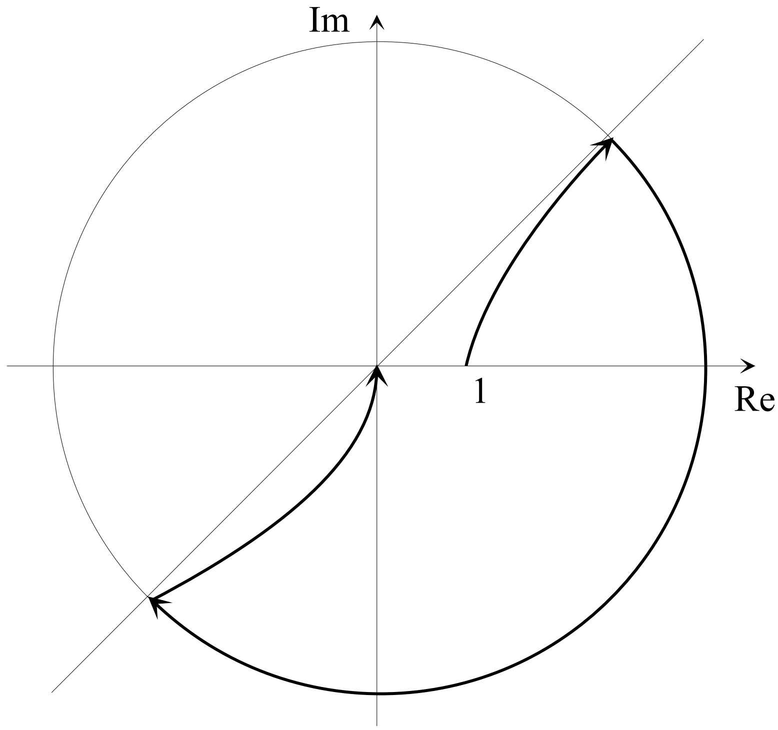


Figure 6.9: Approximate Nyquist contour of $G(s) = \frac{(s+1)}{(s^2+1)}$

The system is thus seen to be stable for all gains k, a fact that could have been deduced from the Routh-Hurwitz table as well.) Note that `matlab` has trouble handling these singular cases, so sketching the Nyquist contour by hand is the only option here!

Exercise 6.3. Discuss the alternative approach of approximating the system above by one with a stable pair of poles close to the imaginary axis, effectively changing the denominator to $s^2 + \epsilon s + 1$. Use `matlab` to picture the limiting contour, and show that it agrees with the one sketched by hand above.

More Examples are given in the Exercise section.

6.6 Gain and Phase Margins

Let us suppose that we have managed to stabilize a system using the feedback gain value k_0. We may be interested in knowing how close the resulting closed-loop system is to instability. We shall discuss in this section two **stability margins**, roughly, measures of proximity to instability (or, from a complementary perspective, of how robustly stable the closed-loop system is.)

The **gain margin** at k_0 gives a measure of how much we can increase the feedback gain without losing stability. The **phase margin** is a measure of how much more additional phase the system can tolerate before losing stability (the extra phase can come, for example, from unmodelled dynamics.)

The assumption for the precise definition of the *gain margin* is that the Nyquist plot crosses the negative real axis at a point to the right of the point $-1/k_0$ (we know the system is stable for the value k_0.) We allow the case where this crossing is in fact the origin.

Definition 6.3. Let the closed-loop system be stable for $k = k_0$ and suppose the nearest crossing of the real axis by the Nyquist plot is at $-1/k_1$, for $k_1 > k_0$ (this may be equal to infinity.)

The **gain margin** at the nominal gain value k_0 is given by

$$\boxed{\mathrm{GM} = 20 \log_{10} \frac{k_1}{k_0}}$$

(The value ∞ is allowed.) The above definiton is traditional and gives the resulting measure in **decibels** (dB); occasionally, by *'gain margin'* one understands simply the ratio $\frac{k_1}{k_0}$ (a number greater than one.) Often, the nominal value is $k_0 = 1$ and the gain margin is then just k_1.

For the definition of the *phase margin*, we suppose that the circle, centred at zero, and of radius $1/k_0$, intersects the Nyquist plot at a point of the lower half-plane. Let $\omega_{PM}(\omega_{PM} > 0)$ be the corresponding frequency.

Definition 6.4. Let the closed-loop system be stable for the gain value $k = k_0$. The **phase margin** at k_0 is defined by

$$\boxed{\phi_{PM} = \pi - \angle G(i\omega_{PM})}$$

where ω_{PM} is defined above.

Example 6.8. The constant gain closed-loop system with open-loop transfer function

$$G(s) = \frac{1}{s(s+1)(s+10)}$$

is stable for the gain value $k = 10$. This can (and should) be checked using the Routh-Hurwitz table, for example.

The Nyquist plot of $G(s)$ has a crossing of the negative real axis, found by setting the imaginary part of $G(i\omega)$ to zero:

$$\Im(G(i\omega)) = \frac{1}{i\omega(i\omega+1)(i\omega+10)} == \frac{-i[(10-\omega^2) - i11\omega]}{\omega[(10-\omega^2)^2 + 11^2\omega^2]}$$

This gives

$$10 - \omega^2 = 0, \text{ or } \quad \omega = \pm\sqrt{10}.$$

Now

$$G(i\sqrt{10}) = -\frac{1}{110},$$

so that the **gain margin** is

$$\mathrm{GM} = 20\log_{10}\frac{110}{10} \simeq 20.83dB.$$

In order to find the phase margin, we set

$$|G(i\omega)| = \frac{1}{10}.$$

This gives

$$\omega^2((10-\omega^2)^2 + 121\omega^2) = 0.$$

Setting $y = \omega^2$, the resulting cubic has the unique positive solution (using `matlab`)

$$y \simeq 0.6153,$$

so that

$$\omega_{PM} \simeq 0.7844$$

and the phase margin is

$$\phi_{PM} \simeq 47.4°.$$

6.7 Bode Plots

In technical reviews of the performance of a stereo component, such as an amplifier, one will find *frequency response plots*, giving the amplification amplitude in decibels as a function of frequency on a logarithmic scale. These are part of the *Bode plot* of the relevant transfer function. They are useful in control theory as well, as they give yet another way of understanding a linear system and they can also be used to extract information about *closed-loop stability*, just as the Nyquist plot did.

Definition 6.5. The **Bode plot** of the transfer function $G(s)$ consists of separate plots of the magnitude in decibels, $20\log_{10}|G(i\omega)|$, and phase $\angle G(i\omega)$ against a logarithmic scale of positive frequencies, $\omega > 0$.

The use of decibels and a logarithmic frequency scale is traditional, but for a good reason: it makes the task of sketching the bode plots very easy, as we now describe. Let us look at the simplest transfer function first.

Example 6.9. The magnitude of the first-order transfer function

$$G(s) = \frac{1}{s+a}, \quad a > 0,$$

decreases monotonically to zero, while the phase goes from zero to $\pi/2$ as $\omega \to \infty$ (see Figure 6.10.)

