PERIODIC ORBITS IN GRAVITATIONAL SYSTEMS

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1 The gravitational N-Body problem

The Newtonian gravitational force is the dominant force in the N-Body systems in the universe, as for example in a planetary system, a planet with its satellites, or a multiple stellar system. The long term evolution of the system depends on the topology of its phase space and the existence of ordered or chaotic regions. The topology of the phase space is determined by the position and the stability character of the periodic orbits of the system (fixed points on a Poincaré map on a surface of section). Islands of stable motion exist around the stable periodic orbits. Chaotic motion appears at the unstable periodic orbits. This makes clear the importance of the periodic orbits in the study of the dynamics of such systems.

In many cases, there is only one massive body, whose gravitational attraction provides the dominant force, as is the case with a planetary system, where the sun is the main attracting body, or a planet surrounded by satellites. In this case the motion of the small bodies (planets or satellites) follow Keplerian orbits, perturbed by the gravitational interaction between the small bodies. This is a nearly integrable dynamical system. In these systems resonances exist between the small bodies in their motion around the massive body. These correspond to periodic motion, and this makes clear the importance of the resonances in the dynamical properties of a nearly integrable system.

The simplest model of a gravitational system is a system of two bodies moving in Keplerian orbits around their common center of mass. This is an integrable system. We consider now a hierarchy of models, starting from the above mentioned integrable system and adding more bodies to the system. We have different models, which are used to study particular systems.

- The restricted three-body problem: Two bodies of finite masses, called *primaries*, revolve around their common center of mass in it circular orbits and a third body with *negligible* mass moves under their gravitational attraction, but does not affect the orbits of the two primaries. In most astronomical applications the second primary has a small mass compared to the first primary, and consequently the motion of the third, massless, body is a perturbed Keplerian orbit. This is a model for the study of an asteroid (Jupiter being the second primary), a trans-Neptunian object (Neptune being the second primary) or an Earth-like planet in an extrasolar planetary system.

- The general three-body problem: Three bodies with finite masses moving under their gravitational attraction. This is a model for a triple stellar system. In many astronomical applications one of the three bodies has a large mass and the other two bodies have small, but not negligible masses. This is a model for an extrasolar planetary system, or a system of two satellites moving around a major planet. In the latter two cases the two small bodies move in perturbed Keplerian orbits.

We will start with the basic elements of gravitational systems in general. Then we will focus out attention to the study of systems of two degrees of freedom, and then extend the results to three or more degrees of freedom. The study will be for a general dynamical system, with particular emphasis on Hamiltonian systems, because the gravitational systems are Hamiltonian.

2 The general N-Body problem

2.1 Basic Equations

Consider N bodies interacting by gravitational forces,

$$F_{ij} = -\frac{Gm_i m_j}{r_{ij}^2}.$$

The position in the inertial frame is determined by the position vectors

$$\vec{r}_i(x_i, y_i, z_i). \quad (i = 1, 2, ...N)$$

We have $3N$ degrees of freedom and the evolution in space is given by the system of differential equations

$$m_i \ddot{\vec{r}}_i = \vec{F}_i,$$

where

$$\vec{F}_i = -\sum_{j=1}^{N} \frac{Gm_i m_j (\vec{r}_i - \vec{r}_j)}{r_{ij}^3} = -\frac{\partial V}{\partial \bar{q}_i}.$$

The system is conservative, and the potential function is

$$V(\vec{r}_m - \vec{r}_n) = -\sum_{ij} \frac{Gm_i m_j}{r_{ij}}. \quad 1 \leq i < j \leq N$$

The gravitational system of N bodies is Hamiltonian and the Hamiltonian function is

$$H = \sum \frac{\vec{p}_i^2}{2m} + V, \quad \vec{p}_i = m_i \dot{\vec{r}}_i$$

We have the following integrals of motion:

- Center of mass

$$\vec{r}_{cm} = \left(\sum m_i \vec{r}_i\right)/m$$

- Linear momentum

$$\vec{p} = \sum m_i \vec{v}_i = constant$$

- Angular momentum
$$\vec{L} = \sum \vec{r}_i \times m_i \vec{v}_i = constant$$
- Energy
$$E = T + V = constant$$

The total momentum of the system is equal to $\vec{p} = m\vec{v}_{cm}$.

We can assume that the total momentum is equal to zero, $\vec{p} = 0$, which implies that the center of mass is at rest, $\vec{v}_{cm} = 0$. Consequently, in the system where the center of mass is at rest, we have $3N - 3$ degrees of freedom.

2.2 Special solutions in the inertial frame

* Equilibrium solution

Consider the set of differential equations that determine the motion of the system:

$$m_i \ddot{\vec{r}}_i = -\frac{\partial V}{\partial \vec{r}_i}.$$

For an equilibrium solution we have

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial y_1} = \frac{\partial V}{\partial z_1} = ... = \frac{\partial V}{\partial z_N} = 0,$$

where V is a homogeneous function of degree -1. By Euler's theorem of homogeneous functions we have

$$\frac{\partial V}{\partial x_1}x_1 + \frac{\partial V}{\partial y_1}y_1 + \frac{\partial V}{\partial z_1}z_1 + ... + \frac{\partial V}{\partial z_N}z_N = -V,$$

from which we readily obtain that at the equilibrium we must have $V = 0$. This is a contradiction, because V is never zero.

*Central Configurations

Consider again the system of differential equations

$$m_i \ddot{\vec{r}}_i = -\frac{\partial V}{\partial \vec{r}_i}.$$

We perform a change of coordinates

$$\vec{r}_i = \phi(t)\vec{a}_i,$$

where $\vec{a}_i$ is a constant and $\phi(t)$ a scalar. The system of differential equations takes the form

$$|\phi|^3 \phi^{-1} \ddot{\phi} m_i \vec{a}_i = \sum_j \frac{m_i m_j (\vec{a}_i - \vec{a}_j)}{||\vec{a}_i - \vec{a}_j||^3}.$$

We note that the right hand side constant is and consequently the left hand side must be also constant:

$$|\phi|^3 \phi^{-1} \ddot{\phi} = \lambda,$$

or

$$\ddot{\phi} = \frac{\lambda \phi}{|\phi|^3},$$

where λ is a constant. This is a differential equation in ϕ, whose solution is

$$\phi = \alpha\, t^{2/3},\ \alpha^3 = 9\lambda/2.$$

The solutions of (A) for some λ are called *central configuration*.

We note now that

$$\lambda m_i \vec{a}_i = \sum_j \frac{m_i m_j (\vec{a}_i - \vec{a}_j)}{\|\vec{a}_i - \vec{a}_j\|^3}, \qquad (A)$$

and summing on i we obtain

$$\sum m_i \vec{a}_i = 0,$$

which implies that the constant vectors $\vec{a}_i$ have origin at the center of mass.

3 Periodic orbits in systems with two degrees of freedom. Autonomous case

3.1 Periodic orbits

Let us consider a dynamical system with two degrees of freedom, defined by the set of two second order differential equations

$$\begin{aligned} \ddot{x}_1 &= F_1(x_1, x_2, \dot{x}_1, \dot{x}_2), \\ \ddot{x}_2 &= F_2(x_1, x_2, \dot{x}_1, \dot{x}_2). \end{aligned} \tag{1}$$

The initial conditions that determine a solution are

$$x_{10},\ \ x_{20},\ \ \dot{x}_{10},\ \ \dot{x}_{20}, \tag{2}$$

and the corresponding solution has the form

$$\begin{aligned} &x_1(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t), \\ &x_2(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t). \end{aligned} \tag{3}$$

The solution is periodic, with period T, if

$$x_i(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t+T) = x_i(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t), \tag{4}$$

for every t.

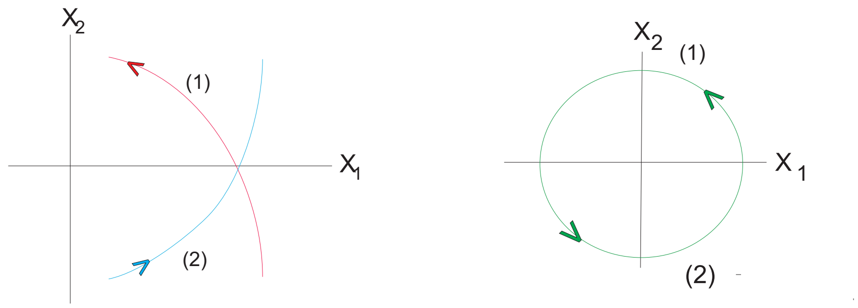

Figure 1: The two symmetric solutions (first panel) and their coincidence to a single orbit (second panel) if the periodicity conditions apply.

3.2 Existence of symmetric periodic orbits

We assume that the differential equations are invariant under the transformation

$$x_1 \to x_1, \;\; x_2 \to -x_2, \;\; t \to -t. \tag{5}$$

This property appears in several models that are of astronomical interest. This means that if

$$x_1(t), \;\; x_2(t)$$

is a solution, then

$$x_1(t), \;\; -x_2(-t) \tag{6}$$

is also a solution. Note that this second solution is the symmetric of the first solution with respect to the x_1-axis. Consequently, if an orbit starts from the x_1 axis perpendicularly, $\dot{x}_{10} = 0$, and crosses again the x_1 axis perpendicularly, $\dot{x}_{10} = 0$, the orbit is closed and consequently is a *symmetric* periodic orbit with respect to the x_1-axis. This is shown in Figure 1.

The initial conditions of a symmetric periodic orbit are

$$x_{10}, \;\; x_{20} = 0, \;\; \dot{x}_{10} = 0, \;\; \dot{x}_{20},$$

which means that a symmetric periodic orbit is determined only by two nonzero initial conditions

$$x_{10}, \; \dot{x}_{20}. \tag{7}$$

From the above we see that the *periodicity conditions* are

$$\begin{aligned} x_2(x_{10}, 0, 0, \dot{x}_{20}; T/2) &= 0, \\ \dot{x}_1(x_{10}, 0, 0, \dot{x}_{20}; T/2) &= 0, \end{aligned} \tag{8}$$

which imply that the orbit starts perpendicularly from the x-axis and crosses again perpendicularly the x-axis after a time interval equal to half the period T. We remark that the second *perpendicular* crossing may take place after several (non perpendicular) crossings from the x_1-axis.

The periodic orbits are not isolated, in general. They belong to families, along which the period varies. A family of symmetric periodic orbits is represented by a continuous curve in the space of initial conditions $x_{10}, \dot{x}_{20}$. This curve is called a *characteristic curve*.

4 Variational equations

A periodic orbit is an orbit which repeats itself for infinite time after a period T. We shall study now the behaviour of the system in the vicinity of a periodic solution by considering perturbed initial conditions, i.e, initial conditions in the vicinity of the initial conditions of the periodic orbit.

We express the system of differential equations (1) as a system of four differential equations of the first order,

$$\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_3 &= F_1(x_1, x_2, x_3, x_4), \\
\dot{x}_4 &= F_2(x_1, x_2, x_3, x_4),
\end{aligned}$$

or, in general

$$\dot{x}_i = f_i(x_1, x_2, x_3, x_4). \quad (i = 1, ...4) \tag{9}$$

Let

$$x_i = x_i(x_{10}, x_{20}, x_{30}, x_{40}; t), \quad (i = 1, ...4) \tag{10}$$

be a solution of the system (9), non periodic in general, corresponding to the initial conditions $x_1(0), x_2(0), x_3(0), x_4(0)$. We consider now new initial conditions, in the vicinity of these initial conditions, of the form

$$x_1(0) + \xi_1(0),\ x_2(0) + \xi_2(0),\ x_3(0) + \xi_3(0),\ x_4(0) + \xi_4(0),$$

where $\xi_i(0)$ are small. The new solution can be expressed in the form

$$x_i'(t) = x_i(t) + \xi_i(t), \quad (i = 1, ...4) \tag{11}$$

where $\xi(t)$ is the deviation vector between the solution (10) and the perturbed solution (11), for the *same time* t, $\xi(t) = x_i'(t) - x_i(t)$. The behaviour of the system in the vicinity of the solution (10) depends on the deviation vector $\xi(t)$.

We assume that the initial perturbation $\xi(0)$ is small, and consequently, for continuity reasons, the deviation $\xi(t)$ should be also small, at least for a finite time interval. For this reason we linearize the system of differential equations (9), to first order terms in the $\xi_i(t)$, by substituting the perturbed solution (11) into the system (9) and keeping only the first order terms in ξ_i. We obtain the system of *variational equations*,

$$\dot{\xi}_i = \sum_{k=1}^{4} p_{ik}\xi_k, \quad p_{ik} = \left(\frac{\partial f_i}{\partial x_k}\right)_{x_i(t)}, \quad (i = 1, ...4) \tag{12}$$

which describes the evolution of the system (9) in the neighborhood of the orbit (10), to first order terms in the deviations. The partial derivatives are computed for the solution $x_i(t)$.

The variational equations (12) are a system of four linear differential equations with time dependent coefficients. If the solution $x(t)$ is T-periodic, then the partial derivatives are also T-periodic. In this latter case the system of variational equations is a *linear system with periodic coefficients*. The theory related to the study of such systems is the *Floquet theory* and some elements of it will be presented in the following sections.

4.1 The fundamental matrix of solutions

The general solution of the linear system (12) is expressed as a linear combination of four linearly independent solutions. In particular, let us consider a 4×4 matrix $\Delta(t)$ whose columns are four linearly independent solutions corresponding to the initial conditions $\Delta(0) = I_4$, where I_4 is the 4×4 unit matrix. This matrix is called *fundamental matrix of solutions* and the general solution of the variational equations is expressed in the form

$$\xi(t) = \Delta(t)\xi(0). \tag{13}$$

A basic property of the matrix $\Delta(t)$ is the Liouville-Jacobi formula (Yakubovich anf Starzhinskii, 1975, Jordan and Smith, 1988)

$$\det \Delta(t) = \det \Delta(0) \exp \int_o^t \mathrm{trace}(P) dt, \tag{14}$$

where P is the matrix of the coefficients of the system of variational equations (12).

