Analysis of multi-variate time series by means of networks

Dimitris Kugiumtzis

January 28, 2015
Spurious cross correlations see [1]: Sec 7.3

Time series of indices (strongly autocorrelated): large cross-correlation

Time series of returns (weakly or no autocorrelated): small cross-correlation

Autocorrelation may cause spurious cross-correlations

⇒ prewhiten the time series to have zero autocorrelation.

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⇒ prewhiten the time series to have zero autocorrelation.
Example: Two independent AR(1) processes

Time series \( \{x_t\}_{t=1}^{n}, \{y_t\}_{t=1}^{n} \) from two independent AR(1) processes:

\[
X_t = 0.95X_{t-1} + \epsilon^X_t \quad Y_t = 0.85Y_{t-1} + \epsilon^Y_t
\]
Example: Two independent AR(1) processes

Time series \( \{x_t\}_{t=1}^n \), \( \{y_t\}_{t=1}^n \) from two independent AR(1) processes:

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$X_t = 0.95X_{t-1} + \epsilon_X^t$  $Y_t = 0.85Y_{t-1} + \epsilon_Y^t$

Prewhitening: 1) Fit AR(p) model to $\{x_t\}_{t=1}^n$ and separately to $\{y_t\}_{t=1}^n$ 2) Take the residuals $\{e_x^t\}_{t=1}^n, \{e_y^t\}_{t=1}^n$. 

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Time series $\{x_t\}^n_{t=1}$, $\{y_t\}^n_{t=1}$ from two independent AR(1) processes:

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Example: Two dependent AR(1) processes - 1

The first AR(1) process drives the second AR(1) process:

\[ X_t = 0.95X_{t-1} + \epsilon_t^X \quad Y_t = 0.5X_{t-1} + 0.85Y_{t-1} + \epsilon_t^Y \]
Example: Two dependent AR(1) processes - 1

The first AR(1) process drives the second AR(1) process:

\[ X_t = 0.95X_{t-1} + \epsilon_t^X \quad \text{\(Y_t = 0.5X_{t-1} + 0.85Y_{t-1} + \epsilon_t^Y\)} \]

After prewhitening, \( r_{X,Y}(\tau), \tau = 1, 2, 3 \) is still statistically significant \( \Rightarrow X_t \) is correlated to \( Y_t + \tau \), but not the opposite \( \Rightarrow \) direction of correlation \( \Rightarrow \) (Granger) causality

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After prewhitening, \( \tau_{X,Y}(\tau) \), \( \tau = 1, 2, 3 \) is still statistically significant \( \Rightarrow \) \( X_t \) is correlated to \( Y_t + \tau \), but not the opposite \( \Rightarrow \) direction of correlation \( \Rightarrow \) (Granger) causality.
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After prewhitening, \( r_{X,Y}(\tau) \), \( \tau = 1, 2, 3 \) is still statistically significant \( \Rightarrow \) \( X_t \) is correlated to \( Y_t + \tau \), but not the opposite \( \Rightarrow \) direction of correlation \( \Rightarrow \) (Granger) causality

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Example: Two dependent AR(1) processes - 1

The first AR(1) process drives the second AR(1) process:

\[ X_t = 0.95 X_{t-1} + \epsilon_t^X \]
\[ Y_t = 0.5 X_{t-1} + 0.85 Y_{t-1} + \epsilon_t^Y \]

After prewhitening, \( r_{X,Y}(\tau) \), \( \tau = 1, 2, 3 \) is still statistically significant.
Example: Two dependent AR(1) processes - 1

The first AR(1) process drives the second AR(1) process:

\[ X_t = 0.95X_{t-1} + \epsilon^X_t \]
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After prewhitening, \( r_{X,Y}(\tau) \), \( \tau = 1, 2, 3 \) is still statistically significant \( \implies X_t \) is correlated to \( Y_{t+\tau} \), but not the opposite
Example: Two dependent AR(1) processes - 1

The first AR(1) process drives the second AR(1) process:

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After prewhitening, \( r_{X,Y}(\tau) \), \( \tau = 1, 2, 3 \) is still statistically significant

\[ \implies X_t \text{ is correlated to } Y_{t+\tau}, \text{ but not the opposite} \]

\[ \implies \text{direction of correlation} \]

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Example: Two dependent AR(1) processes - 1

The first AR(1) process drives the second AR(1) process:

\[ X_t = 0.95X_{t-1} + \epsilon_t^X \quad Y_t = 0.5X_{t-1} + 0.85Y_{t-1} + \epsilon_t^Y \]

After prewhitening, \( r_{X,Y}(\tau) \), \( \tau = 1, 2, 3 \) is still statistically significant
\( \Rightarrow \) \( X_t \) is correlated to \( Y_{t+\tau} \), but not the opposite
\( \Rightarrow \) direction of correlation \( \Rightarrow \) (Granger) causality
The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \quad Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon_t^Y \]
The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \]

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After prewhitening, the statistically significant cross-correlations are for both positive and negative delays, which indicates the interdependence.
The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \]

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After prewhitening, the statistically significant cross-correlations are for both positive and negative delays. This indicates that \( X_t \) is correlated to \( Y_t \) with \(|\tau|\) and to \( Y_t \) with \(|\tau|\), implying interdependence.
Example: Two dependent AR(1) processes - 2

The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon^X_t \quad Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon^Y_t \]

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays. This suggests that \( X_t \) is correlated to \( Y_t \) at \( \tau \) and \( Y_t \) at \( -\tau \), indicating interdependence.

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The two AR(1) processes are inter-dependent:

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The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \]
\[ Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon_t^Y \]

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays. \( X_t \) is correlated to \( Y_t + |\tau| \) and to \( Y_t - |\tau| \).
The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \]
\[ Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon_t^Y \]

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays.

\[ X_t \text{ is correlated to } Y_t + |\tau| \text{ and to } Y_t - |\tau|, \]
\[ \text{interdependence} \]

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The two AR(1) processes are inter-dependent:

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X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \\
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\]

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays.
Example: Two dependent AR(1) processes - 2

The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \]
\[ Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon_t^Y \]

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays

\[ \Rightarrow X_t \text{ is correlated to } Y_{t+|\tau|} \text{ and to } Y_{t-|\tau|}, \]
Example: Two dependent AR(1) processes - 2

The two AR(1) processes are inter-dependent:

\[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon^X_t \quad Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon^Y_t \]

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays

\[ \rightarrow X_t \text{ is correlated to } Y_{t+|\tau|} \text{ and to } Y_{t-|\tau|}, \rightarrow \text{ interdependence} \]
Dynamic Regression and VAR modeling, order 1

Given time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \):

see [1]: Chp 12, [2]: Chp 7

1. Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^n \))

AR(1):

\[
X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t
\]

\( \epsilon_t \sim WN(0, \sigma^2) \)

2. Explain \( X_t \) using past samples from \( X \) and \( Y \).

Dynamic regression model \( X \) at one lag for \( X \) and \( Y \), DR \( X \)(1,1):

\[
X_t = a_{1,0} + a_{1,1} X_{t-1} + a_{1,2} Y_{t-1} + \epsilon_{1,t}
\]

Dynamic regression model \( Y \) at one lag for \( X \) and \( Y \), DR \( Y \)(1,1):

\[
Y_t = a_{2,0} + a_{2,1} X_{t-1} + a_{2,2} Y_{t-1} + \epsilon_{2,t}
\]

3. Join the models for \( X \) and \( Y \) in one, vector variable \( X_t = [X_t, Y_t] \)′.

Vector autoregressive model for \((X, Y)\) of order 1, VAR(1):

\[
\begin{bmatrix}
X_t \\
Y_t
\end{bmatrix} =
\begin{bmatrix}
a_{1,0} & a_{2,0} \\
a_{1,1} & a_{2,1} \\
a_{1,2} & a_{2,2}
\end{bmatrix}
\begin{bmatrix}
X_{t-1} \\
Y_{t-1}
\end{bmatrix} + \begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t}
\end{bmatrix}
\]

and in matrix form

\[
X_t = A_0 + A_1 X_{t-1} + \epsilon_t
\]
Dynamic Regression and VAR modeling, order 1