If we use a *logarithmic* frequency scale, the Bode plot can be sketched using the following two asymptotes: for **small** frequencies ω, $G(i\omega) \simeq \frac{1}{a}$, so the constant line $20\log_{10}(\frac{1}{a})$ is the small-frequency asymptote.

For **large** frequencies, $G(i\omega) \simeq \frac{1}{i\omega}$. Now the db/log plot of $|G(i\omega)| = \frac{1}{\omega}$ is a **line** *of negative slope* -20 db per *'decade'!* (A decade is an increase in frequency by a factor of ten.) This is easy to see, since

$$20\log_{10}(\omega^{-1}) = -20\log_{10}\omega$$

and the *independent variable* is in fact $x = \log_{10}\omega$. So the line $-20x$ has slope -20, which means that if x increases to $x+1$, we lose 20 db. But

$$x + 1 = \log_{10}\omega + 1 = \log_{10}\omega + \log_{10}10 = \log_{10}(10\omega).$$

In addition, the line passes through $20\log_{10}(a^{-1})$ when $\omega = a$ (see Figure 6.11.)

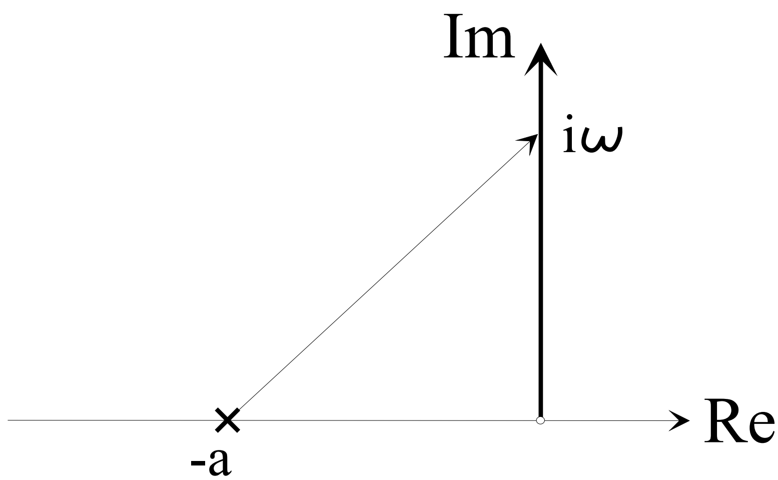


Figure 6.10: Forming $i\omega + a = \frac{1}{G(i\omega)}$

At this frequency $\omega = a$, the **actual** db magnitude is

$$20\log_{10}|G(ia)| = 20\log_{10}\frac{1}{a(i+1)} =$$
$$= 20\log_{10}(a^{-1}) + 20\log_{10}(\frac{1}{\sqrt{2}}) \simeq [20\log_{10}(a^{-1}) - 3](db). \quad (6.3)$$

This approximate 3 decibel drop at the so-called **break frequency** $\omega = a$ is the final ingredient we use to sketch the Bode plot of $G(s)$.

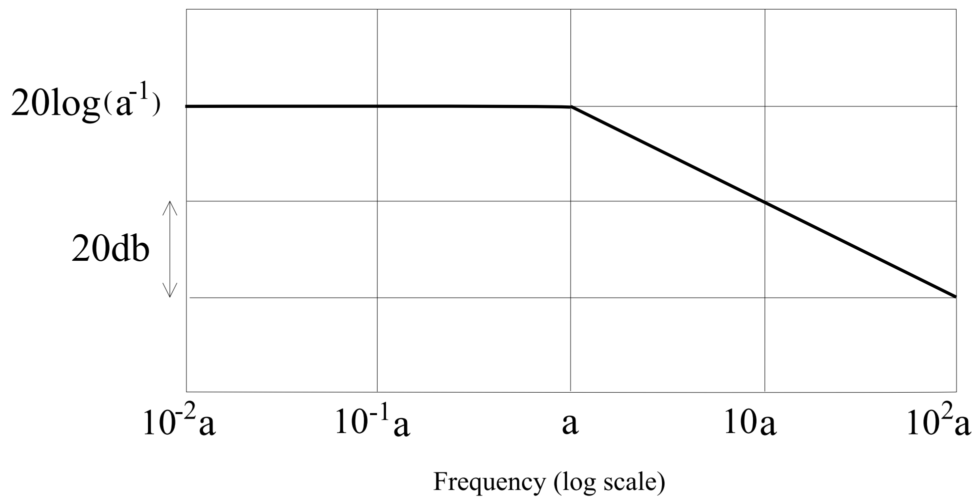


Figure 6.11: Asymptotes for the Bode plot of $G(s) = \frac{1}{s+a}$

6.8 Appendix: The Argument Principle

Let $f(s)$ be a rational function. Consider

$$w = \log f(s).$$

where the (complex) logarithm function is defined by

$$\log s = \log |s| + i\angle(s).$$

Because of the ambiguity in the argument, this function is well defined only in the complex plane with a *cut*, say taking out the negative real axis. The derivative of the logarithm is

$$\frac{d}{ds} \log s = \frac{1}{s},$$

as usual, so that

$$\frac{d}{ds} w = \frac{d}{ds} \log f(s) = \frac{f'(s)}{f(s)}.$$

Since f is a rational function, it is not hard to see that

$$\frac{d}{ds} \log f(s) = \sum_i \frac{k_i}{(s - z_i)} - \sum_j \frac{\ell_j}{(s - p_j)}, \tag{6.4}$$

where $\{z_i\}$ and $\{p_j\}$ are the distinct **zeros** and **poles** of $f(s)$ and $\{k_i\}$ and $\{\ell_j\}$ are their corresponding multiplicities.

Now recall the definition of the complex contour integral

$$\oint_\gamma g(s)ds = \int_0^1 g(\gamma(t))\gamma'(t)dt,$$

where the path γ is defined on the interval $[0, 1]$.

Exercise 6.4. If γ is a unit circle in the complex plane, oriented counter-clockwise,

$$\gamma(t) = e^{i2\pi t},$$

use the above definition of the contour integral to show that

$$\oint_\gamma \frac{ds}{s} = 2\pi i.$$

Generalize this to the case where γ is a circle centred at the complex point p, so that we have

$$\oint_\gamma \frac{ds}{(s-p)} = 2\pi i.$$

The above result can also be shown to hold for **any** path γ encircling the point p *once*, in the counter-clockwise sense.