We shall prove now that the columns of the matrix $\Delta(t)$ are the partial derivatives of the solution $x_i(x_{10}, x_{20}, x_{30}, x_{40}, t)$ with respect to the initial conditions. In particular, we shall prove that the j-th column, $j = 1, ...4$, is given by

$$\begin{pmatrix} \partial x_1/\partial x_{j0} \\ \partial x_2/\partial x_{j0} \\ \partial x_3/\partial x_{j0} \\ \partial x_4/\partial x_{j0} \end{pmatrix}. \tag{15}$$

Proof

The solution $x_i(x_0; t)$ satisfies the system (9),

$$\frac{\partial x_i(x_0, t)}{\partial t} = f_i(x_1(x_0, t), x_2(x_0, t), x_3(x_0, t), x_4(x_0, t)). \;\; (i = 1, ...4)$$

If we apply to the above equations the operator $\partial/\partial x_{j0}$, $j = 1, ...4$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial x_{jo}}\right) = \sum_{k=1}^{4} \left(\frac{\partial f_i}{\partial x_k}\right) \frac{\partial x_k}{\partial x_{j0}}, \;\; (i = 1, ...4) \tag{16}$$

for each x_{j0}. We note that the system (16) is the system of variational equations (12) satisfied by the vector (15). This means that the four vectors (15), for $j = 1, ...4$, are four linearly independent solutions of the variational equations. In addition, we note that $\partial x_i/\partial x_{j0} = \delta_{ij}$ for $t = 0$. Consequently, these four vectors are the four columns of the fundamental matrix of solutions $\Delta(t)$.

4.2 Variational equations for a periodic solution

We assume now that the solution $x(t)$ is T-periodic. Then the system of variational equations corresponding to this periodic solution, is a linear system with *periodic* coefficients.

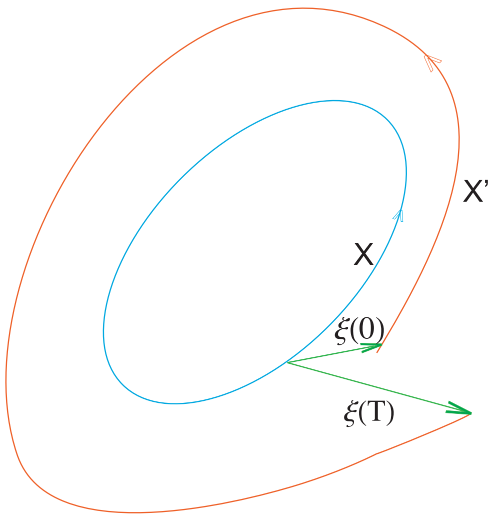

Figure 2: Mapping at integral multiples of the period

We shall prove that the derivative $\dot{x}_i(t)$ of the periodic solution $x_i(t)$ is a solution of the variational equations.

Proof

The solution $x_i(t)$ satisfies the system (9):

$$\dot{x}_i = f_i(x_1, x_2, x_3, x_4). \ (i = 1, ...4)$$

If we apply the operator d/dt we obtain

$$\frac{d}{dt}(\dot{x}_i(t)) = \sum_{j=1}^{4} \left(\frac{\partial f_i}{\partial x_j}\right)_{x_i(t)} \dot{x}_j(t).$$

This is the system of variational equations, for the solution $\xi_i = \dot{x}_i(t)$. So we come to the conclusion that *the variational equations that correspond to a T-periodic orbit have always a T-periodic solution, which is the derivative $\dot{x}_i(t)$ of the periodic solution.*

5 Linear stability of a periodic orbit

Let $x_i(t)$ be a periodic orbit and $x'(t)$ a perturbed orbit, which, to a linear approximation, can be expressed in the form

$$x'_i(t) = x_i(t) + \xi_i(t),$$

where $\xi_i(t)$ is the solution of the variational equations. This latter solution is expressed in the form

$$\xi(t) = \Delta(t)\xi(0), \tag{17}$$

and for $t = T$,

$$\xi(T) = \Delta(T)\xi(0). \tag{18}$$

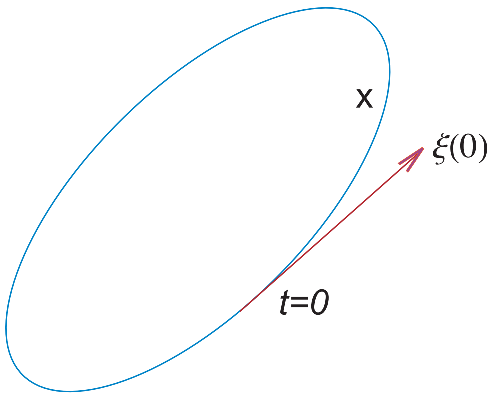

Figure 3: Tangent displacement $\xi(0) = \dot{x}(0)$

From this expression we obtain, by induction,

$$\xi(nT) = [\Delta(T)]^n \xi(0). \tag{19}$$

Equations (18) and (19) give the deviation, to a linear approximation, of the perturbed orbit $x'(t)$ from the periodic orbit $x(t)$ after a time interval equal to n times the period T, due to an initial deviation $\xi(0) = x'(t) - x(t)$ at $t = 0$. In fact Equation (19) is a mapping of the initial deviation $\xi(0)$ at integral multiples of the period T (see Figure 2). This is a linear mapping defined by the matrix $\Delta(T)$. It is clear that the stability of the periodic orbit $x(t)$ depends on the properties of the mapping (19), i.e. on the eigenvalues of the matrix $\Delta(T)$. The matrix $\Delta(T)$ is called the *monodromy matrix*.

5.1 Unit eigenvalues of the monodromy matrix $\Delta(T)$

Existence of a periodic solution $\xi(t)$

Let $\xi(t)$ be a T-periodic solution of the variational equations. We have $\xi(t+T) = \xi(t)$, for any t and consequently, for $t = 0$, $\xi(T) = \xi(0)$. Due to this latter relation, Equation (17) takes the form, for $t = T$, $\xi(0) = \Delta(T)\xi(0)$, and finally

$$(\Delta(T) - I)\, \xi(0) = 0. \tag{20}$$

Thus we come to the conclusion that *if the system of variational equations has a periodic solution, the monodromy matrix has a unit eigenvalue.*

Remark

Note that the system of variational equations that corresponds to a T-periodic solution $x(t)$, has a T-periodic solution $\xi(t) = \dot{x}(t)$. The corresponding eigenvector is $\xi(0) = \dot{x}(0)$, as is seen from Equation (20) and represents a tangent displacement along the periodic orbit (Figure 3).

Existence of an integral of motion

Let $G(x_1, x_2, x_3, x_4) = \textit{constant}$ be an integral of motion. If $x_i(x_o, t)$ is a T-periodic solution, we have the relation

$$G(x_1(x_o,t), x_2(x_o,t), x_3(x_o,t), x_4(x_o,t)) =$$

$$G(x_{1o}, x_{2o}, x_{3o}, x_{4o}).$$

We apply to the above relation the operator $\partial/\partial x_{jo}$, and we obtain

$$\sum_j \left(\frac{\partial G}{\partial x_k}\right)_t \left(\frac{\partial x_k}{\partial x_{jo}}\right)_t = \left(\frac{\partial G}{\partial x_{jo}}\right).$$

We set now $t = T$ and taking into account that

$$\left(\frac{\partial G}{\partial x_k}\right)_{t=T} = \left(\frac{\partial G}{\partial x_k}\right)_{t=0},$$

due to the fact that $x(t)$ is periodic, we obtain

$$(\Delta^{\tau}(T) - I)\,\nabla G = 0, \tag{21}$$

where Δ^{τ} is the transpose of Δ. From this relation we obtain that if $\nabla G \neq 0$ then $\Delta(T)^{\tau}$ has a unit eigenvalue. Thus finally, we come to the conclusion that *if the dynamical system has an integral of motion, which is not stationary along the periodic orbit, the monodromy matrix $\Delta(T)$ has a unit eigenvalue.*

<u>Existence of a unit eigenvalue of $\Delta(T)$</u>

We shall prove that a unit eigenvalue of $\Delta(T)$ implies a T-periodic solution of the variational equations:

Let $\xi(0)$ be the eigenvector corresponding to the unit eigenvalue. We have

$$(\Delta(T) - I)\xi(0) = 0,$$

from which we obtain

$$\xi(0) = \Delta(T)\xi(0).$$

But, according to Equation (18), $\xi(T) = \Delta(T)\xi(0)$, and finally

$$\xi(T) = \xi(0),$$

which implies that $\xi(t)$is periodic. The initial conditions of this periodic orbit is the eigenvector of the monodromy matrix corresponding to this unit eigenvalue.

In general, if there exist two unit eigenvalues with two linearly independent eigenvectors, respectively, then the system of variational equations has two independent periodic solutions.

<u>Remark</u>: We shall prove later that in a Hamiltonian system there are always two unit eigenvalues, but only one eigenvector. So, there is only one periodic solution of the variational equations, corresponding to the double unit eigenvalue.

5.2 Special solutions

Let us assume that the system of variational equations has a special solution with the property

$$\xi(t + T) = \lambda\xi(t). \tag{22}$$

If we set $t = 0$ we obtain

$$\xi(T) = \lambda \xi(0), \tag{23}$$

which means that the initial deviation $\xi(0)$ is mapped, after a period T, to a multiple of it.

Equation (23) is a difference equation, and its solution is of the form

$$\xi(t) = f(t)\lambda^{t/T}, \quad f(t): \ T\text{-periodic}. \tag{24}$$

From Equation (23) and Equation (18), $\xi(T) = \Delta(T)\xi(0)$, we obtain

$$\Delta(T)\xi(0) = \lambda\xi(0)$$

and finally

$$(\Delta(T) - \lambda I)\,\xi(0) = 0.$$

This means that if a solution with the property (22) exists, λ is an eigenvalue of the monodromy matrix and $\xi(0)$ the corresponding eigenvector.

Let us assume now that the monodromy matrix has an eigenvalue λ and let $\xi(0)$ be the corresponding eigenvector. We shall prove that this eigenvector is the initial conditions of a solution with the property (22). We have $\Delta(T)\xi(0) = \lambda\xi(0)$ and also, because of Equation (18), $\Delta(T)\xi(0) = \xi(T)$. From these two relations we obtain

$$\xi(T) = \lambda\xi(0). \tag{25}$$

Equation (17) is valid for any t and setting $t = t + T$ we have,

$$\xi(t+T) = \Delta(t+T)\xi(0). \tag{26}$$

We note now that $\Delta(t+T)$ is a solution matrix (because the matrix p_{ij} of the variational equations is T-periodic) and consequently it satisfies the relation (17), in matrix form (Meyer and Hall, 1992, Yakubovich and Starzhinskii, 1975),

$$\Delta(t+T) = \Delta(t)\Delta(T).$$

Equation (26) takes now the form

$$\xi(t+T) = \Delta(t)\Delta(T)\xi(0). \tag{27}$$

and taking into account the relations (18) and (25) we obtain

$$\xi(t+T) = \lambda\Delta(t)\xi(0),$$

and finally, using (17) we obtain

$$\xi(t+T) = \lambda\xi(t).$$

Thus finally we come to the conclusion that *to each eigenvalue λ of the monodromy matrix $\Delta(T)$ there exists a solution of the form $f(t)\lambda^{t/T}$, where $f(t)$ is T-periodic, whose initial conditions is the eigenvector $f(0)$ corresponding to the eigenvalue λ.*

Characteristic exponents

We define a parameter α as

$$\alpha = \frac{1}{T} \ln \lambda, \tag{28}$$

where the principal value of the logarithm is taken. Then the solution (23) is expressed as

$$\xi(t) = f(t)e^{\alpha t}. \tag{29}$$

The parameter α is called the *characteristic exponent* and is related to the eigenvalue λ by the relation (28). Note that for $\lambda = 1$, it is $\alpha = 0$.

The monodromy matrix $\Delta(T)$ is a 4×4 matrix and it has four eigenvalues. To each eigenvalue there corresponds a solution with the property (22) and consequently it has the form (29) (or (24)).

If there exist four independent eigenvectors to the four eigenvalues of $\Delta(T)$ (this may not be the case if some eigenvalues are multiple), then there exist four independent solutions of the form (29) and the general solution is a linear combination of solutions of the form (29).

From the above is clear that the periodic orbit $x(t)$ is linearly stable if and only if

$$\mathrm{Re}(\alpha) \leq 0. \tag{30}$$

For $\alpha = 0$ ($\lambda = 1$) the solution (29) is T-periodic.

6 Hamiltonian systems

The gravitational systems are Hamiltonian. For this reason, we shall study in this section the special properties that a Hamiltonian system has, in addition to the general properties obtained in the previous sections. We will start with systems with two degrees of freedom.

A Hamiltonian system is defined by the Hamiltonian function

$$H(x_1, x_2, x_3, x_4), \tag{31}$$

where x_1, x_2 are the coordinates and x_3, x_4 the momenta.

The Hamiltonian equations are

$$\begin{aligned}
\dot{x}_1 &= \partial H/\partial x_3, \\
\dot{x}_2 &= \partial H/\partial x_4, \\
\dot{x}_3 &= -\ \partial H/\partial x_1, \\
\dot{x}_4 &= -\ \partial H/\partial x_2,
\end{aligned}$$

or

$$\dot{x} = -J\nabla H, \tag{32}$$

where ∇H is a column vector and J the 4×4 symplectic matrix

$$J = \begin{pmatrix} 0 & -I_2 \\ +I_2 & 0 \end{pmatrix}. \tag{33}$$

Note that $J^{-1} = -J$.

6.1 Variational equations of Hamiltonian systems

The variational equations of a Hamiltonian system (32) have the special form given by

$$\dot{\xi} = -JA\xi, \tag{34}$$

where the elements a_{ij} of the 4×4 matrix A are

$$a_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j}. \ (i, j = 1, ...4) \tag{35}$$

The system (34) is called a *linear Hamiltonian system.* It is easy to see that it can be expressed in the Hamiltonian form (32) with Hamiltonian

$$H = \frac{1}{2}\xi^\tau A\xi = \frac{1}{2}\sum_{i,j=1}^{4} a_{ij}\xi_i\xi_j.$$

From the relations (35) we can verify that the trace of the matrix of the coefficients of the linear Hamiltonian system (34) is equal to zero. Consequently, due to the general property (14), the determinant of the fundamental matrix of solutions $\Delta(t)$ is equal to unity (see also Meyer and Hall, 1992),

$$\det \Delta(t) = 1.$$

For $t = T$ we obtain

$$\det \Delta(T) = 1, \tag{36}$$

from which we see that *the determinant of the monodromy matrix is equal to one.*

As we proved in section 3.1, the fundamental matrix of solutions $\Delta(t)$ is expressed in the form

$$\Delta(t) = \frac{\partial(x_1, x_2, x_{,3}, x_4)}{\partial(x_{10}, x_{20}, x_{30}, x_{40})}. \tag{37}$$

This means that the determinant of the Jacobian of the flow in phase space is equal to one. Consequently, *the volume in phase space is conserved* (Liouville theorem).