Given time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \): see [1]: Chp 12, [2]: Chp 7

1. Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^n \))

\[
X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t
\]
\[\epsilon_t \sim WN(0,\sigma^2)\]

2. Explain \( X_t \) using past samples from \( X \) and \( Y \).

Dynamic regression model \( X \) at one lag for \( X \) and \( Y \), DR\( X \)(1,1):

\[
X_t = a_{1,0} + a_{1,1} X_{t-1} + a_{1,2} Y_{t-1} + \epsilon_{1,t}
\]

Dynamic regression model \( Y \) at one lag for \( X \) and \( Y \), DR\( Y \)(1,1):

\[
Y_t = a_{2,0} + a_{2,1} X_{t-1} + a_{2,2} Y_{t-1} + \epsilon_{2,t}
\]

3. Join the models for \( X \) and \( Y \) in one, vector variable \( X_t = [X_t, Y_t]' \).

Vector autoregressive model for \( (X, Y) \) of order 1, VAR(1):

\[
\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} a_{1,0} \\ a_{2,0} \end{bmatrix} + \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}
\]

and in matrix form

\[
X_t = A_0 + A_1 X_{t-1} + \epsilon_t
\]
Given time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \):

**1.** Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^n \))

**AR(1):** \( X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \quad \epsilon_t \sim WN(0, \sigma^2_\epsilon) \)
Given time series $\{x_t\}_{t=1}^n$, $\{y_t\}_{t=1}^n$: see [1]: Chp 12, [2]: Chp 7

1. Explain $X_t$ using only past samples from $X$ (without using $\{y_t\}_{t=1}^n$)

   **AR(1):**
   
   $$X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \quad \epsilon_t \sim WN(0, \sigma^2_\epsilon)$$

2. Explain $X_t$ using past samples from $X$ and $Y$. 

   **Dynamic regression model $X$ at one lag for $X$ and $Y$:**
   
   $$X_t = a_{1,0} + a_{1,1} X_{t-1} + a_{1,2} Y_{t-1} + \epsilon_{1,t}$$

   **Dynamic regression model $Y$ at one lag for $X$ and $Y$:**
   
   $$Y_t = a_{2,0} + a_{2,1} X_{t-1} + a_{2,2} Y_{t-1} + \epsilon_{2,t}$$

3. Join the models for $X$ and $Y$ in one, vector variable $X_t = [X_t, Y_t]'$.

   **Vector autoregressive model for ($X$, $Y$) of order 1, VAR(1):**
   
   $$[X_t, Y_t] = [a_{1,0}, a_{2,0}] + [a_{1,1}, a_{2,1}] X_{t-1} + [a_{1,2}, a_{2,2}] Y_{t-1} + \epsilon_{1,t}, \epsilon_{2,t}$$

   and in matrix form
   
   $$X_t = A_0 + A_1 X_{t-1} + \epsilon_t$$

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Dynamic Regression and VAR modeling, order 1

Given time series \( \{x_t\}_{t=1}^{n}, \{y_t\}_{t=1}^{n} \): see [1]: Chp 12, [2]: Chp 7

1. Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^{n} \))

   **AR(1):**
   \[
   X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \quad \epsilon_t \sim WN(0, \sigma^2_\epsilon)
   \]

2. Explain \( X_t \) using past samples from \( X \) and \( Y \).

   Dynamic regression model \( X \) at one lag for \( X \) and \( Y \), **DR\( X(1,1) \):**
   \[
   X_t = a_{1,0} + a_{1,1} X_{t-1} + a_{1,2} Y_{t-1} + \epsilon_{1,t}
   \]
Dynamic Regression and VAR modeling, order 1

Given time series \( \{x_t\}_{t=1}^{n}, \{y_t\}_{t=1}^{n} \):

1. Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^{n} \))
   
   **AR(1):** \( X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \quad \epsilon_t \sim WN(0, \sigma_\epsilon^2) \)

2. Explain \( X_t \) using past samples from \( X \) and \( Y \).
   
   Dynamic regression model \( X \) at one lag for \( X \) and \( Y \), \( DR_X(1, 1) \):
   \( X_t = a_{1,0} + a_{1,1} X_{t-1} + a_{1,2} Y_{t-1} + \epsilon_{1,t} \)

   Dynamic regression model \( Y \) at one lag for \( X \) and \( Y \), \( DR_Y(1, 1) \):
   \( Y_t = a_{2,0} + a_{2,1} X_{t-1} + a_{2,2} Y_{t-1} + \epsilon_{2,t} \)

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1. Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^n \))
   AR(1): \( X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \quad \epsilon_t \sim WN(0, \sigma^2_\epsilon) \)

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   Dynamic regression model \( Y \) at one lag for \( X \) and \( Y \), \( DR_Y(1, 1) \):
   \( Y_t = a_{2,0} + a_{2,1} X_{t-1} + a_{2,2} Y_{t-1} + \epsilon_{2,t} \)

3. Join the models for \( X \) and \( Y \) in one, vector variable \( X_t = [X_t, Y_t]' \).
Dynamic Regression and VAR modeling, order 1

Given time series \( \{x_t\}_{t=1}^n \), \( \{y_t\}_{t=1}^n \):

1. Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^n \))
   
   AR(1): \( X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \quad \epsilon_t \sim \text{WN}(0, \sigma_\epsilon^2) \)

2. Explain \( X_t \) using past samples from \( X \) and \( Y \).
   
   Dynamic regression model \( X \) at one lag for \( X \) and \( Y \), \( DR_X(1,1) \):
   
   \( X_t = a_{1,0} + a_{1,1} X_{t-1} + a_{1,2} Y_{t-1} + \epsilon_{1,t} \)

   Dynamic regression model \( Y \) at one lag for \( X \) and \( Y \), \( DR_Y(1,1) \):
   
   \( Y_t = a_{2,0} + a_{2,1} X_{t-1} + a_{2,2} Y_{t-1} + \epsilon_{2,t} \)

3. Join the models for \( X \) and \( Y \) in one, vector variable \( X_t = [X_t, Y_t]' \).
   
   Vector autoregressive model for \( (X, Y) \) of order 1, \( \text{VAR}(1) \):
   
   \[
   \begin{bmatrix}
   X_t \\
   Y_t
   \end{bmatrix} =
   \begin{bmatrix}
   a_{1,0} \\
   a_{2,0}
   \end{bmatrix}
   +
   \begin{bmatrix}
   a_{1,1} & a_{1,2} \\
   a_{2,1} & a_{2,2}
   \end{bmatrix}
   \begin{bmatrix}
   X_{t-1} \\
   Y_{t-1}
   \end{bmatrix}
   +
   \begin{bmatrix}
   \epsilon_{1,t} \\
   \epsilon_{2,t}
   \end{bmatrix}
   \]
Dynamic Regression and VAR modeling, order 1

Given time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \):

1. Explain \( X_t \) using only past samples from \( X \) (without using \( \{y_t\}_{t=1}^n \))

   \[ AR(1): X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \quad \epsilon_t \sim WN(0, \sigma_\epsilon^2) \]

2. Explain \( X_t \) using past samples from \( X \) and \( Y \).

   Dynamic regression model \( X \) at one lag for \( X \) and \( Y \), \( DR_X(1, 1) \):
   \[ X_t = a_{1,0} + a_{1,1} X_{t-1} + a_{1,2} Y_{t-1} + \epsilon_{1,t} \]

   Dynamic regression model \( Y \) at one lag for \( X \) and \( Y \), \( DR_Y(1, 1) \):
   \[ Y_t = a_{2,0} + a_{2,1} X_{t-1} + a_{2,2} Y_{t-1} + \epsilon_{2,t} \]