Theorem 6.3 (The Argument Principle-2). *Let γ be a simple, closed path in the complex plane, oriented counter-clockwise and not passing through any of the zeros and poles of the rational function $f(s)$.*

Let Z_γ and P_γ be the numbers of zeros and poles of $f(s)$ inside γ. Then,

$$\oint_\gamma \frac{f'(s)}{f(s)}ds = 2\pi i(Z_\gamma - P_\gamma).$$

Proof of Theorem 6.3. The proof follows from equation 6.4 and Exercise 6.4. Note that Z_γ and P_γ are **total counts** of poles and zeros inside γ, so that a pole of multiplicity ℓ contributes the positive integer ℓ to P_γ. □

Proof of Theorem 6.2. In light of our discussion of the complex logarithm, it is helpful to rewrite the left-hand side of the equation in the Theorem as

$$\oint_\gamma \frac{f'(s)}{f(s)}ds = \oint_\Gamma dw,$$

which in turn equals

$$\oint_\Gamma dw = w(\gamma(1)) - w(\gamma(0)) = \log f(\gamma(1)) - \log f(\gamma(0)) =$$

$$= \log|f(\gamma(1))| - \log|f(\gamma(0))| + i\left(\angle f(\gamma(1)) - \angle f(\gamma(0))\right).$$

Since the path γ is closed, so is Γ, and we are left with i times the difference in argument between the initial and final point of Γ. But this difference is equal to $(2\pi)\times$ (the net number of encirclements of the origin in the complex plane).

By Theorem 6.3, we thus have

$$\angle f(\gamma(1)) - \angle f(\gamma(0)) = 2\pi(Z_\gamma - P_\gamma)$$

and the Argument Principle 6.2 is proved. (Note that the D-contour γ_D is oriented **clockwise**. The effect of this is to flip the signs of all terms, but the above equality in unaffected.) □

Remark 6.1. In summary, if we map a simple, closed curve γ using a **rational** map $f(s)$, then the *image curve* $\Gamma = f(\gamma)$ **is not necessarily simple any more**: it picks up a *'twist'* (encirclement of the origin) in one direction for every zero enclosed by γ and a twist in the opposite direction for each enclosed pole.

6.9 Summary

- The **Nyquist plot** of a transfer function $G(s)$ is related to the frequency response of the system (response to sinusoids); it is the locus of the image points $G(i\omega)$, for $-\infty < \omega < \infty$.

- The Nyquist plot of a strictly proper $G(s)$ is easy to sketch: as $\omega \to \infty$, the approach to zero is along a ray at an angle of $-(n-m)\pi/2$ (where $G(s)$ has n poles and m zeros); if there are no poles on the imaginary axis, the value at $\omega = 0$ is simply $G(0)$.

  Crossings of the real and imaginary axis are found by setting the imaginary and real part of $G(i\omega)$ to zero.

- The D-**contour** γ_D is a half-disc of large radius centred at 0 and containing a large part of the imaginary axis. The **Nyquist contour** Γ_D is the image of the D-contour under G. If there are no poles on the imaginary axis, it essentially coincides with the Nyquist plot.

- The **Nyquist stability criterion** says that a closed-loop system with gain k is stable if the net number of counter-clockwise encirclements of the point $-1/k$ by the Nyquist contour is exactly equal to the number of unstable open-loop poles. If G is stable, then we require no encirclements of $-1/k$. This can be used to find the range of gains k for CL stability.

- If there are poles on the imaginary axis, a bounded Nyquist contour is obtained if we by-pass these poles by semi-circles of small radius. Care must be exercised to map these properly.

- The **gain margin** is equal to $20 \log_{10} \frac{k_1}{k_0}$, where k_0 is the nominal gain of a stable CL system and k_1 is the largest gain the system will tolerate before it loses stability.

- The **phase margin** ϕ_{PM} is the largest additional phase that can be present in the Nyquist plot (due to unmodelled dynamics, for example), before stability is lost.

6.10 Exercises

1. Sketch the Nyquist plots of the following transfer functions:

   (a) $G(s) = \frac{1}{s^2(1+T_1 s)(1+T_2 s)},\ T_1, T_2 > 0$

   (b) $G(s) = \frac{1}{s^4(s+k)},\ k > 0$

   (c) $G(s) = \frac{1}{s^2+4}$
   Hint: Draw small semi-circles to exclude the imaginary poles.

2. Sketch the Nyquist plots of the open-loop transfer functions

   (a) $G(s) = \frac{1}{(s+1)(s+2)(s+3)}$

   (b) $G(s) = \frac{s+1}{s^2(s+2)}$

   (c) $G(s) = \frac{(s+1)(s+2)}{s^3}$

   Use the Nyquist stability criterion to determine the range of values of the feedback gain k in the corresponding closed-loop systems for which each of the CL systems is stable.

3. A unity feedback system has the second-order OL transfer function

   $$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)},\ 0 < \zeta < 1,\ \omega_n > 0.$$

   Sketch the Nyquist plot.

   Show that the gain margin is always infinite and determine the phase margin ϕ_{PM} as a function of ζ. Find ϕ_{PM} when $\zeta = 0.1, 0.2, 0.4$ and 0.6 and show that, approximately, $\zeta \simeq 0.01\phi_{PM}$, where ϕ_{PM} is in degrees.

4. A unity feedback system has the open-loop transfer function

   $$kG(s) = \frac{k}{s(0.2s+1)(0.05s+1)}.$$

   Use the Nyquist plot of $G(s)$ to find the gain k for which

   (a) the gain margin is equal to 10

   (b) the phase margin is equal to 40°

5. A constant-gain feedback system has the OL transfer function

$$G(s) = \frac{1}{s(2s+1)(4s+1)}.$$

Show, using the Nyquist criterion, that the CL system is stable for $k < 3/4$.

When $k = 1/2$, determine the *approximate* values of the phase and gain margins from the Nyquist plot.

Chapter 7

The View Ahead: State Space Methods

7.1 Introduction

Chapter 8

Review Problems and Suggestions for Mini-Projects

8.1 Review Problems

The following collection typically contains problems that require the use of methods from more than one of the chapters of the course.

1. A system with input u and output x is governed by the equation

   $$\frac{d^3x}{dt^3} + (\alpha + 2)\frac{d^2x}{dt^2} + (4\alpha + 6)\frac{dx}{dt} + 4(\alpha + 3)x = u$$

   for $t > 0$, α being a positive parameter. Find the transfer function $G(s)$, defined as $X(s)/U(s)$.

   Use the Routh criterion to show that when $\alpha > 2$ all the poles of $G(s)$ have real parts less than -1.

   For $\alpha = 2$ find all the poles of $G(s)$ and determine the corresponding impulse response function.

2. The block diagram of a unity feedback system is shown in the figure below. Find the transform $Y(s)$ of the controlled output $y(t)$ in terms of the transforms $R(s)$ and $D(s)$ respectively of the reference input $r(t)$ and the disturbance $d(t)$.