6.2 Symplectic property

An important property of the monodromy matrix of a Hamiltonian system is the symplectic property that we shall prove now. Let ξ and ξ' be two solutions of the variational equations. The following property holds:

$$\xi_1\xi_3' + \xi_2\xi_4' - \xi_3\xi_1' - \xi_4\xi_2' = \text{constant},$$

or, in matrix form,

$$\xi'^\tau \, J \, \xi = \text{constant}, \tag{38}$$

where ξ, ξ' are 4-vectors and J is the symplectic matrix (33).

Proof: The proof can be made by direct substitution.

If we apply the property (38) for the four solutions that are the four columns of the matrix $\Delta(t)$, we obtain the equation

$$\Delta^\tau(t)\ J\ \Delta(t) = J.$$

We set now $t = T$ and obtain

$$\Delta^\tau(T)\ J\ \Delta(T) = J. \tag{39}$$

This is an important property of the monodromy matrix of a Hamiltonian system, which is called the *symplectic* property. Thus we come to the conclusion that *the monodromy matrix of a Hamiltonian system is symplectic.*

6.3 Eigenvalues of a symplectic matrix

We shall see now that the eigenvalues of a symplectic matrix have some special properties. We express the property (39) as

$$\Delta^\tau(T) = J\Delta^{-1}(T)J^{-1},$$

from which we see that the matrix $\Delta^\tau(T)$ is related to the matrix $\Delta^{-1}(T)$ by a similarity transformation. Consequently, they have the same set of eigenvalues. Thus finally, we come to the conclusion that the eigenvalues of the $\Delta(T)$ are in reciprocal pairs. In addition, due to the fact hat the matrix $\Delta(T)$ is real, they are also in complex conjugate pairs.

From the above we see that the four eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the monodromy matrix have the property

$$\lambda_1\lambda_2 = 1, \quad \lambda_3\lambda_4 = 1. \tag{40}$$

We note now that the variational equations correspond to a periodic orbit $x(t)$. So, $\xi(t) = \dot{x}(t)$ is a periodic solution of the variational equations and according to section 4.1, one eigenvalue is equal to one, $\lambda_1 = 1$. Using now relation (40) we come to the conclusion that *the monodromy matrix of a Hamiltonian system corresponding to a periodic orbit has a double unit eigenvalue.*

$$\lambda_1 = 1, \quad \lambda_2 = 1. \tag{41}$$

6.4 Stability

The stability is determined by the two nonzero eigenvalues λ_3, λ_4 of the monodromy matrix. As we proved, these eigenvalues are reciprocal and also complex conjugate, so they are either on the unit circle, or on the real axis, one inside the unit circle and the other outside. A special case is $\lambda_3 = \lambda_4 = +1$ or $\lambda_3 = \lambda_4 = -1$. All these cases are shown in Figures 4-6. It is evident that the orbit is stable only in the case where the two nonzero eigenvalues λ_3, λ_4 are complex conjugate, on the unit circle (Figure 4). If the eigenvalues λ_3, λ_4 are real, the orbit is unstable, because one of them will be larger than $+1$ or smaller than -1.

The stability criteria can be obtained from the elements of the monodromy matrix as follows: The eigenvalues are the roots of the characteristic equation of $\Delta(T)$ and consequently

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \mathrm{trace}\Delta(T),$$

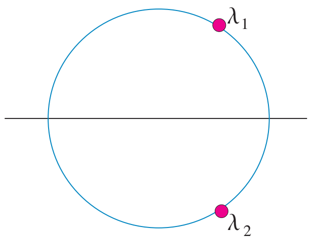

Figure 4: Eigenvalues complex conjugate on the unit circle

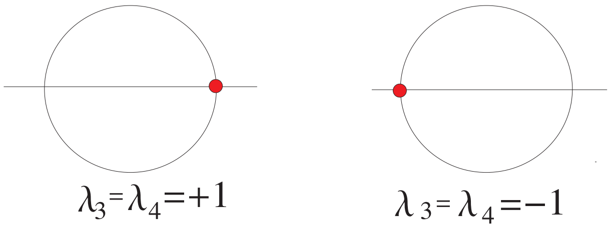

Figure 5: Eigenvalues at the critical points +1 and −1

$$\lambda_1\lambda_2\lambda_3\lambda_4 = \det\Delta(T) = 1.$$

Taking into account that $\lambda_1 = \lambda_2 = 1$ we find that the two nonzero eigenvalues λ_3, λ_4 are the roots of the quadratic equation

$$\lambda^2 - K\lambda + 1 = 0, \tag{42}$$

where

$$K = \mathrm{trace}\Delta(T) - 2. \tag{43}$$

The stability depends on the value of K, which is called the *stability index*. Note that the stability index depends only the trace of the monodromy matrix.

We have the following cases:

- $|K| < 2$, $-4 < \mathrm{trace}\Delta(T) < 4$: eigenvalues $\lambda_{3,4}$ complex conjugate on the unit circle, characteristic exponents $\alpha = \pm i\phi$ (Figure 4).
- $K = 2$, $\mathrm{trace}\Delta(T) = 4$: eigenvalues $\lambda_{3,4} = 1$, characteristic exponents $\alpha = 0$ (Figure 5a).

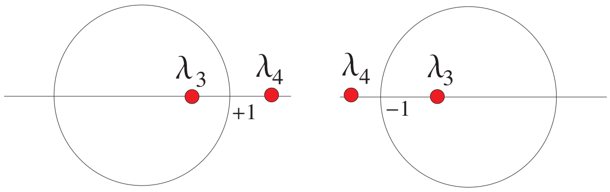

Figure 6: Real eigenvalues inside and outside the unit circle

- $K = -2$, trace$\Delta(T) = 0$: eigenvalues $\lambda_{3,4} = -1$, characteristic exponents $\alpha = 0 \pm i\pi/T$ (Figure 5b).
- $K > 2$, trace$\Delta(T) > 4$: eigenvalues $\lambda_{3,4}$ real and positive, $\lambda_3 = 1/\lambda_4$, characteristic exponents $\pm\alpha$, α real (Figure 6a).
- $K < -2$, trace$\Delta(T) < 0$: eigenvalues $\lambda_{3,4}$ real and negative, $\lambda_3 = 1/\lambda_4$, characteristic exponents $\pm(\alpha \pm i\pi/T)$, α real (Figure 6b).

We have stability if all the eigenvalues are on the unit circle. So, we have stability only in the case that the stability index is $-2 \le K \le 2$.

Asymptotic stability never appears, because it is not possible for the eigenvalues λ_3, λ_4 to be both inside the unit circle. This is also a consequence of the fact that the volume in phase space is conserved.

Let us assume that a periodic orbit is stable, which implies that the eigenvalues λ_3, λ_4 are on the unit circle (Figure 4) and we assume that they are not equal to $+1$ or -1. If a parameter varies, then the eigenvalues λ_3, λ_4 are restricted to move *on the unit circle*, because they must be both inverse, $\lambda_3 = 1/\lambda_4$ and complex conjugate. Consequently, the stability is conserved. However, if λ_3, λ_4 meet at the points $+1$ or -1, then it is possible for them to go outside the unit circle, as shown in Figure 6, and thus generate instability. For this reason the orbits with $\lambda_3 = \lambda_4 = \pm 1$ are called *critical* as far as the stability is concerned.

6.5 Periodic solution of the variational equations corresponding to a first integral of motion

$G(x_1, x_2, x_3, x_4)$: First integral of motion

$$[G, H] = 0 \rightarrow$$

$$\nabla[G, H] = 0 \;\; \rightarrow \;\; [\nabla G, H] + [G, \nabla H] = 0 \;\; \rightarrow$$

$$[\nabla G, H] = [\nabla H, G]$$

The variational equations have the solution

$$\eta = -J\nabla G$$

PROOF

$$\dot{\eta} = -J\frac{d}{dt}\nabla G = -J[\nabla G, H]$$

$$\rightarrow \;\; \dot{\eta} = -J[\nabla H, G]$$

we use

$$[P, Q] = -(\nabla P)^\tau J(\nabla Q)$$

and obtain

$$\dot{\eta} = J\nabla(\nabla H)^\tau J\nabla G$$

$$\dot{\eta} = -J(\nabla(\nabla H)^{\tau})\eta$$

This means that η satisfies the system of variational equations.

Remark

We used the definition that

$$[\nabla P, Q]$$

is the vector

$$\begin{pmatrix} \left[\frac{\partial P}{\partial x_1}, Q\right] \\ \left[\frac{\partial P}{\partial x_2}, Q\right] \\ \left[\frac{\partial P}{\partial x_3}, Q\right] \\ \left[\frac{\partial P}{\partial x_4}, Q\right] \end{pmatrix}$$

The system of variational equations can be expressed as

$$\dot{\xi} = -J\nabla(\nabla)^{\tau}\xi$$

where $\nabla(\nabla)^{\tau}$ is the matrix with elements

$$\partial^2 H/\partial x_i \partial x_j$$

6.6 Solution in the vicinity of a T-periodic orbit $x(t)$

An orbit $x'(t)$ in the vicinity of the periodic orbit $x(t)$ is expressed in the form

$$x'(t) = x(t) + \xi(t), \tag{44}$$

to a linear approximation in the deviations, where $\xi(t)$ is the solution of the variation equations. If $\xi(t)$ is bounded then $x'(t)$ is also bounded.

The properties of the orbit $x'(t)$, in the vicinity of the periodic orbit $x(t)$, are determined from the properties of the solution $\xi(t)$ of the variational equations. In particular, if a T-periodic solution $\xi(t)$ exists, then a new T-periodic orbit $x'(t)$ exists.

We remark that the periodic solution of the variational equations $\xi(t) = \dot{x}(t)$ does not generate a new periodic orbit $x'(t)$, but the same orbit $x(t)$, with a phase shift, because $\xi(0)$ is tangent to the orbit $x(t)$ (Figure 3).

General solution of the variational equations

To each eigenvalue of $\Delta(T)$ there corresponds a solution $\xi(t)$, so we have four linearly independent solutions. However, to the double unit eigenvalue $\lambda_1 = \lambda_2 = 1$, there exists only one unit eigenvector $\xi(0) = \dot{x}(0)$. The two linearly independent solutions corresponding to the double unit eigenvalue are

$$\begin{aligned} \xi^1 &= f_1(t), \\ \xi^2 &= f_2(t) + t\, f_1(t), \end{aligned} \tag{45}$$

where $f_1(t),\ f_2(t)$ are T-periodic.

For the other two eigenvalues $\lambda_3 = \lambda_4$ we have the solutions

- eigenvalues real and positive
  $\xi^{3,4} = f_{3,4}(t)\, e^{\pm\alpha t}$
- eigenvalues real and negative
  $\xi^{3,4} = f_{3,4}(t)\, e^{\pm\alpha t}\, e^{\pm i\pi t/T}$
- eigenvalues complex conjugate on the unit circle $\xi^{3,4} = f_{3,4}(t)\, e^{\pm i\beta t}$

where the functions $f_3(t),\ f_4(t)$ are T-periodic.

The initial conditions for the above four special solutions are $f_i(0),\ i = 1, ...4$ and are given by

$$\begin{aligned} (\Delta(T) - I)\, f_1(0) &= 0, \\ (\Delta(T) - I) f_2(0) &= T f_1(0), \\ (\Delta(T) - \lambda_{3,4} I)\, f_{3,4}(0) &= 0, \end{aligned} \tag{46}$$

The general solution is a linear combination of the above four solutions:

Characteristic exponents real and positive.

$$\begin{aligned} \xi(t) &= c_1 f_1(t) + c_2(f_2(t) + t\, f_1(t)) \\ &\quad + c_3 f_3(t) e^{\alpha t} + c_4 f_4(t) e^{-\alpha t}. \end{aligned} \tag{47}$$

The orbit $x(t)$ is unstable.

Characteristic exponents real and negative.

$$\begin{aligned} \xi(t) &= c_1 f_1(t) + c_2(f_2(t) + t\, f_1(t)) \\ &\quad + c_3 f_3(t) e^{\alpha t} e^{i\pi t/T} + c_4 f_4(t) e^{-\alpha t} e^{-i\pi t/T}. \end{aligned} \tag{48}$$

The orbit $x(t)$ is unstable.

Characteristic exponents complex, on the unit circle.

$$\begin{aligned} \xi(t) &= c_1 f_1(t) + c_2(f_2(t) + t\, f_1(t)) \\ &\quad + c_3 f_3(t) e^{i\beta t} + c_4 f_4(t) e^{-i\beta t}. \end{aligned} \tag{49}$$

The orbit $x(t)$ is stable.

The above expressions give the totality of orbits in the vicinity of the periodic orbit $x(t)$ for all possible cases.

<u>Isoenergetic displacements</u>

Let us consider a periodic orbit $x(t)$, with initial conditions $x(0)$ and a nearby orbit $x'(t) = x(t) + \xi(t)$, with initial conditions $x'(0) = x(0) + \xi(0)$. The displacement vector $\xi(0)$ can be expressed as a linear combination of the four vectors $f(0)$ defined by Equations (46),

$$\xi(0) = c_1 f_1(0) + c_2 f_2(0) + c_3 f_3(0) + c_4 f_4(0), \tag{50}$$

where c_i are arbitrary constants. We shall prove that the displacement $\xi(0)$ is isoenergetic (the energy, i.e. the Hamiltonian, is not changed) if

$$c_2 = 0.$$

Proof

The Hamiltonian is constant along the periodic orbit $x(t)$,

$$H(x_1, x_2, x_3, x_4) = \text{constant}.$$

The change of H due to the displacement $\xi(0)$ is given by

$$\delta H = (\nabla H_0, \xi(0)),$$

where ∇H_0 is computed at the initial conditions $x_i(0)$.