3. Join the models for \( X \) and \( Y \) in one, vector variable \( \mathbf{X}_t = [X_t, Y_t]' \).

   Vector autoregressive model for \( (X, Y) \) of order 1, \( VAR(1) \):
   \[
   \begin{bmatrix}
   X_t \\
   Y_t
   \end{bmatrix} =
   \begin{bmatrix}
   a_{1,0} \\
   a_{2,0}
   \end{bmatrix} +
   \begin{bmatrix}
   a_{1,1} & a_{1,2} \\
   a_{2,1} & a_{2,2}
   \end{bmatrix} \begin{bmatrix}
   X_{t-1} \\
   Y_{t-1}
   \end{bmatrix} +
   \begin{bmatrix}
   \epsilon_{1,t} \\
   \epsilon_{2,t}
   \end{bmatrix}
   
   \]

   and in matrix form
   \[
   \mathbf{X}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{X}_{t-1} + \epsilon_t
   \]
1. AR($p$): $X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$
1. **AR(p):** \( X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t \)

2. **DR\(_X(p_1, q_1)\) for \( X: \)**
   \( X_t = a_0 + a_{1,1} X_{t-1} + \cdots + a_{1,p_1} X_{t-p_1} + b_{1,1} Y_{t-1} + \cdots + b_{1,q_1} Y_{t-q_1} + \epsilon_{1,t} \)
1. **AR(p):** \( X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t \)

2. **DR\(_X(p_1, q_1)\) for \( X: \)**
   \( X_t = a_0 + a_{1,1} X_{t-1} + \cdots + a_{1,p_1} X_{t-p_1} + b_{1,1} Y_{t-1} + \cdots + b_{1,q_1} Y_{t-q_1} + \epsilon_{1,t} \)

   **and DR\(_Y(p_2, q_2)\) for \( Y: \)**
   \( Y_t = b_0 + a_{2,1} X_{t-1} + \cdots + a_{2,p_2} X_{t-p_2} + b_{2,1} Y_{t-1} + \cdots + b_{2,q_2} Y_{t-q_2} + \epsilon_{2,t} \)
Dynamic Regression and VAR modeling, order $p$

1. $\text{AR}(p)$: $X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$

2. $\text{DR}_X(p_1, q_1)$ for $X$: 
   
   $X_t = a_0 + a_{1,1} X_{t-1} + \cdots + a_{1,p_1} X_{t-p_1} + b_{1,1} Y_{t-1} + \cdots + b_{1,q_1} Y_{t-q_1} + \epsilon_{1,t}$

   and $\text{DR}_Y(p_2, q_2)$ for $Y$:

   $Y_t = b_0 + a_{2,1} X_{t-1} + \cdots + a_{2,p_2} X_{t-p_2} + b_{2,1} Y_{t-1} + \cdots + b_{2,q_2} Y_{t-q_2} + \epsilon_{2,t}$

   $p_1, q_1, p_2, q_2$ can all be different
1. **AR(p):** \( X_t = \phi_0 + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t \)

2. **DR_X(p_1, q_1)** for \( X \):
\[
X_t = a_0 + a_{1,1} X_{t-1} + \cdots + a_{1,p_1} X_{t-p_1} + b_{1,1} Y_{t-1} + \cdots + b_{1,q_1} Y_{t-q_1} + \epsilon_{1,t}
\]

and **DR_Y(p_2, q_2)** for \( Y \):
\[
Y_t = b_0 + a_{2,1} X_{t-1} + \cdots + a_{2,p_2} X_{t-p_2} + b_{2,1} Y_{t-1} + \cdots + b_{2,q_2} Y_{t-q_2} + \epsilon_{2,t}
\]

\( p_1, q_1, p_2, q_2 \) can all be different

3. **VAR(p) model for \( (X, Y) \):**
\[
\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} + \begin{bmatrix} a_{1,1} & b_{1,1} \\ a_{2,1} & b_{2,1} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix} + \cdots + \begin{bmatrix} a_{1,p} & b_{1,p} \\ a_{2,p} & b_{2,p} \end{bmatrix} \begin{bmatrix} X_{t-p} \\ Y_{t-p} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}
\]
\[
X_t = A_0 + A_1 X_{t-1} + \cdots + A_p X_{t-p} + \epsilon_t
\]
Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \) from two independent AR(1) processes:

\[ x_t = 0.95 x_{t-1} + \varepsilon_x \]
\[ y_t = 0.85 y_{t-1} + \varepsilon_y \]

DR form for \((x, y)\):
\[ DR_x(1, 0) \text{ and } DR_y(0, 1) \]

VAR form for \((x, y)\):
\[ A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \]

2. The first AR(1) process drives the second AR(1) process:

\[ x_t = 0.95 x_{t-1} + \varepsilon_x \]
\[ y_t = 0.5 x_{t-1} + 0.85 y_{t-1} + \varepsilon_y \]

DR \(x\):
\[ DR_x(1, 0) \]

VAR(1):
\[ A_1 = \begin{bmatrix} 0.95 & 0 \\ 0.5 & 0.85 \end{bmatrix} \]

3. The two AR(1) processes are inter-dependent:

\[ x_t = 1.2 x_{t-1} - 0.5 y_{t-1} + \varepsilon_x \]
\[ y_t = 0.3 y_{t-1} + \varepsilon_y \]

DR \(x\):
\[ DR_x(1, 1) \]

VAR(1):
\[ A_1 = \begin{bmatrix} 1.2 & -0.5 \\ 0 & 0.3 \end{bmatrix} \]
Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^n \), \( \{y_t\}_{t=1}^n \) from two independent AR(1) processes:
\[
X_t = 0.95X_{t-1} + \epsilon_t^X \\
Y_t = 0.85Y_{t-1} + \epsilon_t^Y
\]
Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \) from two independent AR(1) processes:
\[
X_t = 0.95X_{t-1} + \epsilon_t^X \\
Y_t = 0.85Y_{t-1} + \epsilon_t^Y
\]

DR form for \((X, Y)\): \(DR_X(1, 0)\) and \(DR_Y(0, 1)\)

VAR form for \((X, Y)\): VAR(1),
\[
A_1 = \begin{bmatrix}
0.95 & 0 \\
0 & 0.85
\end{bmatrix}
\]

2. The first AR(1) process drives the second AR(1) process:
\[
X_t = 0.95X_{t-1} + \epsilon_t^X \\
Y_t = 0.85X_{t-1} + \epsilon_t^Y
\]

DR \(X(1, 0)\) and DR \(Y(1, 1)\)

VAR(1):
\[
A_1 = \begin{bmatrix}
0.95 & 0 \\
0 & 0.85
\end{bmatrix}
\]

3. The two AR(1) processes are inter-dependent:
\[
X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \\
Y_t = 0.3Y_{t-1} + \epsilon_t^Y
\]

DR \(X(1, 1)\) and DR \(Y(1, 1)\)

VAR(1):
\[
A_1 = \begin{bmatrix}
1.2 & -0.5 \\
0.3 & 1
\end{bmatrix}
\]
Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^{n}, \{y_t\}_{t=1}^{n} \) from two independent AR(1) processes:

\[
X_t = 0.95X_{t-1} + \epsilon_t^X \quad Y_t = 0.85Y_{t-1} + \epsilon_t^Y
\]

DR form for \((X, Y)\): \(DR_X(1, 0)\) and \(DR_Y(0, 1)\)

2. The first AR(1) process drives the second AR(1) process:

\[
X_t = 0.95X_{t-1} + \epsilon_t^X \\
Y_t = 0.5X_{t-1} + 0.85Y_{t-1} + \epsilon_t^Y
\]

DR \((X, Y)\): \(DR_X(1, 0)\) and \(DR_Y(1, 1)\)

VAR(1):

\[
a_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.5 \end{bmatrix}
\]

3. The two AR(1) processes are inter-dependent:

\[
X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \\
Y_t = 0.3Y_{t-1} + \epsilon_t^Y
\]

DR \((X, Y)\): \(DR_X(1, 1)\) and \(DR_Y(1, 1)\)

VAR(1):

\[
a_1 = \begin{bmatrix} 1.2 & -0.5 \\ 0.3 & 0 \end{bmatrix}
\]

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Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^{n}, \{y_t\}_{t=1}^{n} \) from two independent AR(1) processes:
   \[ X_t = 0.95X_{t-1} + \epsilon_t^X \quad Y_t = 0.85 Y_{t-1} + \epsilon_t^Y \]