   The input $r(t)$ is a step function with $r(t) = 10$ for $t > 0$ and zero otherwise. There is also a step function disturbance $d(t) = -2$ for

$t > 0$ and zero otherwise. Let y_{so} be the steady state value of $y(t)$ in the absence of the disturbance. Let y_{sd} be the steady state value of $y(t)$ with the disturbance present. Show that

$$\frac{y_{sd} - y_{so}}{y_{so}} = \frac{24}{25}$$

Find $y(t)$ for $t > 0$ when the disturbance is present, and determine the time after which $y(t)$ remains within 5% of its steady state value y_{ss}. (You may assume that the inverse Laplace transforms of $\frac{s+a}{(s+a)^2+b^2}$ and $\frac{b}{(s+a)^2+b^2}$ are $e^{-at}\cos bt$ and $e^{-at}\sin bt$ respectively.)

3. The block diagram of a unity feedback system is shown in the figure below with $y(t)$ the controlled output, $r(t)$ the reference input and $e(t)$ the error signal.

   Find the Laplace transform $E(s)$ of $e(t)$ in terms of $G(s)$ and the Laplace transform $R(s)$ of $r(t)$.

   Show that the steady state error for a unit step input is given by $e_{ss} = 1/(1+\kappa_0)$, where $\kappa_0 = \lim_{s\to 0} G(s)$.

   For the system with open-loop transfer function

$$G(s) = \frac{K(s+7)}{(s+3)(s^2+2s+2)}$$

   draw the root-locus diagram for the closed-loop system as the non-negative K is increased from zero.

   Show that, for a stable system, e_{ss} is greater than 6/125.

4. A unity feedback control system has the forward path transfer function

$$G(s) = \frac{K(s+4)}{s(s+2)(s+5)(s+7)}\ .$$

   with the gain K non-negative.

   (a) Show that the closed-loop system is stable provided $K < 186$.
   (b) Sketch the root locus diagram assuming that the breakaway point is halfway between the two relevant open-loop real poles.

(c) Determine the value of K which results in an under-damped system with dominant poles in positions corresponding to a mode having damping ratio $\zeta = 1/\sqrt{2}$. Determine the the approximate positions of the other closed-loop poles and establish whether the *dominant pole* assumption is valid.

5. Determine the frequency response function for a system with transfer function
$$G(s) = \frac{1}{s(s+1)(s+4)}$$
and the steady state response of the system to the input $5\cos 3t$.

For the unity feedback system with open-loop transfer function
$$G(s) = \frac{K}{s(s+1)(s+4)} \;\; , \; K > 0$$
sketch the Nyquist plot for the closed-loop system.

Use the Nyquist stability criterion to determine the range of values of K for which the closed-loop system is stable.

Show that, if ϕ_p is the phase margin and ω_p is the corresponding phase margin frequency, then
$$\tan\phi_p = \frac{4-\omega_p^2}{5\omega_p}$$
with $\omega_p \le 2$. (You may assume that $\tan^{-1} u + \tan^{-1} v = \tan^{-1}\left(\frac{u+v}{1-uv}\right)$.)

Hence find the value of K for which $\tan\phi_p = 3/5$.

6. (a) Given the linear time-invariant system
$$\dot{x} = Ax + Bu \;\; , \; y = Cx$$
where $x \in R^n$, $u \in R^m$, $y \in R^l$ and A, B, C are constant matrices of dimension $n \times n$, $n \times m$, and $l \times n$ respectively, define the transition matrix e^{At} and the system transfer matrix.

Determine these matrices for the system with
$$A = \begin{bmatrix} -a & \omega \\ -w & a \end{bmatrix} \;\; , \quad B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \;\; , \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) A model of water pollution in a river is given by the differential equations

$$\begin{aligned}\dot{x}_1 &= -1.5x_1 + 0.25x_2 \\ \dot{x}_2 &= -2x_2 + u\end{aligned}$$

where x_1 is the dissolved oxygen deficity (DOD), x_2 is the biochemical oxygen demand (BOD) and u is the BOD content of the effluent discharged into the river from the effluent treatment plant. Time is measured in days.

The water engineer requires the system to have the dynamics of a second-order plant with damping ratio $\zeta = 0.5$ and to have a settling time T_s to within 5% of equilibrium of 1/2 day. Note that $T_s \approx 3/\zeta\omega_n$, where ω_n is the undamped natural frequency. Calculate the parameters k_1 and k_2 required in the feedback control law

$$u = k_1 x_1 + k_2 x_2$$

to achieve these design requirements.

7. From the definition of the Laplace Transform calculate the transform of $f(t) = t$.

Prove that

$$Ł\left[\frac{df}{dt}\right] = sF(s) - f(0^+) \ .$$

The unit step response of a second-order system is $x(t)$ with

$$X(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \ .$$

Determine $Ł[\dot{x}(t)]$. Hence, show that the peak overshoot occurs at

$$t = \pi/(\beta\omega_n) \ ,$$

where $\beta = \sqrt{1-\zeta^2}$.

Using Laplace transforms, solve the differential equation

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + x(t) = u(t)$$

for the case $x(0) = 2,\ \dot{x}(0) = 0$ and $u(t) = 1$ for $t > 0$.

For $u(t) = \cos(\omega t)$, ω a positive constant, determine the steady state solution $x(t)$ as $t \to \infty$.

8. A unity feedback system has output $x(t)$ and reference input $r(t)$. Find the transforms $X(s)$ and $E(s)$ of $x(t)$ and the error signal $e(t) = r(t) - x(t)$ respectively, in terms of the transform $R(s)$ of $r(t)$ and the forward path transfer function $G(s)$.

   It is desired to design a unity feedback system to meet the following two specifications:

   (a) the poles of the closed loop transfer function lie to the left of $s = -1$

   (b) the steady state error e_{ss} to a unit step input is less than 1/10.

   Show that the closed-loop system with

$$G(s) = \frac{1}{(s + \frac{1}{2})^2}$$

   meets neither of these specifications.

   A pole and a zero are now introduced in the forward path so that the open-loop transfer function becomes

$$G(s) = \frac{K(s + \frac{5}{4})}{(s+3)(s + \frac{1}{2})^2} .$$

   By using the Routh criterion show that the corresponding closed-loop transfer function satisfies specification (a) provided $K > 3$.

   Find the range of K for which both specifications (a) and (b) are satisfied.

9. A unity feedback system with open-loop transfer function $G(s)$ has output $x(t)$, reference input $r(t)$ and $e(t) = r(t) - x(t)$ is the error signal.

   You may assume the final value theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0}\ sE(s) ,$$

if the limit exists.

Show that the steady state error for a unit step input is given by

$$e_{ss} = \frac{1}{1+\kappa} \ ,$$

where

$$\kappa = \lim_{s \to 0} G(s)$$

For the system with open-loop transfer function

$$G(s) = \frac{K}{(s+1)(s+3)(s+6)}$$

sketch the root-locus diagram for the closed-loop system as K is increased from zero.

Show that for a stable system $e_{ss} > 1/15$.

From the root-locus diagram determine *approximately* the value of K giving three closed-loop real poles, two of them repeated (equal), and find the corresponding value of e_{ss}.

10. A unity feedback control system has the forward path transfer function

$$G(s) = \frac{K}{s(s+2)(s^2+4s+5)}$$

For this closed-loop system the gain K is allowed to vary through positive values.