Consider first $c_1 \neq 0$, $c_2 = c_3 = c_4 = 0$. Then $\xi(0) = c_1 f_1(0) = c_1 \dot{x}(0)$ or, due to the Hamiltonian equations, $\dot{x}_i(0) = J\nabla H_0$ and finally

$$\delta H = (\nabla H_0, c_1 J \nabla H_0) = 0. \quad (51)$$

Next, consider $c_3 \neq 0, c_1 = c_2 = c_4 = 0$, i.e. $\xi(0) = c_3 f_3(0)$. The change of the Hamiltonian is

$$\delta H = c_3(\nabla H_0, f_3(0)).$$

We take now into account that

$$\dot{x}(0) = f_1(0) = J\nabla H_0,$$

and

$$\nabla H_0 = J^{-1} f_1(0) = -J f_1(0),$$

and obtain finally

$$\delta H = -c_3(J f_1(0), f_3(0)). \quad (52)$$

We shall prove that

$$(J f_1(0), f_3(0)) = 0. \quad (53)$$

We have

$$f_1(0) = \Delta(T) f_1(0), \quad f_3(0) = \frac{1}{\lambda_3} \Delta(T) f_3(0)$$

and

$$(J f_1(0), f_3(0)) = (J f_1(0)^\tau . f_3(0)) =$$
$$\frac{1}{\lambda_3} f_1(0)^\tau \left(\Delta^\tau(T) J^\tau \Delta(T)\right) f_3(0),$$

or, using $J_\tau = -J$,

$$-f_1^\tau(0) J f_3(0) = -\frac{1}{\lambda_3} f_1^\tau(0) \left(\Delta^\tau(T) J \Delta(T)\right) f_3(0)$$

$$= \frac{1}{\lambda_3} f_1^\tau(0) J f_3(0).$$

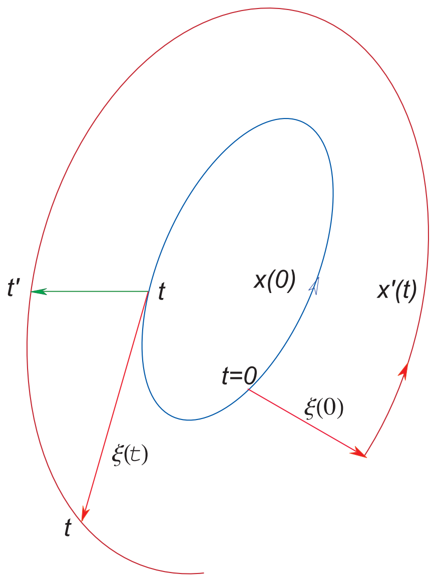

Figure 7: Orbital stability

From this latter relation we obtain

$$\left(1+\frac{1}{\lambda_3}\right) f_1^{\tau}(0) J f_3(0) = 0,$$

and for $\lambda_3 \neq -1$ we have

$$(J f_1(0), f_3(0)) = 0.$$

Thus, from Equation (51) we obtain $\delta H = 0$. A similar proof holds for $\xi(0) = c_4 f_4(0)$. Thus we come to the conclusion that

- Any perturbation which is a linear combination of $f_1(0)$, $f_3(0)$, $f_4(0)$ is isoenergetic.
- For an isoenergetic displacement, no secular term appears in the general solution, as is readily obtained from Equations(47),(48) and (49).
- In order to obtain a change of the energy, we must have a displacement along the vector $f_2(0)$.

6.7 Orbital stability

The stability that we mentioned before refers to the evolution of the deviation vector $\xi(t) = x'(t) - x(t)$ between the perturbed solution $x'(t)$ and the periodic orbit $x(t)$ *at the same time* t. If $\xi(t)$ is bounded, then the periodic orbit is stable. This would happens if all the eigenvalues of the monodromy matrix are on the unit circle in the complex plane. In this case two particles, one on the periodic orbit $x(t)$ and the other on the perturbed orbit $x'(t)$, that start close to each other at $t = 0$, would stay always close. This however is not true in a Hamiltonian system, because there is only one eigenvector corresponding to the double unit eigenvalue and consequently a secular term appears always in the general solution, as can be seen from Equation (49). We remark that this secular term appears if the vector of initial deviation $\xi(0) = x'(0) - x(0)$ has a component along the direction $f_2(0)$.

In order to understand the meaning of the secular term, we consider the initial conditions

$$x'(0) = x(0) + \epsilon f_2(0). \tag{54}$$

The corresponding solution is

$$x'(t) = x(t) + \epsilon f_2(0) + \epsilon t f_1(0),$$

and using $f_1(t) = \dot{x}(t)$ we obtain

$$x'(t) = x(t) + \epsilon t \dot{x}(t) + \epsilon f_2(t), \tag{55}$$

and to a linear approximation in ϵ,

$$x'(t) = x(t + \epsilon t) + \epsilon f_2(t + \epsilon t). \tag{56}$$

We define the time

$$t' = t + \epsilon t,$$

and come finally to the conclusion that

$$x'(t) - x(t') = \text{bounded}.$$

Thus we come to the conclusion that the secular term introduces a phase shift only along the orbit. This means that the two orbits, $x(t)$ and $x'(t)$, considered as geometrical curves, are close to each other (Figure 7). In this case we say that we have *orbital stability*, provided that the eigenvalues λ_3, λ_4 are on the unit circle and consequently the corresponding solution is bounded.

6.8 Families of periodic orbits

Consider the system of differential equations (not Hamiltonian in general)

$$\dot{x}_i = F_i(x_1, x_2, x_3, x_4), \quad (i = 1, ...4) \tag{57}$$

and let $x_{i0} = x_i(0)$ be the initial conditions corresponding to a T-periodic orbit. The solution is expressed in the form $x_i(x_{10}, x_{20}, x_{30}, x_{40}; t)$ and the periodicity conditions are

$$x_i(x_{10}, x_{20}, x_{30}, x_{40}; T) - x_{i0} = 0. \quad (i = 1, ...4) \tag{58}$$

A new periodic orbit with period $T + \delta T$, in the vicinity of the above T-periodic solution exists if the Jacobian determinant of the left-hand-side of Equation (58), computed at the initial conditions x_{i0}, is different from zero (implicit function theorem). In this case the T-periodic orbit $x(t)$ is not isolated, but belongs to a family of periodic orbits along which he initial conditions and the period vary. The Jacobian determinant is

$$|\Delta(T) - I_4|, \tag{59}$$

which however is always equal to zero, because the monodromy matrix $\Delta(T)$ has a unit eigenvalue.

In order to overcome the problem, we keep one initial condition fixed, e.g. x_{20} and vary the other three initial conditions x_{10}, x_{30}, x_{40}. We consider the first three periodicity conditions. The corresponding Jacobian matrix is

$$|\Delta_3(T) - I_3|, \tag{60}$$

where $\Delta_3(T)$ is the matrix obtained from the monodromy matrix $\Delta(T)$ by deleting the second column (because x_{2o} is not considered as variable) and the fourth row (because the fourth periodicity condition is not considered). If this latter Jacobian matrix is not equal to zero (and this is the case in general), a new periodic solution exists in the vicinity of the periodic orbit $x(t)$.

The non vanishing of the matrix (59) secures that the first three periodicity conditions are satisfied for the new initial conditions x'_{10}, x_{20}, x'_{30}, x'_{40}, with the new period T':

$$x_1(T') = x'_{10}, \quad x_2(T') = x_{20}. \quad x_3(T') = x'_{30} \tag{61}$$

From the above we see that the three first periodicity conditions (58) are satisfied. In order for a new periodic orbit to exist, we must prove that the fourth periodicity condition is also satisfied.

We assume that an integral of motion exists

$$G(x_1, x_2, x_3, x_4) = \text{constant}.$$

Let x''_{40} be the value of x_4 after one period T'. We have

$$G(x'_{10}, x_{20}, x'_{30}, x'_{40}) - G(x'_{10}, x_{20}, x'_{30}, x''_{40}) = 0,$$

and by Taylor's theorem,

$$\left(\frac{\partial G}{\partial x_4}\right)_{x'''_{40}} (x'_{40} - x''_{40}) = 0, \tag{62}$$

where the partial derivative is computed for $(x'_{10}, x_{20}, x'_{30}, x'''_{40})$, x'''_{40} being between x'_{40} and x''_{40}. Consequently

$$x''_{4o} = x'_{4o},$$

which implies that the fourth periodicity condition is also satisfied, provided that the partial derivative in (62) is not equal to zero.

We shall find now the new periodic orbit, to a linear approximation. Consider the orbit $x'(t)$, corresponding to the initial conditions

$$x'(0) = x(0) + c_2 f_2(0), \tag{63}$$

for c_2 small. The solution is given by Equation (56), and can be expressed, to a linear approximation in the small parameter c_2, as

$$x'(t) = x((1 + c_2)t) + c_2 f_2((1 + c_2)t).$$

The functions $x(t)$, $f_2(t)$ are T-periodic and consequently the new solution $x'(t)$ is $(T/(1+c_2))$-periodic, and to first order in c_2, with period

$$T' = (1 - c_2)T.$$

From the above we conclude that a family of periodic orbits is obtained, by varying the initial conditions of $x(0)$ along the vector $f_2(0)$, as is seen from Equation (63). We remark that it is along this vector that the energy (Hamiltonian) varies. The initial conditions of all the periodic orbits of the family lie on a curve that is called the *characteristic curve*. The vector $f_2(0)$ is tangent to the characteristic curve. The period and the energy vary along the family.

If the periodic orbits are symmetric with respect to the x_1-axis, then $x_{20} = \dot{x}_{10} = 0$ and the characteristic curve is a curve in the space of two initial conditions only, $x_{10}, \dot{x}_{20}$.

Remark: The procedure described in this section, for the continuation of a periodic orbit, was for a change of the initial conditions, keeping the parameters of the system fixed. In this case the period of the new periodic orbit was, in general, different from the initial period. It may happen however that the system of differential equations (57) depends on a parameter and we wish to study the continuation of the periodic orbit with respect to this parameter. The procedure is the same, but now we fix the period T and study the continuation to a new periodic orbit with the same period.

6.9 Bifurcations of families of periodic orbits in the plane

In the general solution for the displacement $\xi(t)$, there is a term of the form

$$c_3 f_3(t) e^{\sigma t} + c_4 f_4(t) e^{-\sigma t}, \tag{64}$$

where σ is either real or complex, on the unit circle (characteristic exponent). The functions $f_3(t)$, $f_4(t)$ are T-periodic.

The initial conditions that generate this component of the solution are

$$x'(0) = x(0) + c_3 f_3(0) + c_4 f_4(0). \tag{65}$$

Bifurcation with the same period

If for an orbit of the family we have $\sigma = 0$, or $\sigma = i2\pi\nu$, and we select the initial conditions (65) for $\xi(0)$, the perturbed solution

$$x'(t) = x(t) + \xi(t)$$

is also T-periodic.

A new family of periodic orbits bifurcates from the T-periodic orbit $x(t)$ of the original family, starting with the same period. Note that $\sigma = 0$ implies $\lambda_3 = \lambda_4 = 1$, which means that the bifurcation takes place at the point of the original family where the stability changes (critical points with respect to the stability).

Bifurcation with multiple period

Assume now that the characteristic exponent of a T-periodic orbit $x(t)$ of the family of periodic orbits is equal to

$$\sigma = i2\pi\frac{p}{q}.$$

If this periodic orbit $x(t)$ is described q times, then the characteristic exponent is

$$\sigma = i2\pi p \;\rightarrow\; e^{\pm\sigma} = 1.$$

This means that we also have a bifurcation of a new family of periodic orbits, from the T-periodic orbit $x(t)$ of the original family, but now the new periodic orbits have a period equal to qT.

7 Vertical stability of planar periodic orbits

In the previous sections we studied the stability of a planar periodic orbit with respect to perturbations of the initial conditions *in the plane*. We shall study now the stability with respect to perturbations of the initial conditions perpendicular to the plane of motion. This type of stability is called *vertical stability* and completes the study of the stability of a planar periodic orbit.

Consider a dynamical system of three degrees of freedom, of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3), \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3), \\ \dot{x}_3 &= x_3 f_3(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3). \end{aligned} \tag{66}$$

This is the form of the differential equations of many gravitational systems, for example of the 3-dimensional three body problem.

It is easy to verify that the equations (66) admit a planar solution, which we will assume to be periodic:

$$x_1(t),\;\; x_2(t),\;\; x_3(t) = 0, \tag{67}$$

corresponding to the initial conditions

$$x_{10},\;\; x_{20},\;\; x_{30} = 0,\;\; \dot{x}_{10},\;\; \dot{x}_{20},\;\; \dot{x}_{30} = 0.$$

We consider now a small perturbation along the x_3 axis also,

$$x_{10} + \epsilon_1,\;\; x_{20} + \epsilon_2,\;\; x_{30} = 0 + \epsilon_3,\;\; \dot{x}_{10} + \epsilon_4,\;\; \dot{x}_{20} + \epsilon_5,\;\; \dot{x}_{30} = 0 + \epsilon_6,$$

where ϵ_i are small, and we want to study the behaviour of the perturbed solution. We define new variables

$$x_4 = \dot{x}_1,\; x_5 = \dot{x}_2,\; x_6 = \dot{x}_3,$$

and a simple calculation shows that the system of variational equations of the system (66), for the periodic solution (67), breaks into two uncoupled systems: a system in the planar displacements $\xi_1, \xi_2, \xi_4, \xi_5$, corresponding to the variational equations of the planar motion, and a system in the vertical displacements (along the x_3 axis) ξ_3, ξ_6. This latter system is

$$\begin{aligned} \dot{\xi}_3 &= \xi_6, \\ \dot{\xi}_6 &= f_{30}(t)\xi_3, \end{aligned} \tag{68}$$

where the function $f_{30}(t)$ is the T-periodic function,

$$f_3(x_1(t), x_2(t), x_3 = 0, x_4(t), x_5(t), x_6 = 0),$$

computed for the planar T-periodic solution (67). The system (68) is the system of variational equations for the displacements along the x_3 axis. The vertical stability depends on the eigenvalues λ_5, λ_6 of the monodromy matrix $\Delta_2(T)$ of this system.

If the system is Hamiltonian, $\Delta_2(T)$ is symplectic. Then, either λ_5, λ_6 are real, of the form

$$\lambda_{5,6} = e^{\pm\alpha t},$$

and we have *vertical instability*, or they are complex conjugate on the unit circle, of the form

$$\lambda_{5,6} = e^{\pm i\varphi t},$$

and we have *vertical stability.*

The α or $i\varphi$ are the characteristic exponents for the vertical perturbations, (along the x_3 axis).

The solution of the system of vertical variational equations is of the form

$$\xi = c_1 f_1(t) e^{-\sigma t} + c_2 f_2(t) e^{\sigma t}, \tag{69}$$

where ξ is a vector with elements ξ_3, ξ_6, for σ equal to α (vertically unstable) or to $i\varphi$ (vertically stable).

If for a periodic orbit of the planar family we have $\sigma = 0$ (vertically critical orbit), then $\xi_3(t)$ is T-periodic. Then we have a bifurcation of a family of 3-dimensional periodic orbits from this point of the planar family.

8 Extension to three or more degrees of freedom

All the above results concerning the eigenvalues and the stability of a periodic orbit, obtained in systems with two degrees of freedom, can be easily extended to three or more degrees of freedom.