   DR form for \((X, Y)\): DR\(_X(1, 0)\) and DR\(_Y(0, 1)\)

   VAR form for \((X, Y)\): VAR(1),
   \[ X_t = A_0 + A_1 X_{t-1} + \epsilon_t, \]
   \[ A_0 = \emptyset \quad A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \]

2. The first AR(1) process drives the second AR(1) process:
   \[ X_t = 0.95X_{t-1} + \epsilon_t^X \quad Y_t = 0.55 X_{t-1} + 0.85 \epsilon_t^Y \]

   DR\(_X(1, 0)\) and DR\(_Y(1, 1)\)

   VAR(1):
   \[ A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \]

3. The two AR(1) processes are inter-dependent:
   \[ X_t = 1.2 X_{t-1} - 0.5 Y_{t-1} + \epsilon_t^X \quad Y_t = 0.3 X_{t-1} + 0.85 \epsilon_t^Y \]

   DR\(_X(1, 1)\) and DR\(_Y(1, 1)\)

   VAR(1):
   \[ A_1 = \begin{bmatrix} 1.2 & -0.5 \\ 0.3 & 0.85 \end{bmatrix} \]
Examples of DR and VAR

1. Time series \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n from two independent AR(1) processes:
   \[ X_t = 0.95X_{t-1} + \epsilon^X_t \quad Y_t = 0.85Y_{t-1} + \epsilon^Y_t \]

   DR form for \((X, Y)\): \(\text{DR}_X(1, 0)\) and \(\text{DR}_Y(0, 1)\)

   VAR form for \((X, Y)\): VAR(1), \[ X_t = A_0 + A_1X_{t-1} + \epsilon_t, \]
   \[ A_0 = \emptyset \quad A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \]

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Examples of DR and VAR

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   DR form for \((X, Y)\): \( DR_X(1, 0) \) and \( DR_Y(0, 1) \)
   VAR form for \((X, Y)\): \( VAR(1) \),
   \[ X_t = A_0 + A_1X_{t-1} + \epsilon_t, \]
   \[ A_0 = \emptyset, \quad A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \]

2. The first AR(1) process drives the second AR(1) process:
   \[ X_t = 0.95X_{t-1} + \epsilon_t^X, \quad Y_t = 0.5X_{t-1} + 0.85Y_{t-1} + \epsilon_t^Y \]
Examples of DR and VAR

1. Time series \( \{ x_t \}_{t=1}^n, \{ y_t \}_{t=1}^n \) from two independent AR(1) processes:
\[
X_t = 0.95 X_{t-1} + \epsilon_t^X \quad Y_t = 0.85 Y_{t-1} + \epsilon_t^Y
\]
DR form for \((X, Y)\): \( \text{DR}_X(1, 0) \) and \( \text{DR}_Y(0, 1) \)
VAR form for \((X, Y)\): \( \text{VAR}(1) \),
\[
X_t = A_0 + A_1 X_{t-1} + \epsilon_t,
\]
\[
A_0 = \emptyset \quad A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix}
\]

2. The first AR(1) process drives the second AR(1) process:
\[
X_t = 0.95 X_{t-1} + \epsilon_t^X \quad Y_t = 0.5 X_{t-1} + 0.85 Y_{t-1} + \epsilon_t^Y
\]
DR\(X(1, 0)\) and DR\(Y(1, 1)\)
Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \) from two independent AR(1) processes:
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   DR form for \((X, Y)\): \( \text{DR}_X(1, 0) \text{ and } \text{DR}_Y(0, 1) \)
   VAR form for \((X, Y)\): \( \text{VAR}(1), X_t = A_0 + A_1X_{t-1} + \epsilon_t \),
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   DR\(X(1, 0)\text{ and } \text{DR}_Y(1, 1) \)
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Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^{n}, \{y_t\}_{t=1}^{n} \) from two independent AR(1) processes:
   \[ X_t = 0.95X_{t-1} + \epsilon_t^X \quad Y_t = 0.85 Y_{t-1} + \epsilon_t^Y \]
   DR form for \((X, Y)\): \( \text{DR}_X(1, 0) \) and \( \text{DR}_Y(0, 1) \)
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   DR \( \text{DR}_X(1, 0) \) and \( \text{DR}_Y(1, 1) \)
   VAR(1): \[ A_1 = \begin{bmatrix} 0.95 & 0 \\ 0.5 & 0.85 \end{bmatrix} \]

3. The two AR(1) processes are inter-dependent:
   \[ X_t = 1.2X_{t-1} - 0.5 Y_{t-1} + \epsilon_t^X \quad Y_t = 0.6X_{t-1} + 0.3 Y_{t-1} + \epsilon_t^Y \]
Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \) from two independent AR(1) processes:
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X_t = 0.95X_{t-1} + \epsilon_t^X \\
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DR form for \((X, Y)\): \(\text{DR}_X(1, 0)\) and \(\text{DR}_Y(0, 1)\)
VAR form for \((X, Y)\): \(\text{VAR}(1)\),
\[
X_t = A_0 + A_1X_{t-1} + \epsilon_t,
\]
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A_0 = \emptyset, \quad A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix}
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DR\(_X(1, 0)\) and DR\(_Y(1, 1)\)
VAR\((1)\): \(A_1 = \begin{bmatrix} 0.95 & 0 \\ 0.5 & 0.85 \end{bmatrix}\]

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\[
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\]
DR\(_X(1, 1)\) and DR\(_Y(1, 1)\)
Examples of DR and VAR

1. Time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \) from two independent AR(1) processes:
   \[ X_t = 0.95X_{t-1} + \epsilon^X_t \quad Y_t = 0.85Y_{t-1} + \epsilon^Y_t \]
   
   DR form for \((X, Y)\): \( DR_X(1, 0) \) and \( DR_Y(0, 1) \)
   
   VAR form for \((X, Y)\): \( VAR(1) \), \( X_t = A_0 + A_1X_{t-1} + \epsilon_t \),
   \[ A_0 = \emptyset \quad A_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \]

2. The first AR(1) process drives the second AR(1) process:
   \[ X_t = 0.95X_{t-1} + \epsilon^X_t \quad Y_t = 0.5X_{t-1} + 0.85Y_{t-1} + \epsilon^Y_t \]
   
   \( DR_X(1, 0) \) and \( DR_Y(1, 1) \)
   
   \( VAR(1): A_1 = \begin{bmatrix} 0.95 & 0 \\ 0.5 & 0.85 \end{bmatrix} \)

3. The two AR(1) processes are inter-dependent:
   \[ X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon^X_t \quad Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon^Y_t \]
   
   \( DR_X(1, 1) \) and \( DR_Y(1, 1) \)
   
   \( VAR(1): A_1 = \begin{bmatrix} 1.2 & -0.5 \\ 0.6 & 0.3 \end{bmatrix} \)
The DR and VAR models can be extended adding nonlinear terms, e.g. $X_{t-1}^2$ or $X_{t-1} Y_{t-1}$. 

Alternatively, it can be written as $X_t = 1.4 - X_{t-1}^2 + 0.3 X_{t-2}$. This is a nonlinear AR(2) model.

---

see [1]: Chp 11, [2]: Sec 10.3
The DR and VAR models can be extended adding nonlinear terms, e.g. $X_{t-1}^2$ or $X_{t-1}Y_{t-1}$.

... such models get “complicated”!!!
The DR and VAR models can be extended adding nonlinear terms, e.g. $X_{t-1}^2$ or $X_{t-1}Y_{t-1}$.

... such models get “complicated”!!!

They are “complicated” even when there is no random terms $\epsilon_t$. 

Nonlinear dynamical systems

The DR and VAR models can be extended adding nonlinear terms, e.g. \(X_{t-1}^2\) or \(X_{t-1}Y_{t-1}\).