Show that the closed-loop system is stable provided $K < 170/9$.

Verify that the condition for the breakaway point is satisfied by $s = -0.6018$.

Sketch the root-locus diagram and determine K^*, the value of K, which yields an under-damped system with dominant poles in positions corresponding to the damping ratio $\zeta = \sqrt{2}/2$ of a second-order system.

For $K = K^*$ comment whether the dominant pole approximation is valid.

11. Given that, for a constant gain K, a closed-loop system is assessed to be stable by using the Nyquist stability criterion, define the phase and gain margins.

    A unity feedback system has the forward path transfer function

    $$G(s) = \frac{K}{s(s+1)^2} \ .$$

    Sketch the frequency response function and use the Nyquist stability criterion to determine the range of positive values of K for which the system is stable.

    Determine the gain margin and the gain margin frequency when $K = 1$.

    Determine the value of $K > 0$ which will give a stable closed-loop system with a phase margin of 45 degrees, and determine the phase margin frequency.

    An additional delay element with transfer function e^{-sT}, where T is a positive constant, is included in the forward path. Sketch the frequency response function.

    Show how the system stability depends upon the roots of the equation

    $$(1-\omega^2)\cos(\omega T) - 2\omega \sin(\omega T) = 0 \ .$$

12. A Galitzin seismograph has dynamic behaviour satisfying

    $$\begin{aligned} \ddot{x} + 2\eta\dot{x} + \eta^2 x &= \ddot{f}(t) \\ \ddot{y} + 2\eta\dot{y} + \eta^2 y &= \dot{x} \end{aligned}$$

    where y is the mirror displacement, x is the displacement of the pendulum and f is the ground displacement. The parameter η is a positive constant.

    Determine the response $y(t)$ to a unit ground velocity shift $\dot{f}(t) = 1$ for $t > 0$, assuming

    $$f(0) = \dot{f}(0) = x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0 \ .$$

13. From the definition of the Laplace transform, calculate the Laplace Transform of $f(t) = \sin 2\omega t$ for ω a positive constant.

    Using Laplace transforms, solve the differential equation

    $$\frac{d^3x(t)}{dt^3} + 3\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + x(t) = r(t)$$

    for the case $x(0) = 0$, $\dot{x}(0) = 0$ and $r(t) = 1$ for $t > 0$.

    Determine the error as $t \to \infty$ for the unit ramp input $r(t) = t$.

    For the input $r(t) = \cos \omega t$, ω a positive constant, determine the solution $x(t)$ as $t \to \infty$ for the stable system with transfer function

    $$\frac{X(s)}{R(s)} = G(s) \ .$$

    Hence, or otherwise, for the system (13) with input $r(t) = \cos \omega t$, ω a positive constant, determine the solution $x(t)$ as $t \to \infty$.

14. (a) For the system satisfying

    $$\frac{d^3x(t)}{dt^3} + (13 - K)\frac{d^2x(t)}{dt^2} + (26 - K)\frac{dx(t)}{dt} + (41 + 9K)x(t) = u(t)$$

    establish for what values of the positive constant parameter K the system is stable.

    Establish for what values of the positive constant parameter K the system has poles with real parts more negative than -1.

    (b) A unity feedback system has output $y(t)$ and reference input $r(t)$. Find the transforms $Y(s)$ and $E(s)$ of $y(t)$ and the error signal $e(t) = r(t) - y(t)$ respectively, in terms of the transform $R(s)$ of $r(t)$ and the forward path transfer function $G(s)$.

    Draw the block diagram for a 1-DOF feedback system and establish the the relationship between the sensitivity S and the complementary sensitivity T.

15. State the condition for a branch of the root locus to lie on the real axis. Establish the method for determining a breakaway point on the real axis.

A unity feedback control system has the forward path transfer function

$$KG(s) = \frac{s+4}{(s+2)(s+3)(s+5)(s+7)}.$$

For this closed-loop system the gain K is allowed to vary through positive values.

The value $K = 584$ yields a pair of closed-loop poles on the imaginary axis. Determine the location of these poles.

Verify that the condition for the breakaway point is satisfied by $s = -2.5057$.

Draw the root-locus diagram.

Determine K^*, the value of K, which yields an under-damped system with dominant poles in positions corresponding to a damping ratio $\zeta = 0.1$ of a second-order system. For $K = K^*$

16. State the Nyquist Stability Criterion and indicate how one can assess the stability of a unity feedback system when all the poles of the forward path transfer function lie in the left-hand half-plane of the complex lane.

    Given that, for a constant gain K, a closed-loop system is assessed to be stable by using the Nyquist stability criterion, define the gain and phase margins.

    A unity feedback system has the forward path transfer function

$$G(s) = \frac{K}{s(2s+1)(4s+1)}$$

Sketch the frequency response function and use the Nyquist stability criterion to determine the range of positive values of K for which the system is stable.

For $K = 0.5$ determine the gain margin and the gain margin frequency.

If an additional delay element with transfer function e^{-3sT}, where T is a positive constant, is included in the forward path, sketch the frequency response function.

17. The linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$ and A, B, C are constant matrices of dimension $n \times n$, $n \times m$ and $p \times n$ respectively. Show that the transition matrix e^{At} is given by

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] \ .$$

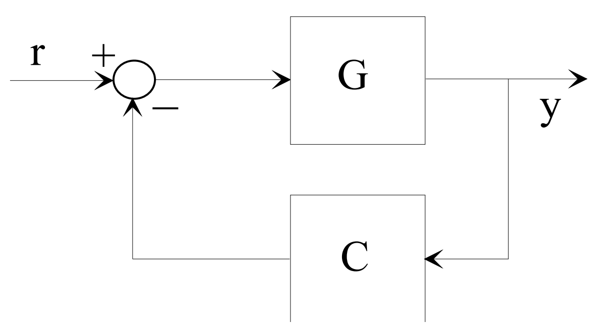
Determine e^{At} for the system with

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \ , \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ , \quad C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $u(t) = \sin 2\omega t$ and $x(0) = (0 \ \ 0)^T$ calculate $Y(s)$.

Without solving for k, show how one can determine the linear feedback control $u(t) = -k^T x$ which yields a closed-loop system with negative real eigenvalues at $-a$ and $-b$.

18. (a) Derive the closed-loop transfer function of the feedback configuration shown in Figure 1.