In a Hamiltonian system the monodromy matrix is a $2n \times 2n$ symplectic matrix, and the eigenvalues are in reciprocal pairs (because of the symplectic property), and in complex conjugate pairs (because the elements of the matrix are real).

There are the following possibilities for the eigenvalues:

- Complex conjugate on the unit circle, $e^{\pm i\phi}$ (Figure 8): STABILITY
- Real, on the real axis, in reciprocal pairs (positive or negative), λ, $1/\lambda$ (Figure 9): INSTABILITY
- Complex, inside and outside the unit circle, in reciprocal and in complex conjugate pairs (Figure 10), $Re^{i\phi}$, $Re^{-i\phi}$, $R^{-1}e^{i\phi}$, $R^{-1}e^{-i\phi}$: COMPLEX INSTABILITY

Note that in three, or more, degrees of freedom we have a new type of instability, *complex instability*, which cannot appear in systems with two degrees of freedom.

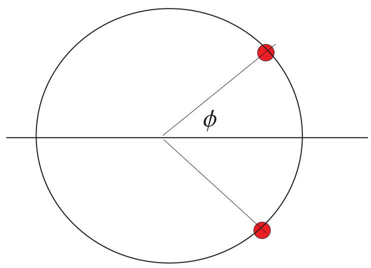

Figure 8: Position of the eigenvalues on the unit circle: Stability

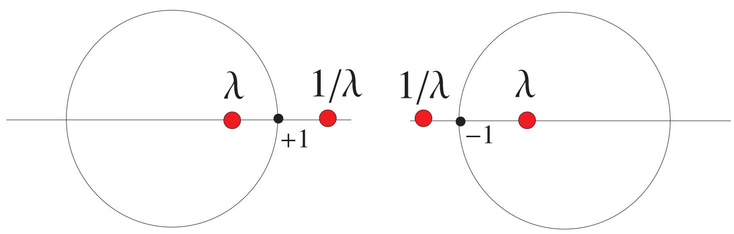

Figure 9: Position of the eigenvalues on the real axis: Instability

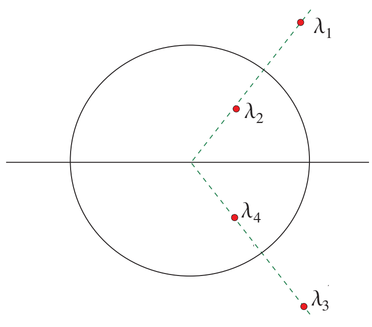

Figure 10: Position of the eigenvalues inside and outside the unit circle in reciprocal pairs and in complex conjugate pairs: Complex instability

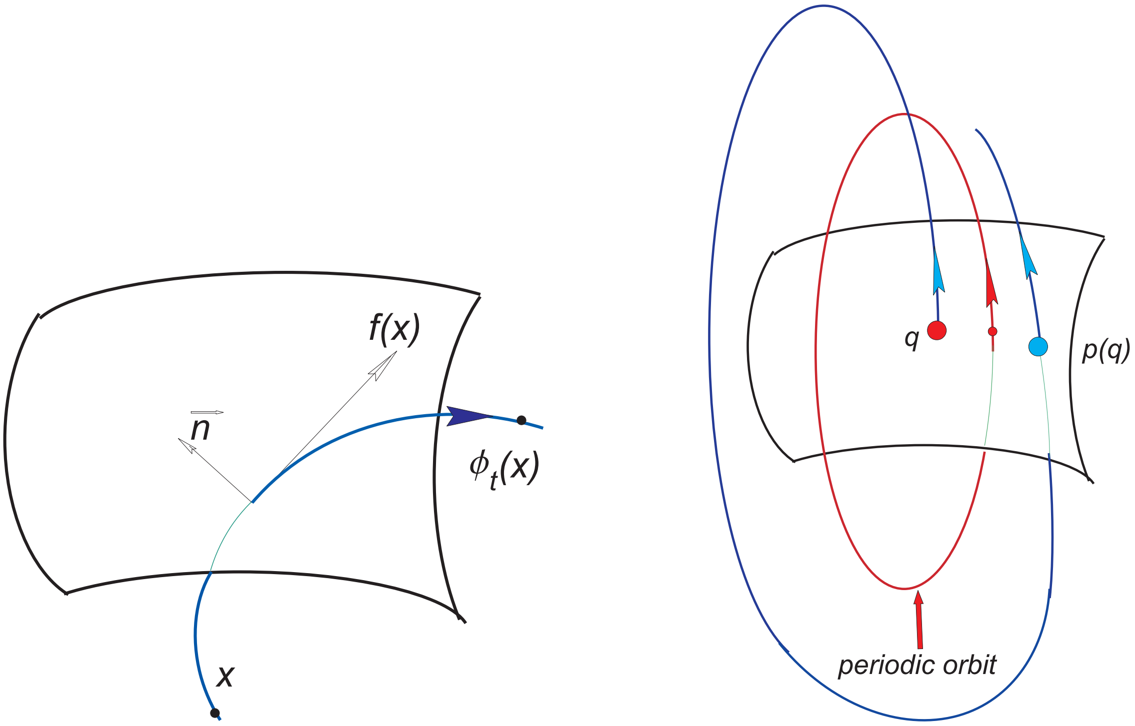

Figure 11: (a):The surface of section. (b) The Poincaré map on the surface of section

9 Poincaré map

The Poincaré map is a method to transform the continuous flow in n-dimensional phase space to an equivalent discrete flow (map) in a phase space of $(n-1)$-dimensions (or $(n-2)$-dimensions for Hamiltonian flows).

Let us consider the dynamical system in $\mathcal{R}^n$

$$\dot{x} = f(x), \tag{70}$$

where x, $f(x)$ are vectors in $\mathcal{R}^n$ and call $\phi_t(x)$ the flow defined by Equation (70). We consider a *surface of section* Σ,

$$\Sigma \subset \mathcal{R}^n : \quad (n-1) - \text{dim}$$

and we assume that the flow is transverse. This means that the velocity vector $f(x)$ of the flow is not tangent to the surface, i.e. $f(x) \cdot n(x) \neq 0$, where $n(x)$ is the tangent unit vector to the surface Σ (Figure 11a).

The Poincaré map, Figure 11b, is the map

$$q \to p(q), \;\text{ where }\; p(q) = \phi_\tau(q),$$

where τ is the time from the point of intersection q to the next point $p(q)$. We note that

- The consecutive points on the surface of section, q, $p(q)$, .. define accurately the state of the system.

- $p(q)$ is a continuous function of q.
- If $\bar{x}(t)$ is a T-periodic orbit (Figure 11b), the corresponding Poincaré map is a *fixed point* (maybe multiple).
- It can be proved that the stability of the invariant point is the same as the stability of the corresponding periodic orbit. The fixed point has the same set of eigenvalues, except the unit eigenvalue (which corresponds to the periodic orbit).

10 Poincaré map in Hamiltonian systems

We consider the canonical equations

$$\dot{q} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial q, \tag{71}$$

where

$$q, p \subset \mathcal{R}^n$$

and consider as surface of section Σ the $(2n-2)$-dimensional surface defined by

$$H = h, \quad f(q,p) = 0 \ \ \text{(for example } q_2 = 0\text{)}. \tag{72}$$

The intersections of the continuous Hamiltonian flow in the $2n$-dimensional phase space, defined by Equation (71), with the surface of section defined by Equation (72) transforms the continuous flow to an equivalent discrete flow (map), on a $(2n-2)$-dimensional surface of section (Figure 12).

10.1 Hamiltonian systems with two degrees of freedom

We will study in particular Hamiltonian systems with two degrees of freedom with Hamiltonian $H(q_1, q_2, p_1, p_2)$. We define

$$x_1 = q_1, \ x_2 = q_2, \ x_3 = p_1, \ x_4 = p_2,$$

and consider the surface of section

$$H(x_1, x_2, x_3, x_4) = h, \quad x_2 = 0, \ \text{ with } x_4 > 0.$$

The map is on the two-dimensional space $x_1\, x_3$. The values of x_1, x_3 define exactly the state, because $x_2 = 0$ and x_4 is obtained from $H = h$, $x_4 > 0$.

The map is two-dimensional and can be expressed as

$$x_1 = g_1(x_1, x_3),$$

$$x_3 = g_2(x_1, x_3).$$

The map is symplectic.

The fixed points of the map correspond to the periodic orbits of the Hamiltonian system.

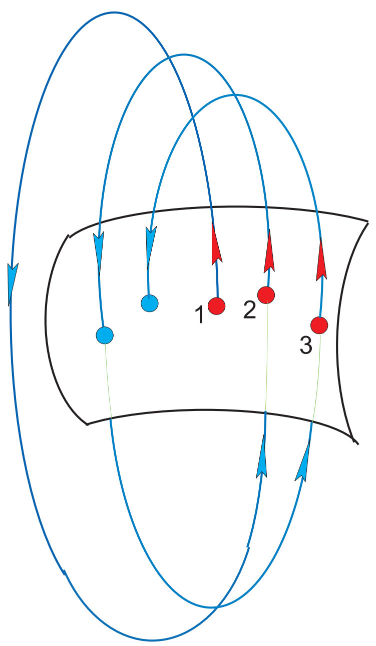

Figure 12: The consecutive points of intersection

The set of eigenvalues of a fixed point is the same as the set of the eigenvalues of the monodromy matrix of the periodic orbit, minus the two unit eigenvalues.

The stability of a fixed point is the same as the stability of the periodic orbit. Note that the shift along the perturbed orbit (due to the double eigenvalue), compared to the periodic orbit (see Figure 7), does not produce instability to the fixed point, so the stability of the fixed point of the Poincaré map is in fact *orbital stability*.

10.2 Invariant curves

The consecutive points of the map may lie on a smooth curve (ordered motion), or be scattered (chaotic motion).

Let us assume that another first integral of motion exists, independent of H =constant,

$$G(x_1, x_2, x_3, x_4) = c.$$

Then all the consecutive points of the map lie on smooth invariant curves. This can be proved as follows: Let (x_1, x_3) be a point of the map on the two-dimensional surface of section. We have $x_2 = 0$ and x_4 is expressed in terms of x_1, x_3, through the energy integral $H = h$, as

$$x_4 = x_4(x_1, x_2 = 0, x_3).$$

The points x_1, x_3 satisfy also the integral

$$G(x_1, x_2 = 0, x_3, x_4(x_1, x_2 = 0, x_3)) = c,$$

which is a relation of the form

$$F(x_1, x_3) = 0,$$

and this implies that the consecutive points (x_1, x_3) of the map lie on a smooth curve.

11 Change of the stability as a parameter varies. Theory of Krein

- The stability of a periodic orbit is determined by the eigenvalues of the monodromy matrix $\Delta(T)$.
- A periodic orbit belongs to a monoparametric family of periodic orbits, along which the period T varies.
- The monodromy matrix depends on the period T.
- This implies that the monodromy matrix varies along the family of periodic orbits.
- The monodromy matrix is symplectic. Consequently, we have a symplectic perturbation of $\Delta(T)$ as we move along the family of periodic orbits.
- Can we have a change from stability to instability along the family?
- What are the dangerous points where a change of stability can take place?

Theory of Krein

The set of eigenvalues of the monodromy matrix are arranged in reciprocal pairs and in complex conjugate pairs.

How do they change under a symplectic perturbation?

Let us assume that all eigenvalues are on the unit circle: STABILITY

Can they move outside the unit circle, and thus generate instability, as the monodromy matrix is perturbed by a symplectic perturbation? (as it happens when we move along the family of periodic orbits).

The eigenvalues that are on the unit circle move, due to the symplectic perturbation of $\Delta(T)$, obeying the Hamiltonian properties (reciprocal pairs, complex conjugate pairs).

Instability is generated only when two eigenvalues move outside the unit circle.

For continuity reasons, this can take place only when two eigenvalues meet on the unit circle.

Does the meeting of two eigenvalues generate instability?

NO, this happens only in some cases.

When is this meeting dangerous?

Let λ be an eigenvalue of the monodromy matrix on the unit circle (multiple in general), and g the corresponding set of eigenvectors. We define the indefinite scalar product $\langle g, g\rangle$ as

$$\langle g, g\rangle = i(Jg, g) = ig^* Jg$$

where i is the imaginary unit and J is the symplectic matrix, defined before. Note that this indefinite scalar product is always real.

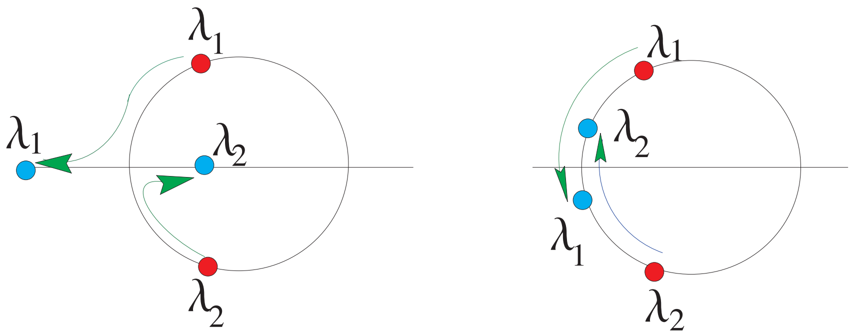

Figure 13: The displacement of the eigenvalues on the unit circle as a parameter varies

The eigenvalue λ is defined as of the first kind, second kind or mixed kind according to

$$\langle g, g \rangle = \left\{ \begin{array}{ll} \text{positive for all } g: & \text{first kind} \\ \text{negative for all } g: & \text{second kind} \\ \text{positive or negative:} & \text{mixed kind} \end{array} \right\}$$

The eigenvalues of the first kind or the second kind are called definite.

- Instability is generated only by a meeting of eigenvalues of DIFFERENT kinds,
- of through a multiple eigenvalue of MIXED kind.
- STRONG STABILITY: All eigenvalues are on the unit circle and are definite.

In the planetary problem (in the model of the restricted or the general three body problem), the meeting of eigenvalues and the generation of instability takes place at the resonance cases.

12 Near-integrable systems

Let us consider the Hamiltonian system

$$H = H_0(q_1, q_2, p_1, p_2) + \epsilon H_1(q_1, q_2, p_1, p_2),$$

where H_0 is integrable.