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\[ X_t = 1.4 - X_{t-1}^2 + 0.3 X_{t-2}. \]

\[ X_t = 1.4 - X_{t-1}^2 + X_{t-1}Y_{t-1}. \]

\[ X_t = 1.4 - X_{t-1}^2 + 0 \]... a nonlinear AR(2) model.

\[ H\text{enon map} \]

\[ X_t = 1.4 - X_{t-1}^2 + Y_{t-1}. \]

\[ X_t = 1.4 - X_{t-1}^2 + 0.3 Y_{t-1}. \]

\[ X_t = 1.4 - X_{t-1}^2 + 0.3 X_{t-2}Y_{t-1}. \]

\[ X_t = 1.4 - X_{t-1}^2 + 0.3 X_{t-2}Y_{t-2}. \]


see [1]: Chp 11, [2]: Sec 10.3

⇒ These are models for **nonlinear dynamical systems**
The DR and VAR models can be extended adding nonlinear terms, e.g. $X_{t-1}^2$ or $X_{t-1}Y_{t-1}$.

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$\implies$ These are models for nonlinear dynamical systems ... and chaos
The DR and VAR models can be extended adding nonlinear terms, e.g. $X_{t-1}^2$ or $X_{t-1}Y_{t-1}$.

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Henon map

$$X_t = 1.4 - X_{t-1}^2 + Y_{t-1} \quad Y_t = 0.3X_{t-1}$$
The DR and VAR models can be extended adding nonlinear terms, e.g. $X_{t-1}^2$ or $X_{t-1}Y_{t-1}$.

... such models get “complicated” !!!

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### Henon map

$$X_t = 1.4 - X_{t-1}^2 + Y_{t-1} \quad Y_t = 0.3X_{t-1}$$

Alternatively, it can be written as:

$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2}$$
The DR and VAR models can be extended adding nonlinear terms, e.g. $X_{t-1}^2$ or $X_{t-1}Y_{t-1}$.

... such models get “complicated”!!!

They are “complicated” even when there is no random terms $\epsilon_t$

$\implies$ These are models for nonlinear dynamical systems ... and chaos

**Henon map**

$$X_t = 1.4 - X_{t-1}^2 + Y_{t-1}, \quad Y_t = 0.3X_{t-1}$$

Alternatively, it can be written as

$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2}$$

... a nonlinear AR(2) model.
Example: Two independent Henon maps, linear measures

Time series \( \{x_t\}_{t=1}^n, \{y_t\}_{t=1}^n \), \( n = 300 \) from two independent Henon maps:

\[
X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} \quad Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2}
\]
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\[Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2}\]

Alternating autocorrelation, zero cross-correlation (correctly!)

After prewhitening, zero autocorrelation, zero cross-correlation
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Alternating autocorrelation, zero cross-correlation (correctly!)

The time series $X$ and $Y$

Autocorrelation of the two time series

Cross-correlation of the original time series
Example: Two independent Henon maps, linear measures

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Analysis of multi-variate time series by means of networks
Example: Two independent Henon maps, linear measures

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Alternating autocorrelation, zero cross-correlation (correctly!)

After prewhitening, zero autocorrelation, zero cross-correlation

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Analysis of multi-variate time series by means of networks
Example: Independent Henon maps, nonlinear measures

Delayed mutual
information $I_X(\tau)$
and $I_Y(\tau)$ and
cross mutual
information
$I_{XY}(\tau)$
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

![Delayed mutual information of the original time series](image)
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

Significant delayed mutual information (for small lags),
Example: Independent Henon maps, nonlinear measures

Delayed mutual information \( I_X(\tau) \) and \( I_Y(\tau) \) and cross mutual information \( I_{XY}(\tau) \)

Significant delayed mutual information (for small lags), Insignificant cross mutual information?
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

Significant delayed mutual information (for small lags), Insignificant cross mutual information ?

$I_X(\tau), I_Y(\tau)$ and $I_{XY}(\tau)$ after prewhitening
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

Significant delayed mutual information (for small lags), Insignificant cross mutual information?

$I_X(\tau), I_Y(\tau)$ and $I_{XY}(\tau)$ after prewhitening
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

Significant delayed mutual information (for small lags), Insignificant cross mutual information?

$l_X(\tau), l_Y(\tau)$ and $l_{XY}(\tau)$ after prewhitening
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

Significant delayed mutual information (for small lags), insignificant cross mutual information?

$\hat{I}_X(\tau), \hat{I}_Y(\tau)$ and $\hat{I}_{XY}(\tau)$ after prewhitening

Smaller but still significant delayed mutual information (for small lags),
Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

Significant delayed mutual information (for small lags), Insignificant cross mutual information?

$I_X(\tau), I_Y(\tau)$ and $I_{XY}(\tau)$ after prewhitening

Smaller but still significant delayed mutual information (for small lags), Insignificant cross mutual information?
Example: Two dependent Henon maps - 1

\[ X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} \]
\[ Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.2(Y_{t-1}^2 - X_{t-1}^2) \]
Example: Two dependent Henon maps - 1

\[ X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} \]
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The time series X and Y

After prewhitening, zero autocorrelation, significant cross-correlation at \( \tau = 0 \)
Example: Two dependent Henon maps - 1

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The time series X and Y

Autocorrelation of the two time series
Example: Two dependent Henon maps - 1

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The time series $X$ and $Y$

Autocorrelation of the two time series

Cross-correlation of the original time series
Example: Two dependent Henon maps - 1

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Alternating autocorrelation, significant cross-correlation at \( \tau = 0 \)
Example: Two dependent Henon maps - 1

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Alternating autocorrelation, significant cross-correlation at \( \tau = 0 \)

After prewhitening, zero autocorrelation, significant cross-correlation at \( \tau = 0 \)
Example: Two dependent Henon maps - 1

Significant $I(X(\tau), Y(\tau))$ for $\tau \geq 0$, $X_t$ is "correlated" to $Y_{t+\tau}$

Significant $I(X(\tau), Y(\tau))$, Small $I(XY(\tau))$ for $\tau \geq 0$, is it significant?
Example: Two dependent Henon maps - 1

Significant $I_X(\tau), I_Y(\tau)$ for $\tau \geq 0$, $X_t$ is "correlated" to $Y_{t+\tau}$.

Significant $I_{XY}(\tau)$ for $\tau \geq 0$, is it significant?
Significant $I_X(\tau)$, $I_Y(\tau)$,
Example: Two dependent Henon maps - 1

Significant $I_X(\tau)$, $I_Y(\tau)$, Significant $I_{XY}(\tau)$ for $\tau \geq 0$, $X_t$ is “correlated” to $Y_{t+\tau}$
Example: Two dependent Henon maps - 1

Significant \( I_X(\tau), I_Y(\tau) \),

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Significant $I_{XY}(\tau)$ for $\tau \geq 0$, $X_t$ is “correlated” to $Y_{t+\tau}$

Significant $I_X(\tau)$, $I_Y(\tau)$,
Small $I_{XY}(\tau)$ for $\tau \geq 0$, is it significant?
Example: Two dependent Henon maps - 2

\[ X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2) \]

\[ Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2) \]
Example: Two dependent Henon maps - 2

\[ X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2) \]
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The time series X and Y
Example: Two dependent Henon maps - 2

\[ X_t = 1.4 - X_{t-1}^2 + 0.3 X_{t-2} + 0.14 (X_{t-1}^2 - Y_{t-1}^2) \]
\[ Y_t = 1.4 - Y_{t-1}^2 + 0.3 Y_{t-2} + 0.08 (Y_{t-1}^2 - X_{t-1}^2) \]

The time series X and Y

Autocorrelation of the two time series

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\]

Alternating autocorrelation, alternating cross-correlation
Example: Two dependent Henon maps - 2

\[ X_t = 1.4 - X^2_{t-1} + 0.3X_{t-2} + 0.14(Y^2_{t-1} - Y^2_{t-1}) \]
\[ Y_t = 1.4 - Y^2_{t-1} + 0.3Y_{t-2} + 0.08(X^2_{t-1} - X^2_{t-1}) \]

Alternating autocorrelation, alternating cross-correlation
Example: Two dependent Henon maps - 2

\[ X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2) \]
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Alternating autocorrelation, alternating cross-correlation
Example: Two dependent Henon maps - 2

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X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2) \\
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\]

The time series \(X\) and \(Y\)

Autocorrelation of the two time series

Cross-correlation of the original time series

Alternating autocorrelation, alternating cross-correlation

The prewhitened time series \(X\) and \(Y\)

Autocorrelation of the prewhitened time series

Cross-correlation of the prewhitened time series
Example: Two dependent Henon maps - 2

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X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2) \\
Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)
\]

The time series X and Y

Autocorrelation of the two time series

Cross-correlation of the original time series

Alternating autocorrelation, alternating cross-correlation

The prewhitened time series X and Y

Autocorrelation of the prewhitened time series

Cross-correlation of the prewhitened time series

After prewhitening, zero autocorrelation, significant cross-correlation at \(\tau = 0\)
Example: Two dependent Henon maps - 2

Significant \( I_X(\tau), I_Y(\tau), \) for \( \tau < 0, \tau \geq 0, \) \( X_t \) is "correlated" to \( Y_{t+|\tau|} \) and \( Y_{t-|\tau|} \)

Significant \( I_{XY}(\tau) \) for \( \tau \geq 0, \) is it significant?
Example: Two dependent Henon maps - 2

Significant $I(X(\tau), Y(\tau))$ for $\tau < 0$, $\tau \geq 0$, $X_t$ is "correlated" to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$.