Give the definitions of open-loop and closed-loop poles and zeros. Derive an expression for the sensitivity S of the closed-loop system to variations in the transfer function G, where S is defined by

$$S = \frac{\Delta Y}{\Delta G} \cdot \frac{G}{Y}$$

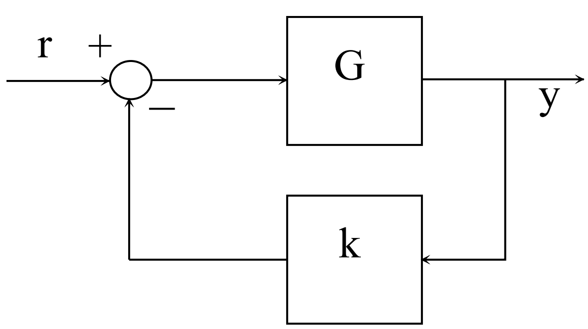
(b) The input $u(t)$ and output $y(t)$ of a linear, time-invariant system are related by the differential equation

$$4\ddot{y} + 10\dot{y} + 4y = 5\dot{u} + 20u.$$

i. Derive an expression for the Laplace transform $Y(s)$ of the output y in terms of the Laplace transform of the input $U(s)$ and initial condition data.
ii. If all initial conditions are zero, find the response $y(t)$ to a unit step input.

19. In the simple gain feedback configuration shown in the figure, the open-loop transfer function is

$$G(s) = \frac{(4s^2 + s + 6)}{(s^3 + s^2 + s + 1)}.$$



(a) By using the Routh-Hurwitz criterion or otherwise show that the open-loop system is marginally stable. Find the poles of G.

(b) Find the range of gains k, $k \geq 0$, for which the closed-loop system can be stabilized.

(c) Find the critical value of the gain k, $k > 0$, such that the closed-loop system has a pair of poles on the imaginary axis. Find these poles. *Note that we insist on $k > 0$ since we already know that two open-loop poles are on the imaginary axis.*

For this critical k, perform the partial fractions expansion of the closed-loop transfer function to find the output $y(t)$ when the input is a unit impulse function, in other words the Laplace transform of the input is equal to 1.

20. The transfer function

$$G(s) = \frac{(s+2)}{(s^2+9)(s^2+6s+5)}$$

is connected in the feedback configuration of Figure 2 above.

(a) Draw the root locus. Determine the number of branches and find the asymptotes and their centre and also the angles of departure for the pair of complex open-loop poles.

(b) Explain the definition and use of phase margin and angle margin by reference to a Nyquist plot.

21. (a) State the Nyquist stability criterion in the general case of an open-loop system with possibly unstable poles.

Sketch the D-contour in the complex plane and give an outline of why the Nyquist criterion is true.

(b) Sketch the Nyquist plots of

i.
$$G(s) = \frac{1}{s(s+4)}$$

ii.
$$G(s) = \frac{s+8}{s(s+1)(s+3)(s+5)}$$

iii.
$$G(s) = \frac{1}{(s+2)^2(s^2+2s+2)}$$

(c) For the system of part (b)(ii), find the range of values of the gain k for which the closed-loop system is stable.

22. (a) Use the Routh-Hurwitz criterion to factor the polynomials

i. $s^4 + s^3 - 2s^2 - 3s - 3$

ii. $s^5 + 3s^4 + 4s^3 + 4s^2 + 3s + 1$.

In both cases, give the complete set of roots.

(b) Draw the root locus of the feedback system of Figure 2, with

$$G(s) = \frac{1}{s(s^2 + 2s + 2)(s - 1)}.$$

Determine the number of branches, the asymptotes and their centre and also the relevant angles of departure. You can use the approximation that the breakaway point lies halfway between the two real open-loop poles.

23. (a) Convert the following single-input, single-output open-loop system into state-space form

$$G(s) = \frac{s + 2}{s(s + 3)}.$$

Check that the state space system A, B, C that you have obtained yields the correct transfer function $G(s)$.

(b) Given the linear time-invariant state space system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where the dimensions of the constant matrices A, B and C are $n \times n$, $n \times m$ and $p \times n$ respectively, define the state transition matrix e^{At} and, assuming that A is diagonalizable, show that e^{At} is never singular. Also give the formula for the transfer function matrix.

Find the state transition matrix for the system with

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}.$$

24. A linear system is described by the system of linear differential equations

$$\begin{aligned} \ddot{x} + 3\dot{x} + 2x &= \dot{u} + 4u \\ \ddot{y} + 4\dot{y} + 3y &= 2x + u. \end{aligned}$$

   (a) Assuming zero initial conditions, find the transfer function $G(s)$ relating the input u to the output y. Find the poles and zeros of this system and assess its stability.

   (b) Draw a block diagram involving the variables u, x and y and the linear system given by the differential equations above.

   (c) Find the impulse response $g(t) = \mathcal{L}^{-1}(G(s))$.

   (d) If we now close the loop by setting

$$u = -ky + r,$$

   with r the reference input, give the block diagram of the feedback system and compute the closed-loop transfer function $H(s)$ from r to y.

25. (a) Draw the root locus diagram of the feedback system shown in Figure 1 where the open-loop transfer function is

$$G(s) = \frac{s+5}{(s+2)(s^2+2s+2)}.$$

   Find the centre and the angles of the asymptotes and the angles of departure from the complex pole pair. Also find the value of the gain k at crossover to the right-half plane and the pure-imaginary closed-loop poles for this value of k.

   (b) If the reference input r is a unit step, $r(t) = h(t)$, find the closed-loop transfer function $H(s)$ and the transfer function from r to the error $e = r - ky$. Hence find the steady-state value of the output y and of the error e, as a function of the gain k.

26. (a) The input u to a linear system with transfer function

$$G(s) = \frac{1}{(s+1)(s^2+2s+2)}$$

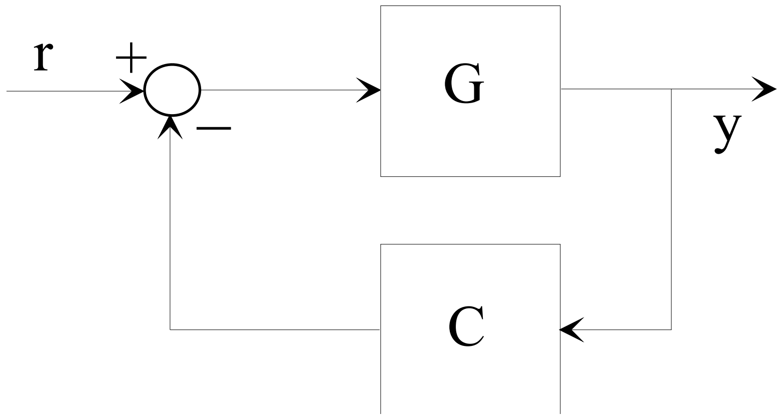


Figure 8.1: A constant gain feedback system

is the sinusoid $u(t) = 5\cos 3t$.

Find the output $y(t)$ and show that the asymptotic output waveform is also sinusoidal with the same frequency as u. Verify that the amplitude of the output sinusoid equals $5|G(3i)|$.

(b) Use the Routh-Hurwitz criterion to determine the stability of each of the polynomials below:

i. $s^4 + 2s^3 + 3s^2 + 4s + 5$

ii. $s^4 + s^3 - 2s^2 - 3s - 3$.

In the second case, find the complete set of roots of the polynomial.

27. (a) The linear system

$$G(s) = \frac{s}{s^2 + \omega^2}, \quad \omega > 0$$

is placed in the forward path of the constant-gain feedback configuration of Figure 1.