Action-angle variables of the integrable part:

$$J_1,\ J_2,\ \theta_1,\ \theta_2$$

The Hamiltonian takes the form

$$H = H_0(J_1, J_2) + \epsilon H_1(J_1, J_2, \theta_1, \theta_2).$$

The flow of the unperturbed Hamiltonian $H_0(J_1, J_2)$ is given by the equations

$$J_1 = J_{10},\ \ J_2 = J_{20},\ \ \dot{\theta}_1 = n_1,\ \ \dot{\theta}_2 = n_2,$$

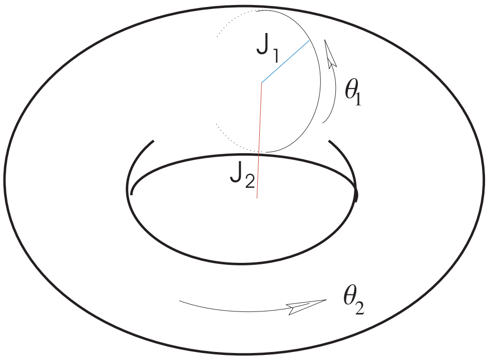

Figure 14: The 2-torus

where the frequencies n_1, n_2 are given by

$$n_1 = \partial H_0/\partial J_1, \;\; n_2 = \partial H_0/\partial J_2.$$

The flow takes place on a 2-torus, with constant radii J_1, J_2 and constant frequencies n_1, n_2. The unperturbed motion is on a 2-torus

We define now a Poincaré map on the surface of section

$$H_0(J_1, J_2) = h, \;\; \theta_2 = 0 \mod 2\pi,$$

(or equivalently, $J_2 = J_{20}$ and $\theta_2 = 0$).

The unperturbed mapping can be obtained analytically as

$$\begin{array}{rcc} J_1 \rightarrow & & J_1, \\ \theta_1 \rightarrow & & \theta_1 + 2\pi\frac{n_1}{n_2}. \end{array}$$

This is a mapping in the variables J_1 θ_1 and depends on the parameter h (or J_2).

At each iteration the angle θ_1 changes by

$$\delta\theta_1 = 2\pi\frac{n_1}{n_2}.$$

The angle $\delta\theta_1$ is called the rotation angle.

Depends on the action J_1 and the parameter h (or J_2).

The map is a twist map.

The map is usually presented in the Poincaré variables .

$$X = \sqrt{2J_1}\cos\theta_1, \;\; Y = \sqrt{2J_1}\sin\theta_1.$$

The surface of section on the 2-torus and the circular invariant curves of the unperturbed motion, in X, Y variables.

All consecutive points of the map lie on smooth invariant curves that are circles with radii $\rho = \sqrt{2J_1}$. The rotation angle $\delta\theta_1$ as well as the ratio $\frac{n_1}{n_2}$ varies along the radius ρ.

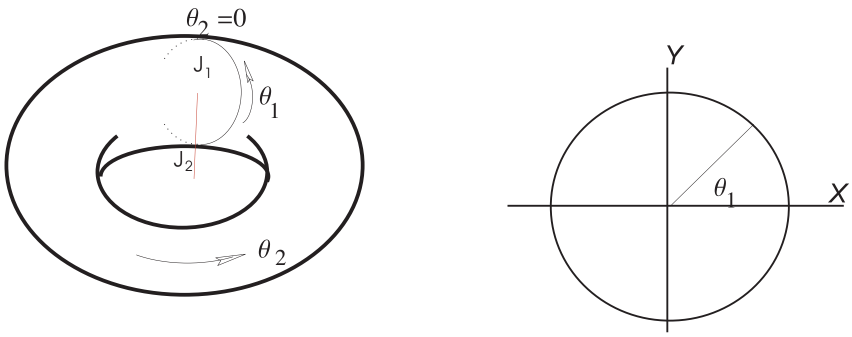

Figure 15: The mapping in action-angle variables

There are certain radii that are *resonant*: the ratio of the frequencies is rational,

$$\frac{n_1}{n_2} = \frac{s}{r}, \quad \text{or} \quad \Delta\theta_1 = 2\pi\frac{s}{r}.$$

All points on a resonant invariant curve (circle) are r-multiple fixed points: The point comes to the initial position after r rotations along the angle θ_2 on the 2-torus. ($\delta\theta_1 = 2\pi s$ and $\delta\theta_2 = 2\pi r$).

The unperturbed mapping can be obtained from the generating function

$$F = J_{1,n+1}\theta_{1,n} + G_0(J_{1,n+1}),$$

where G_0 is defined by

$$\partial G_0/\partial J_1 = 2\pi n_1/n_2,$$

by making use of the equations

$$J_{1,n} = \frac{\partial F}{\partial \theta_{1,n}}, \quad \theta_{1,n+1} = \frac{\partial F}{\partial J_{1,n+1}}.$$

The mapping is symplectic.

We assume now that $\epsilon \neq 0$ in the Hamiltonian H. Then, for sufficiently small ϵ, we have a perturbed twist map

$$\begin{aligned} J_1 &= J_1 + \epsilon(...) \\ \theta_1 &= \theta_1 + 2\pi\frac{n_1}{n_2} + \epsilon(...). \end{aligned}$$

The perturbed map is obtained from a generating function

$$F = J_{1,n+1}\theta_{1,n} + G_0(J_{1,n+1}) + \epsilon G_1(J_{1,n+1}, \theta_{1,n}).$$

What happens to the unperturbed circular invariant curves when $\epsilon > 0$?

- Non-resonant invariant curve: survives as closed invariant curve (KAM theorem).
- Resonant invariant curve: Out of the infinite set of r-multiple fixed points, only a finite (even) number survive, half stable and half unstable (Poincaré-Birkhoff fixed point theorem).

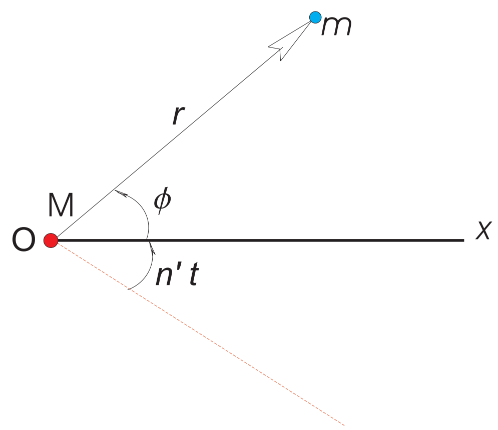

Figure 16: The rotating x-axis

13 The gravitational two-body problem

Relative motion of two point masses M, m:

$$\ddot{\vec{r}} = -\frac{G(M+m)}{r^2}\,\vec{e}_r$$

Equivalent to central motion of a body of unit mass around a body of mass $M+m$.

If m is negligible, compared to M,

$$\ddot{\vec{r}} = -\frac{GM}{r^2}\,\vec{e}_r$$

Potential function

$$V(r) = -\frac{GM}{r}$$

<u>The two-body problem in a rotating frame</u>

Polar coordinates r, ϕ

The angle ϕ is with respect to a rotating axis, which rotates with constant angular velocity n'.

Lagrangian (per unit mass):

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2(n'+\dot{\phi})^2\right) + \frac{GM}{r}$$

The momenta are

$$p_r = \dot{r}$$
$$p_\phi = r^2(\dot{\phi}+n')$$

Hamiltonian function (in rotating frame)

$$H_o = \frac{p_r^2}{2} + \frac{p_\phi^2}{2r^2} - n' p_\phi - \frac{GM}{r}$$

In terms of the elements of the orbit

$$H_o = -\frac{GM}{2a} - n' p_\phi,$$

$$p_\phi = \sqrt{GMa(1-e^2)}.$$

<u>Circular orbits</u>

Radius r_o, arbitrary

Periodic (circular) solution (rotating frame)

$$r = r_o,\ p_r = 0,\ \dot{\phi} = n - n',\ p_\phi = n r_o^2$$

where

$$n = \frac{p_\phi^2}{r_o^2}$$

is the angular velocity of the circular velocity (in the inertial frame).

The following relations hold (circular solution):

$$\frac{p_{\phi o}^2}{r_o^3} = \frac{GM}{r_o^2} \ \rightarrow\ \frac{GM}{r_o^3} = n^2.$$

The period of the circular orbit in the rotating frame is

$$T = \frac{2\pi}{(n-n')}$$

13.1 Variational equations

$$\dot{\xi} = A\xi$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -n^2 & 0 & 0 & 2p_{\phi o}/r_o^3 \\ -2p_{\phi o}/r_o^3 & 0 & 0 & 1/r_o^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues of A

$$\rho_1 = 0,\ \rho_2 = 0,\ \rho_{3,4} = \pm in$$

Solution

$$\xi(t) = e^{At}\xi(0)$$

Monodromy matrix

$$\Delta(T) = e^{AT}; \quad T = 2\pi/(n - n')$$

Eigenvalues of the monodromy matrix

$$\lambda_1 = 1, \; \lambda_2 = 1, \; \lambda_{3,4} = e^{\pm i\varphi},$$

$$\varphi = \frac{2\pi}{(1 - \frac{n'}{n})}$$

For the circular orbit we have

$$r_o = (GM)^{1/3} n^{-2/3} \;\rightarrow\; n = (GM)^{1/2} r_o^{-3/2}$$

The monodromy matrix is

$$\Delta(T) = \begin{pmatrix} \cos nT & \frac{1}{n}\sin nT & 0 & \frac{\alpha}{n^2}(1 - \cos nT) \\ -n\sin nT & \cos nT & 0 & \frac{\alpha}{n^2}\sin nT \\ 0 & 0 & 1 & \frac{\alpha^2}{n^3}\sin nT + \beta T \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\alpha = 2p_\phi / r_o^3, \quad \beta = 1/r_o^2 - \alpha^2/n^2$$

(evidently, the same eigenvalues)

13.2 Reduction to one degree of freedom

elimination of the angle ϕ: motion on the radius

Differential equations

$$\begin{aligned} \dot{r} &= p_r \\ \dot{p}_r &= p_\phi^2/(r^3) - GM/r^2 \end{aligned}$$

where p_ϕ is a parameter.

Note that the energy integral exists.

If we compute the monodromy matrix for the circular solution (fixed point) $r = r_o$, $p_r = 0$, we find he eigenvalues

$$\lambda_{3,4} = e^{\pm i\varphi}, \quad \varphi = \frac{2\pi}{(1 - \frac{n'}{n})}$$

The eigenvalues $e^{\pm i\varphi}$ are associated to the energy integral. They are not equal to unity because the integral is stationary along the circular orbit.

The double unit eigenvalue is associated to the angular momentum integral.

Note that $\lambda_{3,4} = 1$ if $n' = 0$.

14 Action-angle variables of the two-body problem

Delaunay variables

$$\hat{J}_1 = L, \quad \hat{\theta}_1 = M,$$
$$\hat{J}_2 = G, \quad \hat{\theta}_2 = \omega,$$

with Hamiltonian

$$\hat{H}_o = -\frac{(GM)^2}{2L^2}$$

and

$$L = \sqrt{GMa}, \quad G = \sqrt{GMa(1-e^2)}$$

Canonical change of variables to rotating frame:

by the generating function (time dependent)

$$F_2 = J_1\hat{\theta}_1 + J_2(\hat{\theta}_1 + \hat{\theta}_2) - J_2\lambda',$$

$$\lambda' = n't + \omega'$$

Action-angle variables in rotating frame

$$\begin{array}{ll} J_1 = L - G, & \theta_1 = M = \lambda - \omega, \\ J_2 = G, & \theta_2 = \lambda - \lambda', \end{array}$$

where where ω is the argument of pericenter and $\lambda = M + \omega$ is the mean longitude. The Hamiltonian is

$$H_o = -\frac{(GM)^2}{2L^2} - n'G$$

Note that these action-angle variables are the action-angle variables we obtain if we start from the canonical variables r, p_r, θ, p_θ.

In terms of the elements of the orbit, the Hamiltonian is expressed in the form

$$H_o = -\frac{GM}{2a} - n'\sqrt{GMa(1-e^2)}$$

This is constant and is the energy integral

$$H_o = h = \text{constant}$$

15 Families of periodic orbits in the rotating frame

Two-body problem. Rotating frame xOy. Angular velocity n'.

Circular orbits

A Keplerian circular orbit in the inertial frame is periodic (also circular) in the rotating frame, with period

$$T = \frac{2\pi}{n - n'}$$

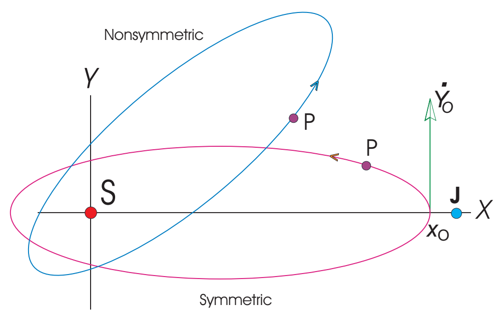

Figure 17: The resonant elliptic orbits for different orientations

where

$$T^c = \frac{2\pi}{n}$$

is the period in the inertial frame.

A circular orbit in xOy with radius $r_o = a$ exists, for any a. Consequently, a family of circular periodic orbits exists.

Parameter along the family: the semimajor axis a, or the frequency n.

The circular family is represented by a smooth curve, in the space $h - r_o$, given by

$$-\frac{GM}{2a} - n'\sqrt{GMa} = h$$

obtained from the energy integral for $e = 0$.

<u>*Elliptic orbits*</u>

An elliptic orbit in the inertial frame is periodic in the rotating frame only if it is resonant

$$\frac{n}{n'} = \frac{p}{q} = \text{rational}$$

which means that the semimajor axis must be given by

$$\frac{(GM)^{1/2}a^{-3/2}}{n'} = \frac{p}{q}$$

For a fixed resonance p/q, corresponding to a fixed semimajor axis $a_{p/q}$, the elliptic orbits belong to a two parametric family of periodic orbits, with e, ω as parameters.

An elliptic orbit is periodic for arbitrary orientation, defined by ω, and arbitrary eccentricity.

In general, an elliptic orbit is not symmetric with respect to the rotating x-axis, contrary to the circular orbits, which are symmetric.

In the space $h - r_o$, where $r_o = a_{p/q}(1 - e)$ is the pericenter distance, an elliptic family is represented by a smooth curve, given by the energy integral

$$-\frac{GM}{2a} - n'\sqrt{GMa(1 - e^2)} = h$$

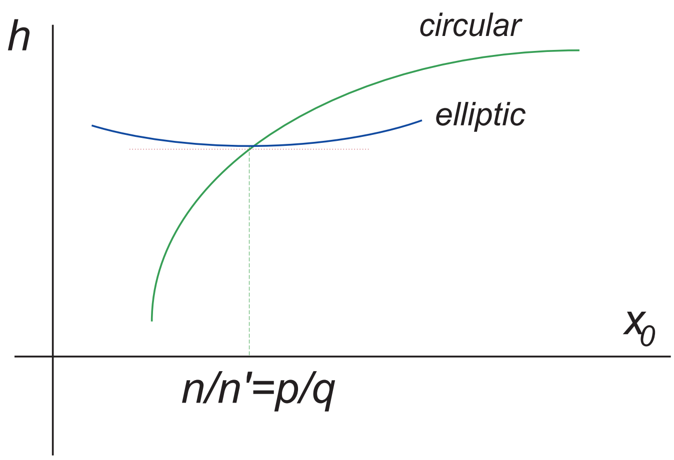

Figure 18: The families of circular and of resonant elliptic periodic orbits

Note that this presentation is not unique: a point on the elliptic family represents all the elliptic resonant orbits with the same eccentricity, but arbitrary orientation ω.