Significant $I(X(\tau), Y(\tau))$, Small $I_{XY}(\tau)$ for $\tau \geq 0$, is it significant?

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Example: Two dependent Henon maps - 2

Significant $I_X(\tau)$, $I_Y(\tau)$,
Example: Two dependent Henon maps - 2

Significant $I_X(\tau)$, $I_Y(\tau)$.

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Significant $I_X(\tau)$, $I_Y(\tau)$,
Small $I_{XY}(\tau)$ for $\tau \geq 0$, is it significant?
Example: Two dependent Henon maps - 2, large $n$

The same but for $n = 4000$

Significant $I_X(\tau), I_Y(\tau),$
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Significant $I_X(\tau), I_Y(\tau)$,

Small $I_{XY}(\tau)$ for $\tau \geq 0$
Example: Two dependent Henon maps - 2, large $n$

The same but for $n = 4000$

Significant $I_X(\tau), I_Y(\tau),$
Significant $I_{XY}(\tau)$ for $\tau < 0, \tau \geq 0$, $X_t$ is “correlated” to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$

Significant $I_X(\tau), I_Y(\tau),$
Small $I_{XY}(\tau)$ for $\tau \geq 0$ ... but also for $\tau < 0$
Example: VAR model, $K = 3$

$$
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} =
\begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix}
\begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots +
\begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
$$

$$X_t = [X_{1,t}, X_{2,t}, X_{3,t}]'$$

$$X_t = A_1 X_{t-1} + \epsilon_t$$
Example: VAR model, $K = 3$

\[
\begin{bmatrix}
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\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
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X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \\
X_t = A_1 X_{t-1} + \epsilon_t
\]

The original time series

Autocorrelation of the time series
Example: VAR model, $K = 3$

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\epsilon_{2,t} \\
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Example: VAR model, \( K = 3 \)

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\begin{bmatrix}
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+ \cdots + 
\begin{bmatrix}
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\epsilon_{2,t} \\
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\[X_t = [X_{1,t}, X_{2,t}, X_{3,t}]'
X_t = A_1 X_{t-1} + \epsilon_t\]
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\begin{bmatrix}
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\begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
\]

\[X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \quad X_t = A_1 X_{t-1} + \epsilon_t\]
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$
and $N$ time series $\{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^n$

Candidate similarity measures $\text{sim}(i, j)$ for any observed $X_i, X_j$ (without or after prewhitening):

1. Delayed cross correlation $r_{X_i X_j}(\tau)$
2. Delayed cross mutual information $I_{X_i X_j}(\tau)$

What $\tau$ to choose?

1. $\tau = 0$ correlation of $X_i, t$ and $X_j, t$

2. $\tau > 0$ correlation of $X_i, t$ and $X_j, t + \tau$, $X_i$ influences the evolution of $X_j$

3. $\tau < 0$ correlation of $X_i, t$ and $X_j, t - |\tau|$, $X_j$ influences the evolution of $X_i$

$X_i$ influences the evolution of $X_j$ $\Rightarrow$ $X_i$ (Granger) causes $X_j$

There are other measures more appropriate to measure Granger causality.

Dimitris Kugiumtzis  
Analysis of multi-variate time series by means of networks
Similarity measure for time series network

\( N \) variables (nodes) \( X_1, X_2, \ldots, X_N \)
and \( N \) time series \( \{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^n \)

Candidate similarity measures \( \text{sim}(i,j) \) for any observed \( X_i, X_j \) (without or after prewhitening):

1. \( \tau = 0 \) correlation of \( X_i, t \) and \( X_j, t \)
2. \( \tau > 0 \) correlation of \( X_i, t \) and \( X_j, t + \tau \), \( X_i \) influences the evolution of \( X_j \)
3. \( \tau < 0 \) correlation of \( X_i, t \) and \( X_j, t - |\tau| \), \( X_j \) influences the evolution of \( X_i \)

\( X_i \) influences the evolution of \( X_j \) \( \implies \) \( X_i \) (Granger) causes \( X_j \)

There are other measures more appropriate to measure Granger causality.
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$

and $N$ time series $\{x_1,t, x_2,t, \ldots, x_N,t\}_{t=1}^n$

Candidate similarity measures $\text{sim}(i, j)$ for any observed $X_i, X_j$ (without or after prewhitening):

1. delayed cross correlation $r_{X_i X_j}(\tau)$
Similarity measure for time series network

\(N\) variables (nodes) \(X_1, X_2, \ldots, X_N\)
and \(N\) time series \(\{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^n\)

Candidate similarity measures \(\text{sim}(i,j)\) for any observed \(X_i, X_j\) (without or after prewhitening):

1. delayed cross correlation \(r_{X_iX_j}(\tau)\)
2. delayed cross mutual information \(I_{X_iX_j}(\tau)\)

\(r_{X_iX_j}(\tau)\) and \(I_{X_iX_j}(\tau)\) are measures of the linear and non-linear dependence between \(X_i\) and \(X_j\) at a lag \(\tau\).
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$
and $N$ time series $\{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^n$

Candidate similarity measures $\text{sim}(i,j)$ for any observed $X_i, X_j$ (without or after prewhitening):

1. delayed cross correlation $r_{X_iX_j}(\tau)$
2. delayed cross mutual information $I_{X_iX_j}(\tau)$

What $\tau$ to choose?
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$
and $N$ time series $\{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^n$

Candidate similarity measures $\text{sim}(i, j)$ for any observed $X_i, X_j$ (without or after prewhitening):

1. delayed cross correlation $r_{X_i X_j}(\tau)$
2. delayed cross mutual information $I_{X_i X_j}(\tau)$

What $\tau$ to choose?

1. $\tau = 0$ correlation of $X_{i,t}$ and $X_{j,t}$
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$ and $N$ time series $\{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^{n}$

Candidate similarity measures $\text{sim}(i, j)$ for any observed $X_i, X_j$ (without or after prewhitening):

1. delayed cross correlation $r_{X_i X_j}(\tau)$
2. delayed cross mutual information $I_{X_i X_j}(\tau)$

What $\tau$ to choose?

1. $\tau = 0$ correlation of $X_{i,t}$ and $X_{j,t}$
2. $\tau > 0$ correlation of $X_{i,t}$ and $X_{j,t+\tau}$, $X_i$ influences the evolution of $X_j$
Similarity measure for time series network

\( N \) variables (nodes) \( X_1, X_2, \ldots, X_N \) and \( N \) time series \( \{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^n \)

Candidate similarity measures sim\((i,j)\) for any observed \( X_i, X_j \) (without or after prewhitening):

1. delayed cross correlation \( r_{X_iX_j}(\tau) \)
2. delayed cross mutual information \( I_{X_iX_j}(\tau) \)

What \( \tau \) to choose?