Find the closed-loop transfer function $H(s)$.

Show that, for some value of the gain $k^* > 0$, the closed-loop poles for $0 < k < k^*$ lie on a circle in the complex plane. Find this k^* and also the centre and radius of the circle.

It is desirable to have the closed-loop poles with damping ratio $\zeta = 0.5$. Find the gain k to accomplish this.

(b) State the Nyquist stability criterion for a system that may have unstable open-loop poles.

With reference to a Nyquist plot, explain the definitions and use of **gain** and **phase margins**.

28. (a) Draw the root locus of the feedback system of Figure 1 with

$$G(s) = \frac{1}{(s^2+2s+5)(s^2+6s+10)}.$$

Determine the number of branches, the centre and angles of the asymptotes, the relevant angles of departure and the critical value of the gain at crossover.

(b) Draw the Nyquist plot of the feedback system with

$$G(s) = \frac{5(s+10)}{(s+1)(s+2)(s+3)}.$$

For what positive values of the gain is the system stable?

29. A linear system consists of two linear sub-systems $G_1(s)$ and $G_2(s)$, with u being the input to $G_1(s)$, whose output x is the input to $G_2(s)$. The output of $G_2(s)$ is y. The linear differential relations defining the two systems are:

$$\begin{aligned} \dddot{x} - 5\ddot{x} + \dot{x} - 5x &= 2u \\ \ddot{y} + 5\dot{y} + 6y &= 6\dot{x} + 3x. \end{aligned}$$

(a) Assuming zero initial conditions, derive the overall transfer function

$$G(s) = \frac{Y(s)}{U(s)}.$$

(b) Determine the zeros and poles of the system $G(s)$. You can use the Routh-Hurwitz Table to help with the factorization of the cubic. Is this system stable?

(c) Use the partial fractions expansion of $G(s)$ to obtain the impulse response of the system.

30. (a) Give the definition of the sensitivity function $S(s)$ and show that

$$S(s) = \frac{1}{1 + G(s)C(s)}$$

for a control system with open-loop transfer function $G(s)$ and controller $C(s)$ in the feedback loop.

(b) Sketch the Nyquist diagram of the constant-gain feedback system with

$$G(s) = \frac{6}{(s+1)(s^2+2s+2)}.$$

Determine the range of gains k giving closed-loop stability.

(c) Find the gain and phase margins, GM and ϕ_{PM}, and the corresponding frequencies ω_{GM} and ω_{PM} for the above system.

[Note that the only positive root of the cubic equation $y^3 + y^2 + 4y - 32 = 0$ *is approximately equal to* 2.5045.*]*

31. The open-loop system

$$G(s) = \frac{s^2+4}{(s^2+3s+2)(s-p)}$$

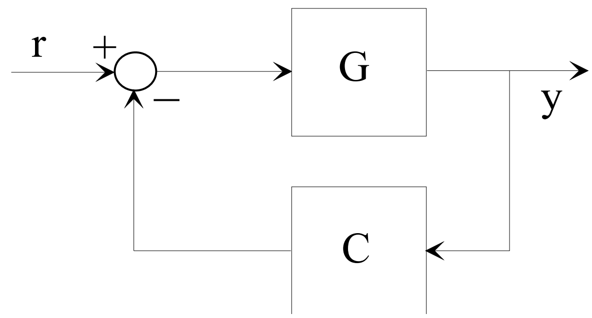
is unstable for $p > 0$. This problem will show that, *using constant-gain feedback, it is possible to obtain a stable closed-loop system only for certain values of the open-loop pole* p.

(a) For $p = 0.5$, give the Routh-Hurwitz Table for the closed-loop stability of the constant-gain feedback system and show that the closed-loop system can be made stable for gain k in a certain interval. Determine this interval.

(b) Draw the root locus diagram for the above $G(s)$. Make sure you compute all the relevant angles of departure. You can assume that the break point is approximately at the mid-point of the relevant interval.

(c) Now give the RH Table for $p = 2$ and show that it is **not** possible to stabilize this system using constant-gain feedback. Find the number of unstable closed-loop poles as the gain k varies.

32. (a) Derive the closed-loop transfer function of the feedback system shown. If $G(s) = n(s)/d(s)$ and the controller $C(s) = n_c(s)/d_c(s)$, obtain an expression for $H(s)$ in terms of n, d, n_c, d_c. Define the closed-loop zeros and poles of this feedback system.



Obtain the equation for the closed-loop poles when a controller of the form

$$C(s) = a + \frac{b}{s} + cs$$

is used. The terms of this PID controller are called *proportional, integral and derivative control*, respectively.

Using any suitable method, find a and b for a PI controller (with $c = 0$) to stabilize the system

$$G(s) = \frac{1}{s^2 + 2s - 3}.$$

(b) Use the partial fractions expansion method to find the step response of the system

$$G_1(s) = \frac{(s+3)}{(s+1)(s+2)^2[(s+1)^2+4]}.$$

33. A linear system describing the evolution of the two variables $x_1(t)$ and $x_2(t)$ with fixed initial conditions $(x_1(0) = -1, x_2(0) = 1, \dot{x}_2(0) = -2)$ and with control action $u(t)$ is given, in the transform domain, by

$$\begin{aligned}
(s+1)X_1(s) + (s+3)X_2(s) &= sU(s) \\
sX_1(s) + (s^2+2s+2)X_2(s) &= (s-1),
\end{aligned}$$

where $X_1(s) = \mathcal{L}(x_1(t))$, $X_2(s) = \mathcal{L}(x_2(t))$ and $U(s) = \mathcal{L}(u(t))$.

(a) Determine the system of linear differential equations satisfied by $x_1(t)$, $x_2(t)$ and $u(t)$ (note that it is not necessary to know the initial conditions.)

(b) Find the Laplace transforms $X_1(s)$ and $X_2(s)$ of the responses $x_1(t)$ and $x_2(t)$ to a step input, $u(t) = h(t)$.

Are the rational functions you found strictly proper?

If the measured output is $y(t) = x_2(t)$, use the Routh-Hurwitz Table to factorize the denominator polynomial for $X_2(s)$ and hence find $y(t)$, for $t \geq 0$.

8.2 Mini-Projects

8.2.1 Ship Steering Control System

You wish to design an effective ship steering control system, basically an autopilot for two-dimensional motion. Consider the effect of a rudder deflection δ on the angular position ψ relative to an arbitrary, but fixed, reference direction (such as due north), as shown in Fig. 1. The relationship between δ and ψ is given by what is known as Nomoto's equation and may be expressed as

$$\frac{\dot{\psi}}{\delta} = \frac{-K(1+T_3 s)}{(1+T_1 s)(1+T_2 s)}$$

It should be noted that a fixed δ results in a fixed rotation rate $\dot{\psi}$. The dynamics of rotation indicate that the rotation rate asymptotically approaches a constant value, so that, if the ship were moving in a straight line and a constant rudder rotation occurred, the ship would spiral in toward a circular path, which then gives the vessel a constant rate of rotation, as shown in Fig. 2. We may also model the steering gear as a simple first order lag, as shown in Fig. 3, where θ is the wheel rotation causing the rudder rotation δ.

