THE LJUNGGREN EQUATION REVISITED

DRAZIOTIS KONSTANTINOS

Abstract. We study the Ljunggren equation $Y^2 + 1 = 2X^4$ using the method "multiplication by 2" of Chabauty [2].

1. Introduction

In [5], Ljunggren proved that the only positive integral solutions of diophantine equation

$$L_2 : Y^2 + 1 = 2X^4$$

are $(X,Y) = (1,1), (13, 239)$. Since the proof was quite complicated, Mordell asked if one could find a simpler proof. In [8] Tzanakis and Steiner gave a simpler proof using the theory of Baker. A second proof was given by Chen [3], using the Thue-Siegel method combined with Pade approximation on algebraic functions. In this paper we solve this equation with another method. Our approach is inspired by Chabauty [2] and uses the group structure of an elliptic curve and the multiplication by 2-map. This method is used by Poulakis [6] and later by Bugeaud [1] to obtain an upper bound for the height of the integral points.

2. The integral solutions of $L_2$

The proof consists of two parts. The first uses the group structure of the elliptic curve and the second is a reduction to a unit equation in a certain quartic number field.

To solve the equation $L_2$ it is enough to solve $E_2$, where

$$E_2 : F(X,Y) = Y^2 - (X^3 - 2X) = 0.$$  

Let $P = (a,b) \in E_2(\mathbb{Z})$. Suppose that $a$ is not zero. Then we set $a = 2x^2$, $b = 2xy$ and we deduce that $(x, y) \in L_2(\mathbb{Z})$. We assume that

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$|a| \geq 2$. Let $R = (s, t)$ be a point of $E_2$ over the algebraic closure $\overline{\mathbb{Q}}$ of \(\mathbb{Q}\), such that $2R = P$. By [7, chapter 3, p.59], we have

$$a = \frac{(s^2 + 2)^2}{4s(s^2 - 2)}$$

and so $s$ is a root of the polynomial

$$\Theta_a(S) = S^4 - 4aS^3 + 4S^2 + 8aS + 4.$$ 

The roots of $\Theta_a(S)$ are:

$$a \pm \sqrt{a^2 - 2} \pm \sqrt{2a^2 \pm 2a\sqrt{a^2 - 2}},$$

where the first $\pm$ coincides with the third. Put $L = \mathbb{Q}(s)$. Since $a = 2x^2$, we have $a^2 - 2 = 4x^4 - 2 = 2y^2$ and so $L = \mathbb{Q}(\sqrt{2x^2 \pm y\sqrt{2}})$. Also, $\mathbb{Q}(\sqrt{2}) \subset L$ and $N_{L/\mathbb{Q}}(2x^2 \pm y\sqrt{2}) = 2$. It follows that the only prime dividing the discriminant of $L$ is 2. So the only prime ramified in $L$ is 2. Furthermore, from [4, Chapter 9, Proposition 9.4.1, p.461] $L$ is a totally real quartic extension of $\mathbb{Q}$. So from Jones list$^1$ or the database$^2$ of Jürgen Klüners and Gunter Malle, we conclude that $L = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$.

The element $s_\pm = \frac{s \pm \sqrt{7}}{2}$ is a root of the polynomial with integer coefficients:

$$\lambda(S) = (1/256) \text{res}_W(\Theta_a(2S \mp W), W^2 - 2) = S^8 - 4aS^7 + \cdots + 1,$$

where $\text{res}_W(\cdot, \cdot)$ denotes the resultant of two polynomials with respect to $W$. Thus $s_\pm$ is a unit in $L$. So $u = \frac{s + \sqrt{7}}{2}$ and $v = \frac{\sqrt{7} - s}{2}$ satisfy the unit equation $u + v = \sqrt{2}$ in $L$. The algorithm of Wildanger [9] which is implemented in the computer algebra system Magma$^3$ V2.10-22, gives the solutions of this unit equation in $L$, which are listed in table 1 where we have put $[a_1 \ a_2 \ a_3 \ a_4] = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3$, with $\theta = \sqrt{2 + \sqrt{2}}$.

We substitute to (1) each solution of the unit equation and we check if it gives an integer. Thus, it follows that $a = 2, 338$. So, for $|a| \geq 2$, the solutions of $E_2$ are $(X, Y) = (2, \pm 2), (338, \pm 6214)$ and for $|a| < 2$, are $(X, Y) = (0, 0), (-1, \pm 1)$. So $L_2(\mathbb{Z}) = \{(\pm 1, \pm 1), (\pm 13, \pm 239)\}$.


$^2$http://www.mathematik.uni-kassel.de/~klueners/minimum/minimum.html

$^3$http://magma.maths.usyd.edu.au/magma
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Table 1: The solutions of the unit equation

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References


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