1. \( \tau = 0 \) correlation of \( X_{i,t} \) and \( X_{j,t} \)
2. \( \tau > 0 \) correlation of \( X_{i,t} \) and \( X_{j,t+\tau}, \) \( X_i \) influences the evolution of \( X_j \)
3. \( \tau < 0 \) correlation of \( X_{i,t} \) and \( X_{j,t-|\tau|}, \) \( X_j \) influences the evolution of \( X_i \)
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$
and $N$ time series \{\(x_{1,t}, x_{2,t}, \ldots, x_{N,t}\)\} \(t=1\)

Candidate similarity measures \(\text{sim}(i,j)\) for any observed $X_i, X_j$ (without or after prewhitening):

1. delayed cross correlation $r_{X_iX_j}(\tau)$
2. delayed cross mutual information $I_{X_iX_j}(\tau)$

What $\tau$ to choose?

1. $\tau = 0$ correlation of $X_{i,t}$ and $X_{j,t}$
2. $\tau > 0$ correlation of $X_{i,t}$ and $X_{j,t+\tau}$, $X_i$ influences the evolution of $X_j$
3. $\tau < 0$ correlation of $X_{i,t}$ and $X_{j,t-|\tau|}$, $X_j$ influences the evolution of $X_i$

$X_i$ influences the evolution of $X_j$ $\Rightarrow$ $X_i$ (Granger) causes $X_j$
Similarity measure for time series network

$N$ variables (nodes) $X_1, X_2, \ldots, X_N$
and $N$ time series $\{x_{1,t}, x_{2,t}, \ldots, x_{N,t}\}_{t=1}^n$

Candidate similarity measures $\text{sim}(i,j)$ for any observed $X_i, X_j$ (without or after prewhitening):

1. delayed cross correlation $r_{X_i X_j}(\tau)$
2. delayed cross mutual information $I_{X_i X_j}(\tau)$

What $\tau$ to choose?

1. $\tau = 0$ correlation of $X_{i,t}$ and $X_{j,t}$
2. $\tau > 0$ correlation of $X_{i,t}$ and $X_{j,t+\tau}$, $X_i$ influences the evolution of $X_j$
3. $\tau < 0$ correlation of $X_{i,t}$ and $X_{j,t-|\tau|}$, $X_j$ influences the evolution of $X_i$

$X_i$ influences the evolution of $X_j \implies X_i$ (Granger) causes $X_j$

There are other measures more appropriate to measure Granger causality.
Example: VAR model, $K = 3$

\[
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} = \begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix} \begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots + \begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
\]

\[
X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \quad X_t = A_1 X_{t-1} + \epsilon_t
\]
Example: VAR model, $K = 3$

$$
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} = \begin{bmatrix} 0.95 & -0.5 & -0.3 \\ 0 & 0.85 & 0.3 \\ 0 & 0 & 0.9 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots + \begin{bmatrix} \epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
$$

$$
X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \quad X_t = A_1 X_{t-1} + \epsilon_t
$$

Cross correlation matrix $R(\tau)$
Example: VAR model, $K = 3$

$$
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} =
\begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix}
\begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots +
\begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
$$

$$X_t = [X_{1,t}, X_{2,t}, X_{3,t}]'$$

$$X_t = A_1X_{t-1} + \epsilon_t$$

Cross correlation matrix $R(\tau)$

$$R(0) =
\begin{bmatrix}
-0.00 & 0.01 \\
-0.00 & 0.11 \\
0.01 & 0.11
\end{bmatrix}$$
Example: VAR model, $K = 3$

\[
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} = \begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix} \begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots + \begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
\]

\[
X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' 
\]

\[
X_t = A_1 X_{t-1} + \epsilon_t
\]

Cross correlation matrix $R(\tau)$

\[
R(0) = \begin{bmatrix}
-0.00 & 0.01 \\
0.00 & 0.11 \\
0.01 & 0.11
\end{bmatrix}
\]

\[
R(1) = \begin{bmatrix}
0.05 & -0.05 \\
-0.39 & 0.01 \\
-0.40 & 0.20
\end{bmatrix}
\]
Example: VAR model, $K = 3$

\[
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} = \begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix} \begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots + \begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
\]

\[
\mathbf{x}_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \quad \mathbf{x}_t = A_1\mathbf{x}_{t-1} + \epsilon_t
\]

Cross correlation matrix $R(\tau)$

\[
R(0) = \begin{bmatrix}
-0.00 & 0.01 \\
-0.00 & 0.11 \\
0.01 & 0.11
\end{bmatrix}
\]

\[
R(1) = \begin{bmatrix}
-0.39 & 0.01 \\
-0.40 & 0.20
\end{bmatrix}
\]

\[
R(2) = \begin{bmatrix}
-0.09 & 0.04 \\
-0.20 & -0.03 \\
-0.12 & -0.02
\end{bmatrix}
\]
Example: VAR model, $K = 3$

\[
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} = \begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix} \begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots + \begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
\]

\[
X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' = A_1X_{t-1} + \epsilon_t
\]

Cross correlation matrix $R(\tau)$

\[
R(0) = \begin{bmatrix}
-0.00 & 0.01 \\
-0.00 & 0.11 \\
0.01 & 0.11
\end{bmatrix}
\]

\[
R(1) = \begin{bmatrix}
0.05 & -0.05 \\
-0.39 & 0.01 \\
-0.40 & 0.20
\end{bmatrix}
\]

\[
R(2) = \begin{bmatrix}
-0.09 & 0.04 \\
-0.20 & -0.03 \\
-0.12 & -0.02
\end{bmatrix}
\]

Adjacency matrix threshold $\pm 2/\sqrt{n} = \pm 0.11$
Example: VAR model, $K = 3$

\[
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} = \begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix}
\begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots + \begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
\]

\[
X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \quad X_t = A_1X_{t-1} + \epsilon_t
\]

Cross correlation matrix $R(\tau)$

\[
R(0) = \begin{bmatrix}
-0.00 & 0.01 \\
-0.00 & 0.11 \\
0.01 & 0.11
\end{bmatrix}
\]

\[
R(1) = \begin{bmatrix}
-0.39 & 0.01 \\
-0.40 & 0.20
\end{bmatrix}
\]

\[
R(2) = \begin{bmatrix}
-0.09 & 0.04 \\
-0.20 & -0.03 \\
-0.12 & -0.02
\end{bmatrix}
\]

Adjacency matrix

threshold $\pm 2/\sqrt{n} = \pm 0.11$

\[
A(0) = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
Example: VAR model, $K = 3$

\[
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} =
\begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix}
\begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots +
\begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
\]

\[X_t = [X_{1,t}, X_{2,t}, X_{3,t}]'\]

\[X_t = A_1 X_{t-1} + \epsilon_t\]

Cross correlation matrix $R(\tau)$

\[
R(0) =
\begin{bmatrix}
-0.00 & 0.01 \\
-0.00 & 0.11 \\
0.01 & 0.11
\end{bmatrix}
\]

\[
R(1) =
\begin{bmatrix}
-0.39 & 0.01 \\
-0.40 & 0.20 \\
-0.09 & 0.04
\end{bmatrix}
\]

\[
R(2) =
\begin{bmatrix}
-0.20 & -0.03 \\
-0.12 & -0.02
\end{bmatrix}
\]

Adjacency matrix threshold $\pm 2/\sqrt{n} = \pm 0.11$

\[
A(0) =
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
A(1) =
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\]

Dimitris Kugiumtzis  
Analysis of multi-variate time series by means of networks
Example: VAR model, $K = 3$

$$
\begin{bmatrix}
X_{1,t} \\
X_{2,t} \\
X_{3,t}
\end{bmatrix} =
\begin{bmatrix}
0.95 & -0.5 & -0.3 \\
0 & 0.85 & 0.3 \\
0 & 0 & 0.9
\end{bmatrix}
\begin{bmatrix}
X_{1,t-1} \\
X_{2,t-1} \\
X_{3,t-1}
\end{bmatrix} + \cdots +
\begin{bmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t} \\
\epsilon_{3,t}
\end{bmatrix}
$$

$$
X_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \quad X_t = A_1 X_{t-1} + \epsilon_t
$$