An elliptic periodic orbit in the rotating frame is also periodic in the inertial frame.

The resonant families of periodic orbits bifurcate from the family of circular orbits, at those points corresponding to the resonant values of the radius $a = a_{p/q}$.

Note that along the circular family the value of the semimajor axis varies, and consequently the ratio n/n' varies and passes through resonant (rational) values. It is at these points that we have a bifurcation to an elliptic family.

The families of circular and elliptic orbits. The tangent to the elliptic family at the bifurcation point is parallel to the x-axis.

Period of a circular orbit: The period T of a circular orbit with radius $r_o = a$ in the rotating frame, corresponding to an angular velocity n, is

$$T = \frac{2\pi}{n - n'}.$$

Period of an elliptic orbit: A resonant $n/n' = p/q$ elliptic periodic orbit in the rotating, is also periodic in the inertial frame. Its period should be an integral multiple of the period $T' = 2\pi/n'$ of the rotating frame. Consequently the period of the elliptic periodic orbit is

$$T^{el} = (p - q)T = T'q = T^c p$$

Along the family of elliptic orbits the period, and consequently the resonance, is constant, corresponding to the particular resonance.

16 Position of the eigenvalues along the family of circular orbits

- We keep n' fixed.

- Along the family n/n' varies.
- Two eigenvalues are fixed, equal to unity, $\lambda_1,\ \lambda_2 = 1$.
- The eigenvalues $e^{\pm i\varphi}$, $\varphi = 2\pi n/(n-n')$ change position on the unit circle.
- They meet on the unit circle at the point $+1$ at the resonance

$$n/n' = (\nu+1)/\nu,\ \ \nu 1 = 1, 2, 3, ...$$

- They meet on the unit circle at the point -1 at the resonance

$$n/n' = (2\nu+1)/(2\nu-1),\ \ \nu = 1, 2, 3, ...$$

17 The motion of the two-body problem in action-angle variables

Hamiltonian

$$H_o = -\frac{(GM)^2}{2(J_1+J_2)^2} - n' J_2$$

Solution

$$\begin{aligned} J_1 &= J_{1o}, & \theta_1 &= M = n(t-\tau), \\ J_2 &= J_{2o}, & \theta_2 &= \lambda - \lambda', \end{aligned}$$

where

$$\begin{aligned} J_1 = L - G &= \sqrt{GMa}\left(1 - \sqrt{1-e^2}\right), \\ J_2 = G &= \sqrt{GMa(1-e^2)}, \end{aligned}$$

and $\lambda = M + \omega$.

The motion is on a 2-torus, with fixed radii $J_{1o},\ J_{2o}$.

For a circular orbit: $e = 0 \to J_{1o} = 0$.

17.1 Poincaré map in action-angle variables

Define the map on the surface of section

$$H_o = h,\ \ \theta_2 = 0 \mod 2\pi$$

or, equivalently, on the surface of section

$$J_2 = J_{2o},\ \ \theta_2 = 0 \mod 2\pi$$

The mapping is in the variables $J_1,\ \theta_1$

$$\begin{aligned} \theta_1 &\to & \theta_1 + 2\pi \tfrac{n}{n-n'}, \\ J_1 &\to & J_1. \end{aligned}$$

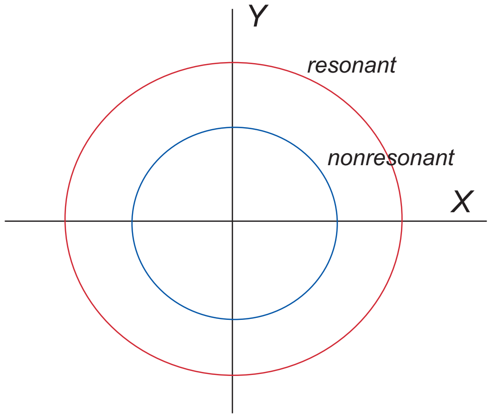

Figure 19: The resonant and non resonant invariant curves

We present it in the Poincaré variables (canonical) $X = \sqrt{2J_1}\cos\theta_1, \;\; Y = \sqrt{2J_1}\sin\theta_1$

The Poincareé map and the surface of section in $X\,Y$ variables. We use the surface of section $J_2 = J_{2o}$, $\theta_2 = 0$.

- All invariant curves are circles with constant radii $\sqrt{2J_1}$, where $J_1 = \sqrt{GMa} - J_{2o}$, corresponding to a certain value of a.
- On a particular invariant curve, with radius $\sqrt{2J_1}$, corresponds a certain eccentricity, obtained from $J_1 = \sqrt{GMa}(1 - (\sqrt{1-e^2})$.
- As J_1 increases, the semimajor axis increases. Consequently the ratio n/n' varies, and passes through resonant values.
- The point $J_1 = 0$ is a fixed point corresponding to a circular orbit ($e = 0$).
- The resonant invariant curves correspond to the resonant elliptic periodic orbits, in the rotating frame.
- The non resonant invariant curves correspond to elliptic orbits in the inertial frame, which however are not periodic in the rotating frame.
- The angle θ_1 on the invariant curve defines the orientation ω. (At $t = 0$ and $\theta_2 = \lambda - \lambda' = 0$, and for $\lambda' = 0$ we have $\lambda = 0$, $\rightarrow$ $nt + \omega = 0$, $\rightarrow$ $M = -\omega$, $\rightarrow$ $\theta_1 = -\omega$.)

Resonant invariant curves

A resonant $n/n' = p/q$ elliptic periodic orbit is a multiple fixed point on the resonant invariant curve. It comes to the initial position after $(p-q)$ iterations:

The angle θ_1 changes during one iteration by:

$$\Delta\theta_1 = 2\pi\frac{1}{1 - \frac{n'}{n}} = 2\pi\frac{p}{p-q}$$

The mapping after $p-q$ iterations:

$$\theta_1 \to \theta_1 + 2\pi \frac{1}{1-\frac{n'}{n}}(p-q) = \theta_1 + 2\pi p$$
$$J_1 \to J_1$$

All points on this invariant curve are fixed points.

A linear analysis shows that the two eigenvalues of a fixed point are equal to 1.

This means that the periodic elliptic orbits in the rotating frame have two pairs of unit eigenvalues.

18 Continuation of periodic orbits

From the two-body problem to:

- The restricted 3-body problem
  - circular
  - elliptic
- The planetary problem

These are nearly integrable dynamical systems. The unperturbed part is the two-body problem:

$$H = H_o + \epsilon H_1$$

How are the periodic orbits of H_o and their stability change when $\epsilon \neq 0$?

19 The restricted three-body problem

19.1 Circular model, planar orbits

Consider the Sun and Jupiter as primaries revolving around their center of mass in circular orbits. We will study orbits of the small body in the plane.

Define a rotating frame xOy with O the center of mass of the Sun and Jupiter and the x-axis along the line Sun-Jupiter. The System xOy rotates with constant angular velocity n'.

The motion of a small body A, with negligible mass, is given, in the usual normalized units

$$G = 1, \;\; m_{sun} = 1-\mu, \;\; m_J = \mu, \;\; n' = 1,$$

by the Hamiltonian

$$H = \left[\frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - n' p_\phi - \frac{1-\mu}{r}\right] + \mu H_1$$

The first part is the Hamiltonian H_o of the two body problem. The small parameter is μ.

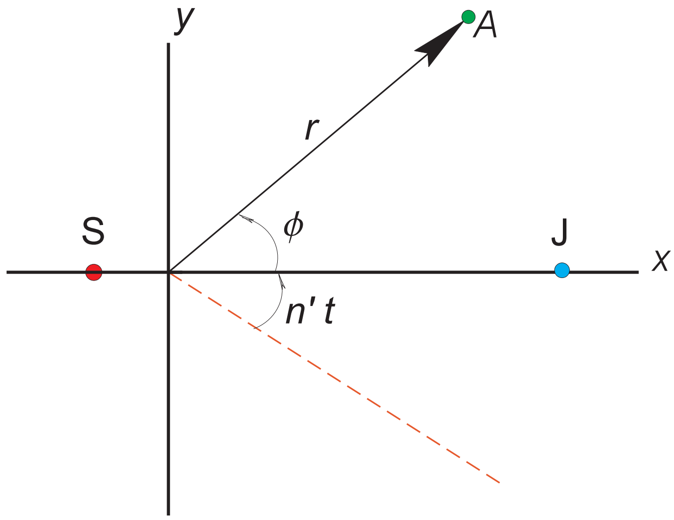

Figure 20: The rotating frame

19.2 Continuation to $\mu > 0$

Family of nearly circular orbits

The continuation is possible if

$$|\Delta_3(T) - T_3| \neq 0$$

From the monodromy matrix:

$$|\Delta_3(T) - T_3| = (1 - \cos nT) \times \text{a constant}$$

This is equal to zero if $nT = 2\pi\nu$, i.e. at the resonances $n/n' = (\nu + 1)/\nu$.

- All the circular orbits that are not at the resonance $n/n' = 2/1, 3/2, ...,$ are continued as nearly circular orbits in the rotating frame.
- The resonant circular orbits $n/n' = 2/1$, $3/2$, ... are not continued as periodic orbits in the rotating frame.
- At these resonances, a gap appears and the single unperturbed family of circular orbits breaks into a set of disconnected families of periodic orbits.

Family of nearly elliptic orbits

- The elliptic unperturbed orbits that are periodic in the rotating frame are all resonant.
- Their semimajor axis is such that $n/n' = p/q$.
- For each resonance, and a fixed eccentricity e, they are represented by a resonant invariant curve on a Poincaré map in the action-angle variables.

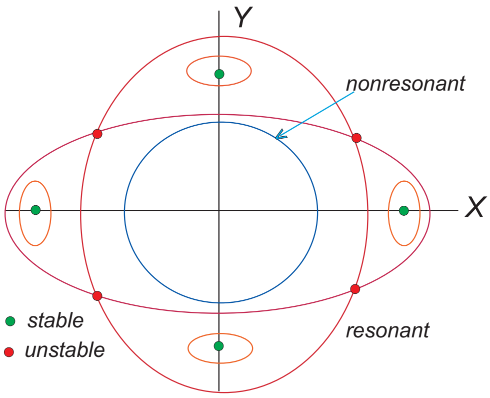

Figure 21: Out of the infinity of resonant fixed points, only a finite number survives the perturbation.

- There is an infinity of resonant invariant curves, for the same resonance p/q, and different eccentricities, on different Poincaré surfaces of section.
- Each resonant invariant curve corresponds to an infinity of elliptic orbits, for arbitrary orientation ω.

What happens when $\mu > 0$?

Poincaré-Birkhoff fixed point theorem:

- Out of the infinite set of (multiple in general) fixed points, only a finite number survive (usually only two).
- Half of them are stable and half unstable.
- If the system has symmetries (as the restricted problem) , usually the symmetric periodic orbits survive (but not always!).
- The resonant fixed points that survive correspond to monoparametric families of elliptic periodic orbits, in the rotating frame.
- These families bifurcate from the circular family, at the corresponding circular resonant orbits.
- They are in most cases symmetric with respect to the rotating x-axis.

Note that a resonant circular periodic orbit, at the resonance $n/n' = p/q$ has the pair of characteristic exponents

$$\sigma = i\frac{2\pi}{1-\frac{n'}{n}} = i2\pi\frac{p}{p-q}$$

and if it is described $p-q$ times, it becomes critical with respect to the stability, and according to the general theory we have a bifurcation of a new family of periodic orbits.

<u>The stability along the families</u>

All periodic orbits, for $\mu = 0$ have two pairs of unit eigenvalues. One pair is the unit pair $\lambda_1 = \lambda_2 = 1$. The other pair λ_3, λ_4 is always on the unit circle, not necessarily at the points $+1$ or -1.

It is $\lambda_3 = \lambda_4 = 1$ if the orbit is resonant elliptic.

When $\mu > 0$ the unit pair λ_1, λ_2 survives as unit pair, because of the existence of the Jacobi (energy) integral.

The stability will be determined from the second pair λ_3, λ_4 of eigenvalues.

The dangerous points, where instability can be generated, by displacing the eigenvalues λ_3, λ_4 outside the unit circle, are the points $+1$ and -1.

When the eigenvalues λ_3, λ_4 meet at these points, and only at these, they can move outside the unit circle and thus generate instability.

Stability of elliptic orbits

The elliptic orbits evolve, when $\mu > 0$, both to stable and unstable nearly elliptic periodic orbits, in the rotating frame (Poincaré-Birkhoff fixed point theorem).

This means that the eigenvalues λ_3, $\lambda_4 = 1$ of the unperturbed ($\mu = 0$) resonant elliptic orbit may move outside the unit circle for one of the continued orbits, but there is always another another periodic orbit where the eigenvalues move on the unit circle, and thus preserve the stability.

Stability of circular orbits

Along the family of circular orbits ($\mu = 0$) the eigenvalues λ_3, λ_4 depend on the ratio n'/n:

$$\lambda_{3,4} = e^{\pm i\varphi}, \quad \varphi = i\frac{2\pi}{1 - \frac{n'}{n}}$$

The eigenvalues meet at $+1$ on the unit circle when $n/n' = 2/1, 3/2, ...$ However these orbits are not continued as circular orbits.

The eigenvalues meet also on the unit circle at the point -1 when $n/n' = 3/1, 5/3, ...$ These orbits are continued as nearly circular orbits when $\mu > 0$ and according to the theory of Krein they can move outside the unit circle and thus generate instability.

Conclusion

A small unstable region is generated on the family of circular orbits, at the resonances $n/n' = 3/1, 5/3, ..,$ when $\mu > 0$.

The eigenvalues meet on the unit circle at -1 and move outside the unit circle, generating instability.

At both ends of this unstable region the periodic orbits are critical and consequently there is a bifurcation of a family of resonant periodic orbits, 3/1, 5/3, ... These are the elliptic periodic orbits that bifurcate from this resonance.