A simple control system may now be proposed, involving either a computer or a human on the bridge, in which the actual ship direction obtained from a compass is compared to that desired, and corrective action results in a rotation of the wheel θ. This system is shown in Fig. 4. Considering typical data for a medium-size oil tanker 950 ft long weighing 150,000 dead weight tons travelling at 10.8 knots (17.28 ft/sec)

$$T_1 = 23.71\,\text{sec},\ T_2 = -2446.4\,\text{sec},\ T_3 = 35.69\,\text{sec},\ K = 0.474\,\text{sec}^{-1},\ T_G = 11\,\text{sec}$$

Plot the root locus for the system and establish, rather surprisingly, that the system is unstable! It is a little known fact that most large ships are unstable. What this means is that if the ship were travelling in a straight line and the wheel fixed in the corresponding position, the ship would eventually veer off course. Because the time constant associated with the instability is very large (tens of minutes), an attentive helmsman will correct for the small course deviations as they occur. However, the helmsman is attempting to control an unstable system.

You should attempt to design a better controller that not only stabilizes the steering system, but also meets the following performance requirements:

1. The ship should not have more than a 5% overshoot in its response ψ to a step change in the desired angular position ψ.

2. The 2% settling time to the above step input should occur within a time equivalent to at most five ship lengths at the design speed.

Based upon the standard second order system approximation, check that the desired damping ratio ζ corresponding to the first requirement and the desired settling time $T_s \approx 4/(\zeta\omega_n)$ (ω_n undamped frequency) imply two dominant closed-loop complex poles lying in the unshaded region in Fig. 5.

In order to make the ship unconditionally stable, you need to introduce an extra open-loop zero. The proposed system with additional feedback is shown in Fig. 6 with design parameters K_1 and K_2. **Show that the open-loop transfer function is**

$$GH(s) = \frac{0.0237K_1(35.69s+1)}{638046s^4 + 84653s^3 + s^2(2411 + 0.846K_2) + s(0.024K_2 - 1)}$$

Consider the two values $K_2 = 500 + 2\mu$ and $K_2 = 5000 + 2\mu$ where μ =numerical value of the first letter of your first name + numerical value of the first letter of your surname where $A = 1$, $Z = 26$.

Using the root locus technique establish which of these values of K_2 should be selected. Choose a suitable design value of K_1 indicating your reasons by plotting closed-loop step responses. Check that the approximation of the fourth order system by a second order system is reasonable.

You may find the MATLAB commands listed below useful.

```
[z, K]=rlocus(numopen,denopen) [z,
K]=rlocus(numopen,denopen,logspace(-2,2,500)) plot(z), grid
[numclose, denclose]=cloop(numopen,denopen,-1)
step(numclose,denclose), xlabel('t'), ylabel('Step'), grid help
series help cloop help feedback help rlocus help step help print
```

8.2.2 Stabilization of Unstable Transfer Function

This project studies the problem of stabilization of a system with an unstable real pole. Stabilization, as we have discussed at the beginning of our course, is an important aim of feedback control. It is not always easy to stabilize an unstable system and this project deals with some of the problems and shortcomings of stabilization.

Historically, the real world systems on which control was applied have tended to be stable; in this case, improved performance, robustness etc were the primary aims of control. Aviation has been a tremendously important area of application for control system theory. Control is needed in flight both for the actual control of the flight path and for the control of the aircraft engines. With the advent of high-speed computers, the possibility of implementing complicated control strategies in real-time using on-board computers has arisen; this is known by the name *'fly-by-wire'*. It is a very recent and extraordinary idea to try to use 'fly-by-wire' control to fly aircrafts that are **inherently unstable**, in other words 'planes that would be unstable in the absence of control. One may well wonder why it is desirable to do this; the answer is that there are certain crucial benefits even in instability, namely an unstable aircraft will 'bunk' more quickly than a stable one, and this ability to swerve quickly and abruptly is crucial for a fighter aircraft –and this, rather than commercial carriers is, fortunately for us, the type of aircraft where this control strategy is applied to. (The way instability is used to evade an enemy is as follows: in normal flight, the airplane is rendered stable –and robust– by control; at the crucial time, however, control is switched temporarily off, so that an abrupt swerve is precipitated because of the instability; then, control is restored before instability becomes catastrophic and the fighter flies off —this actually works and is implemented in last generation high-tech 'planes.)

Now we, of course, do not have the tools to analyze the multivariable control of unstable aircraft, so we discuss a more modest situation. Consider the system with an unstable pole at $s = 1.5$ given below

$$g(s) = \frac{1}{(s-1.5)(s^2+s+1)}.$$

This system is placed in a feedback loop with gain k (so $h(s) = k$). The closed-loop transfer function is therefore

$$\frac{g(s)}{1+kg(s)}$$

Draw the root locus to convince yourself that this system is unstable for all values of k. Note that the excess number of poles over zeros is three.

Show that the above system cannot be stabilized by a controller of any of the following types:

1. $h(s) = \frac{k}{s+p}$ for any p.
2. $h(s) = \frac{k}{s^2+as+b}$ for any a and b.
3. $h(s) = \frac{k(s+z)}{s^2+as+b}$ for any a, b and z.
4. $h(s) = \frac{k(s+z)}{s+p}$ for any pole-zero pair $-z$, $-p$.

The Routh-Hurwitz criterion is useful here.

After our repeated failures, we now come to two methods for stabilizing the system g. The first uses the controller

$$h(s) = \frac{k(s-1.5)}{s+p}$$

to **cancel** the unstable pole. *Discuss the shortcomings of this method in the light of uncertainties in the values of the system parameters.*

A second stabilizing controller is one that has a lot of zeros to bring the asymptotes back to the left-half plane. An example is the controller

$$h(s) = k(s^2 + 2s + 1)$$

Show that the resulting root locus gives a stable system.

Unfortunately, the implementation of such a controller means that we have to differentiate the output y twice, a process that, in the presence of output noise, for example, is very unreliable.

We now come to a more realistic stabilizing controller. This takes the form

$$h(s) = \frac{k(s^2 + b_1 s + b_0)}{s^2 + a_1 s + a_0}.$$

Design a stabilizing controller of this form. Place the two controller zeros close to the two oscillatory poles of $g(s)$ and the two controller poles 'far enough' on the negative real axis (but not too far as there is a trade-off). The resulting system is stable for only a range of gain values. Using the Routh-Hurwitz criterion, or otherwise, find this range of stablizing gains.

Support all your arguments with appropriate Root Locus plots or Routh-Hurwitz Tables.

Give your conclusions about the difficulties of stabilizations in this case, the way success was achieved and the limitations of the control procedure.

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