Cross correlation matrix $R(\tau)$

$R(0) =
\begin{bmatrix}
-0.00 & 0.01 \\
-0.00 & 0.11 \\
0.01 & 0.11
\end{bmatrix}$

$R(1) =
\begin{bmatrix}
-0.39 & 0.01 \\
-0.40 & 0.20
\end{bmatrix}$

$R(2) =
\begin{bmatrix}
-0.20 & 0.04 \\
-0.12 & -0.02
\end{bmatrix}$

Adjacency matrix threshold $\pm 2/\sqrt{n} = \pm 0.11$

$A(0) =
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}$

$A(1) =
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}$

$A(2) =
\begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}$

Dimitris Kugiumtzis

Analysis of multi-variate time series by means of networks
**Example: VAR model, $K = 5$**

$$\mathbf{X}_t = [X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t}, X_{5,t}]' \quad \mathbf{X}_t = A_1 \mathbf{X}_{t-1} + \epsilon_t$$

$$A_1 = \begin{bmatrix}
-0.95 & 0.2 & -0.3 & 0.4 & -0.8 \\
0 & -0.2 & -0.3 & -0.4 & 0.9 \\
0 & 0 & -0.1 & -0.1 & 0.8 \\
0 & 0 & 0 & -0.8 & -0.9 \\
0 & 0 & 0 & 0 & 0.8 \\
\end{bmatrix}$$
Example: VAR model, $K = 5$

$$X_t = [X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t}, X_{5,t}]'$$

$$X_t = A_1 X_{t-1} + \epsilon_t$$

$$A_1 = \begin{bmatrix}
-0.95 & 0.2 & -0.3 & 0.4 & -0.8 \\
0 & -0.2 & -0.3 & -0.4 & 0.9 \\
0 & 0 & -0.1 & -0.1 & 0.8 \\
0 & 0 & 0 & -0.8 & -0.9 \\
0 & 0 & 0 & 0 & 0.8 \\
\end{bmatrix}$$

$$R(0) = \begin{bmatrix}
-0.62 & -0.47 & 0.40 & 0.07 \\
-0.62 & 0.58 & -0.36 & 0.05 \\
-0.47 & 0.58 & -0.42 & 0.04 \\
0.40 & -0.36 & -0.42 & -0.04 \\
0.07 & 0.05 & 0.04 & -0.04 \\
\end{bmatrix}$$
Example: VAR model, $K = 5$

\[
X_t = [X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t}, X_{5,t}]' \quad X_t = A_1X_{t-1} + \epsilon_t
\]

\[
A_1 = \begin{bmatrix}
-0.95 & 0.2 & -0.3 & 0.4 & -0.8 \\
0 & -0.2 & -0.3 & -0.4 & 0.9 \\
0 & 0 & -0.1 & -0.1 & 0.8 \\
0 & 0 & 0 & -0.8 & -0.9 \\
0 & 0 & 0 & 0 & 0.8
\end{bmatrix}
\]

\[
R(0) = \begin{bmatrix}
-0.62 & -0.47 & 0.40 & 0.07 \\
-0.62 & 0.58 & -0.36 & 0.05 \\
-0.47 & 0.58 & -0.42 & 0.04 \\
0.40 & -0.36 & -0.42 & -0.04 \\
0.07 & 0.05 & 0.04 & -0.04
\end{bmatrix}
\]

\[
R(1) = \begin{bmatrix}
-0.01 & 0.03 & -0.04 & 0.02 \\
0.04 & -0.12 & 0.09 & 0.02 \\
0.29 & -0.12 & 0.20 & 0.02 \\
-0.53 & 0.48 & 0.26 & 0.06 \\
0.49 & -0.53 & -0.62 & 0.65
\end{bmatrix}
\]
Example: VAR model, $K = 5$

$$X_t = [X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t}, X_{5,t}]' \quad X_t = A_1 X_{t-1} + \epsilon_t$$

$$A_1 = \begin{bmatrix}
-0.95 & 0.2 & -0.3 & 0.4 & -0.8 \\
0 & -0.2 & -0.3 & -0.4 & 0.9 \\
0 & 0 & -0.1 & -0.1 & 0.8 \\
0 & 0 & 0 & -0.8 & -0.9 \\
0 & 0 & 0 & 0 & 0.8
\end{bmatrix}$$

$$R(0) = \begin{bmatrix}
-0.62 & -0.47 & 0.40 & 0.07 \\
-0.62 & 0.58 & -0.36 & 0.05 \\
-0.47 & 0.58 & -0.42 & 0.04 \\
0.40 & -0.36 & -0.42 & -0.04 \\
0.07 & 0.05 & 0.04 & -0.04
\end{bmatrix} \quad A(0) = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$R(1) = \begin{bmatrix}
-0.01 & 0.03 & -0.04 & 0.02 \\
0.04 & -0.12 & 0.09 & 0.02 \\
0.29 & -0.12 & 0.20 & 0.02 \\
-0.53 & 0.48 & 0.26 & 0.06 \\
0.49 & -0.53 & -0.62 & 0.65
\end{bmatrix}$$
Example: VAR model, $K = 5$

$$X_t = [X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t}, X_{5,t}]' \quad X_t = A_1X_{t-1} + \epsilon_t$$

$$A_1 = \begin{bmatrix}
-0.95 & 0.2 & -0.3 & 0.4 & -0.8 \\
0 & -0.2 & -0.3 & -0.4 & 0.9 \\
0 & 0 & -0.1 & -0.1 & 0.8 \\
0 & 0 & 0 & -0.8 & -0.9 \\
0 & 0 & 0 & 0 & 0.8 \\
\end{bmatrix}$$

$$R(0) = \begin{bmatrix}
-0.62 & -0.47 & 0.40 & 0.07 \\
-0.62 & 0.58 & -0.36 & 0.05 \\
-0.47 & 0.58 & -0.42 & 0.04 \\
0.40 & -0.36 & -0.42 & -0.04 \\
0.07 & 0.05 & 0.04 & -0.04 \\
\end{bmatrix} \quad A(0) = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$R(1) = \begin{bmatrix}
-0.01 & 0.03 & -0.04 & 0.02 \\
0.04 & -0.12 & 0.09 & 0.02 \\
0.29 & -0.12 & 0.20 & 0.02 \\
-0.53 & 0.48 & 0.26 & 0.06 \\
0.49 & -0.53 & -0.62 & 0.65 \\
\end{bmatrix} \quad A(1) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}$$
Example: World market indices see [3]: Chp14, [4]

Detect information flow between stock indices
Detect information flow between stock indices

- A linear measure: cross correlation for $\tau = 0$ (correlation coefficient)
Detect information flow between stock indices

- A linear measure: cross correlation for $\tau = 0$ (correlation coefficient)
- A nonlinear measure: **transfer entropy** (in essence it is the conditional cross mutual information).
Example: World market indices see [3]: Chp14, [4]

Detect information flow between stock indices

- A linear measure: cross correlation for $\tau = 0$ (correlation coefficient)
- A nonlinear measure: transfer entropy (in essence it is the conditional cross mutual information).

Indices correlation coefficient transfer entropy


<table>
<thead>
<tr>
<th>Americas</th>
<th>1</th>
<th>MERV</th>
<th>Argentina</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>BVSP</td>
<td>Brazil</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>GSPTSE</td>
<td>Canada</td>
</tr>
<tr>
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Example: World market indices

Draw the network of “outgoing” transfer entropy and “incoming” transfer entropy.
Draw the network of “outgoing” transfer entropy and “incoming” transfer entropy.

Fig. 4: (Color online) Minimum spanning tree for (a) the outgoing transfer entropy and (b) the incoming transfer entropy. The minimum spanning tree is drawn by Pajek
Example: World market indices

Draw the network of “outgoing” transfer entropy and “incoming” transfer entropy.

GSPC (Standard and Poor 500) is the information source of the system.
Draw the network of “outgoing” transfer entropy and “incoming” transfer entropy.

- **GSPC** (Standard and Poor 500) is the information source of the system
- **AORD** (Australian index) is the information receiver

![Network Diagram](image_url)