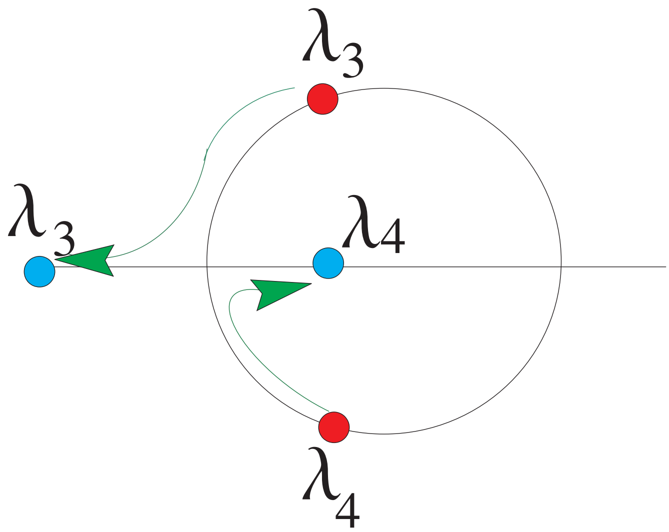

Figure 22: The evolution to instability at the 3/1, 5/3,.. resonance

19.3 Circular model, three-dimensional orbits

As shown, there exist, in the plane $x_1\, x_2$, families of circular periodic orbits and families of resonant elliptic orbits, for $\mu > 0$.

Which orbits can be continued as periodic orbits in the three-dimensional space $x_1 x_2 x_3$?

This can happen only at those points of the family which are critical with respect to the vertical stability.

It is only at these points that the vertical deviations $\xi_3(t)$ of a perturbed orbit have a period equal to the period of the planar periodic orbit.

Remark: Along a resonant family of elliptic periodic orbits, the vertical stability index is very close to critical. (It is exactly critical on the unperturbed elliptic family).

Depending on the particular resonance, such a critical point may or may not exist.

Three-dimensional symmetric periodic orbits exist: There are two types:

Initial conditions

Type 1

$$x_{1o}, \dot{x}_{1o} = 0, x_{2o} = 0, \dot{x}_{2o}, x_{3o}, \dot{x}_{3o} = 0$$

Type 2

$$x_{1o}, \dot{x}_{1o} = 0, x_{2o} = 0, \dot{x}_{2o}, x_{3o} = 0, \dot{x}_{3o}$$

19.4 Elliptic model, planar orbits

Assume the orbit of Jupiter to be elliptic with eccentricity e_J (elliptic problem).

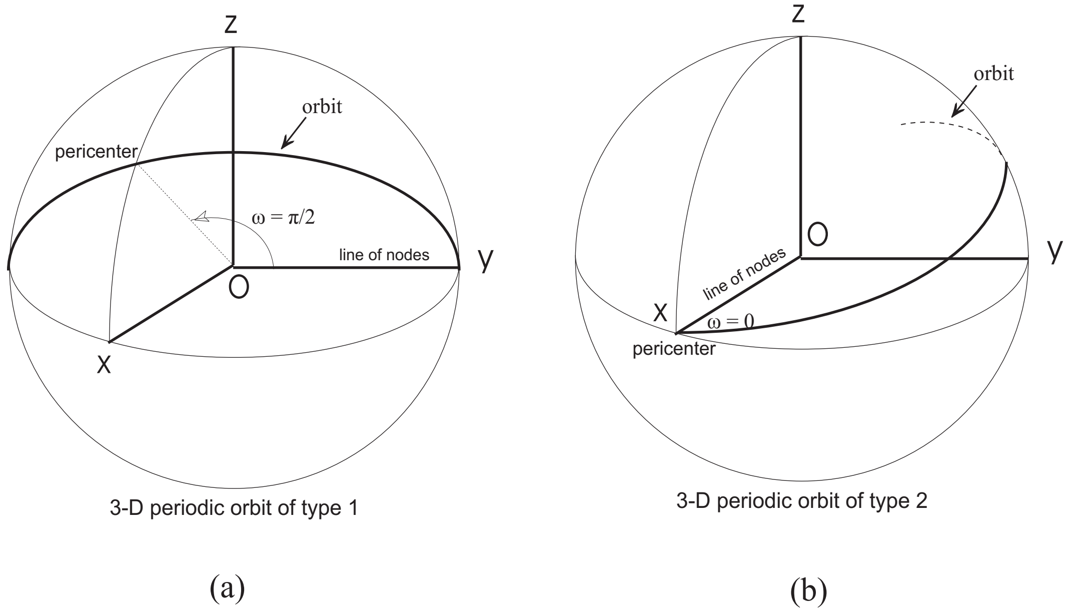

Figure 23: (a) The geometric configuration of a 3-D symmetric periodic orbit of *Type 1* ($\omega = \pi/2$). (b) The geometric configuration of a 3-D symmetric periodic orbit of *Type 2* ($\omega = \pi$).

Consider an orbit on the family of nearly circular orbits or on the family of nearly elliptic orbits of the circular planar model for $\mu > 0$ (circular orbit of Jupiter).

If the eccentricity of Jupiter is nonzero, can this orbit be continued as a planar periodic orbit of the elliptic model?

This can happen only if its period is a multiple of the period of Jupiter.

For a fixed value $e_J > 0$ there exists only one periodic orbit. We have a family by varying e_J.

Continuation from the family of circular orbits

The period of a circular orbit ($\mu = 0$) is

$$T = \frac{2\pi}{n - n'}$$

and the period of Jupiter is

$$T_J = \frac{2\pi}{n'}$$

where n' is its mean angular velocity. We have

$$T = \frac{T_J}{\frac{n}{n'} - 1}.$$

At the resonance $n/n' = p/q$ we have

$$T = T_J \frac{q}{p - q}.$$

and if this orbit is described $p-q$ times, the period T^* of this multiple orbit is an integral multiple of T_J,

$$T^* = q\,T_J.$$

If $n/n' \neq 2/1, 3/2, ..$ the region around this resonant orbit is continued to $\mu > 0$. On the continued family there exists an orbit which, if described $p-q$ times, has a period exactly equal to $q\,T_J$. This means that a bifurcation from the circular family $\mu > 0$ to a family of the elliptic problem can take place close to a resonance.

Continuation from the family of nearly elliptic orbits

Consider a family of $n/n' = p/q$ resonant periodic orbits of the circular planar problem, for $\mu = 0$. The period all along the family is constant, equal to $(p-q)T$, or

$$T = q\,T_j.$$

This family is continued, when $\mu > 0$, to two families of elliptic periodic orbits, differing in phase. Along each family the eccentricity increases, starting from zero values.

For continuity reasons, the period along the continued family is close to $q\,T_J$. If at a certain point, corresponding to a value e of the eccentricity, it happens to be exactly equal to $q\,T_J$, then a bifurcation to the elliptic problem can take place.

The numerical computations have shown that in certain resonances, for example 2/1, 3/1, such bifurcation points do exist, at quite large values of the eccentricity. But in other resonances, for example 7/3, such bifurcation points do not exist.

19.5 Elliptic model, three-dimensional orbits

Continuation from planar orbits

Consider a family of planar periodic orbits of the elliptic model, along which the eccentricity e_J of Jupiter varies. If along this family the vertical stability index is critical, we have a bifurcation of a family of three-dimensional periodic orbits of the elliptic model.

Continuation from 3-D orbits

Consider now a family of three-dimensional periodic orbits of the circular model. If along this family the period is a multiple of the period of Jupiter, then we have a bifurcation of a family of three-dimensional periodic orbits of the elliptic problem.

20 The planetary problem Planar orbits

Consider the families of circular and of resonant elliptic periodic orbits of the planar circular restricted three-body problem . The mass of the Sun is $m_o = 1-\mu$, the mass of Jupiter is $m_1 = \mu$ and the mass of the small body is $m_2 = 0$.

It can be proved that all these periodic orbits can be continued to $m_2 \neq 0$, as symmetric periodic orbits.

We call these orbits, periodic orbits of the planetary type.

These are periodic orbits of the general three-body problem, in a rotating frame xOy, whose x-axis is the line Sun-P_1, the origin O is the center of mass of these two bodies and the y-axis is in the orbital plane of the Sun and P_1.

The planet P_1 moves always on the x-axis, and the planet P_2 moves in the xOy plane. There are three degrees of freedom, on the rotating frame, in the variables

$$x_1, \dot{x}_1, x_2, y_2, \dot{x}_2, \dot{y}_2$$

The position of the rotating frame in space is determined by the angle θ of the rotating x-axis with a fixed direction. The value of $\dot{\theta}$ is obtained from the angular momentum integral.

The dominant mass is the mass m_o of the sun and the two small masses m_1, m_2 are the masses of the two planets P_1, P_2 , respectively.

The initial conditions of a symmetric periodic orbit of the general planar three-body problem are

$$x_{1o}, \dot{x}_{1o} = 0, x_{2o}, y_{2o} = 0, \dot{x}_{2o} = 0, \dot{y}_{2o}$$

A symmetric periodic orbit is determined by the nonzero initial conditions

$$x_{1o}, \ , x_{2o}, \ \dot{y}_{2o}$$

The families of the periodic orbits of the planetary type are represented by smooth curves in the three-dimensional space of initial conditions.

They are similar to the families of the restricted three-body problem:

There exist families of circular orbits of both planets, along which the two planets describe nearly circular orbits. Gaps appear along the circular family, at the resonances $n_1/n_2 = 2/1, 3/2, ...$

At these gaps the circular family is continued as a resonant elliptic family, along which the ratio n_1/n_2 of the frequencies of the two planets is a rational number, equal to $2/1, 3/2, ...$, respectively.

There are also resonant families of elliptic orbits of the two planets. They bifurcate from the circular family, at the point where the ratio n_1/n_2 of the frequencies of the two circular planetary orbits is rational.

At the bifurcation point on the circular family, one pair of eigenvalues of the monodromy matrix is equal to $+1$. Very close to it, a second pair is also equal to $+1$ (these two pairs are both equal to $+1$ on the same orbit of the unperturbed family of circular orbits). Consequently, there are two different resonant elliptic periodic orbits, for a fixed resonance (different from 2/1, 3/2, ..) which start very close to each other.

Along a resonant elliptic family, the ratio n_1/n_2 of the frequencies of the two planets has a constant rational value p/q.

Along a resonant family the eccentricities e_1, e_2 of the two planets increase, starting from zero values.

An unstable region appears on the circular family, at the resonances $3/1, 5/3, ...$

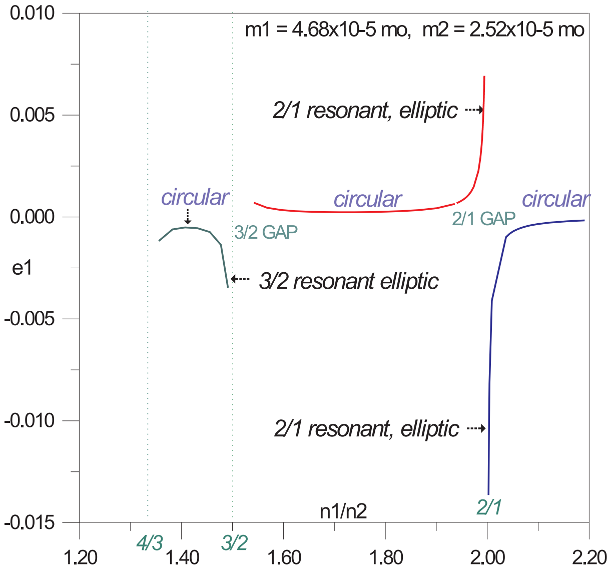

Figure 24: The gaps at the resonances 2/1, 3/2, 4/3,. ..

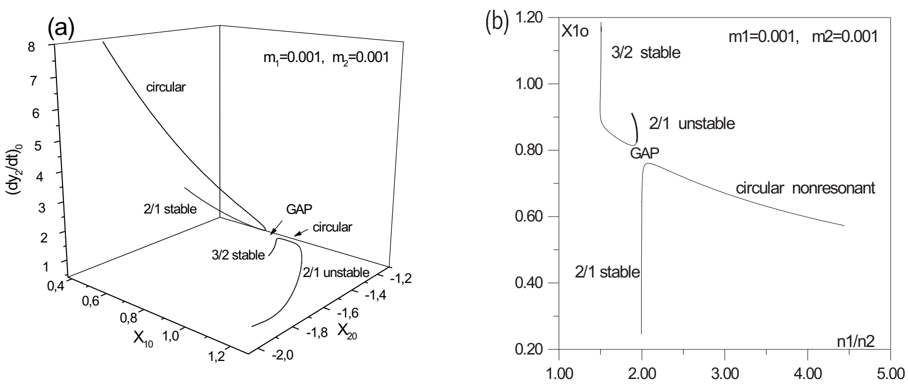

Figure 25: Families of circular and elliptic periodic orbits of the general three body problem, of the planetary type, for $m_1 = 0.001$, $m_2 = 0.001$ and $m_{sun} = 0.998$. (a) The families in the x_{10}, x_{20}, $\dot{y}_{20}$ space. (b) The families in the *resonance* - x_0 space.

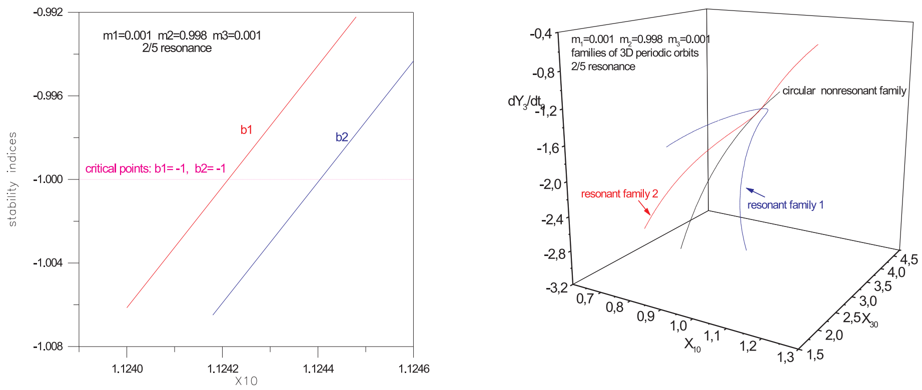

Figure 26: (a)The bifurcation points on the circular family, (b)The two resonant families at the 5/2 resonance

The resonant elliptic families may be stable or unstable, depending on the phase and on the ratio m_1/m_2 of the masses, and on the ratio $(m_1+m_2)/m_o$.

The 5/2 resonance

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