of the time. A larger number of races would be necessary to have better agreement with the past experience. Therefore we ran the program to simulate 1000 races with our four horses. Although very tired after all these races, they performed in a manner quite consistent with our estimates of their abilities. Acorn won 29.8 percent of the time, Balky 39.4 percent, Chestnut 19.5 percent, and Dolby 11.3 percent of the time.

The program **GeneralSimulation** uses this method to simulate repetitions of an arbitrary experiment with a finite number of outcomes occurring with known probabilities. □

Historical Remarks

Anyone who plays the same chance game over and over is really carrying out a simulation, and in this sense the process of simulation has been going on for centuries. As we have remarked, many of the early problems of probability might well have been suggested by gamblers' experiences.

It is natural for anyone trying to understand probability theory to try simple experiments by tossing coins, rolling dice, and so forth. The naturalist Buffon tossed a coin 4040 times, resulting in 2048 heads and 1992 tails. He also estimated the number π by throwing needles on a ruled surface and recording how many times the needles crossed a line (see Section 2.1). The English biologist W. F. R. Weldon[1] recorded 26,306 throws of 12 dice, and the Swiss scientist Rudolf Wolf[2] recorded 100,000 throws of a single die without a computer. Such experiments are very time-consuming and may not accurately represent the chance phenomena being studied. For example, for the dice experiments of Weldon and Wolf, further analysis of the recorded data showed a suspected bias in the dice. The statistician Karl Pearson analyzed a large number of outcomes at certain roulette tables and suggested that the wheels were biased. He wrote in 1894:

> Clearly, since the Casino does not serve the valuable end of huge laboratory for the preparation of probability statistics, it has no scientific *raison d'être*. Men of science cannot have their most refined theories disregarded in this shameless manner! The French Government must be urged by the hierarchy of science to close the gaming-saloons; it would be, of course, a graceful act to hand over the remaining resources of the Casino to the Académie des Sciences for the endowment of a laboratory of orthodox probability; in particular, of the new branch of that study, the application of the theory of chance to the biological problems of evolution, which is likely to occupy so much of men's thoughts in the near future.[3]

However, these early experiments were suggestive and led to important discoveries in probability and statistics. They led Pearson to the *chi-squared test,* which

[1] T. C. Fry, *Probability and Its Engineering Uses,* 2nd ed. (Princeton: Van Nostrand, 1965).

[2] E. Czuber, *Wahrscheinlichkeitsrechnung,* 3rd ed. (Berlin: Teubner, 1914).

[3] K. Pearson, "Science and Monte Carlo," *Fortnightly Review*, vol. 55 (1894), p. 193; cited in S. M. Stigler, *The History of Statistics* (Cambridge: Harvard University Press, 1986).

is of great importance in testing whether observed data fit a given probability distribution.

By the early 1900s it was clear that a better way to generate random numbers was needed. In 1927, L. H. C. Tippett published a list of 41,600 digits obtained by selecting numbers haphazardly from census reports. In 1955, RAND Corporation printed a table of 1,000,000 random numbers generated from electronic noise. The advent of the high-speed computer raised the possibility of generating random numbers directly on the computer, and in the late 1940s John von Neumann suggested that this be done as follows: Suppose that you want a random sequence of four-digit numbers. Choose any four-digit number, say 6235, to start. Square this number to obtain 38,875,225. For the second number choose the middle four digits of this square (i.e., 8752). Do the same process starting with 8752 to get the third number, and so forth.

More modern methods involve the concept of modular arithmetic. If a is an integer and m is a positive integer, then by $a \pmod{m}$ we mean the remainder when a is divided by m. For example, $10 \pmod 4 = 2$, $8 \pmod 2 = 0$, and so forth. To generate a random sequence $X_0, X_1, X_2, \ldots$ of numbers choose a starting number X_0 and then obtain the numbers X_{n+1} from X_n by the formula

$$X_{n+1} = (aX_n + c) \pmod{m} ,$$

where a, c, and m are carefully chosen constants. The sequence $X_0, X_1, X_2, \ldots$ is then a sequence of integers between 0 and $m - 1$. To obtain a sequence of real numbers in $[0, 1)$, we divide each X_j by m. The resulting sequence consists of rational numbers of the form j/m, where $0 \le j \le m - 1$. Since m is usually a very large integer, we think of the numbers in the sequence as being random real numbers in $[0, 1)$.

For both von Neumann's squaring method and the modular arithmetic technique the sequence of numbers is actually completely determined by the first number. Thus, there is nothing really random about these sequences. However, they produce numbers that behave very much as theory would predict for random experiments. To obtain different sequences for different experiments the initial number X_0 is chosen by some other procedure that might involve, for example, the time of day.[4]

During the Second World War, physicists at the Los Alamos Scientific Laboratory needed to know, for purposes of shielding, how far neutrons travel through various materials. This question was beyond the reach of theoretical calculations. Daniel McCracken, writing in the *Scientific American*, states:

> The physicists had most of the necessary data: they knew the average distance a neutron of a given speed would travel in a given substance before it collided with an atomic nucleus, what the probabilities were that the neutron would bounce off instead of being absorbed by the nucleus, how much energy the neutron was likely to lose after a given

[4]For a detailed discussion of random numbers, see D. E. Knuth, *The Art of Computer Programming,* vol. II (Reading: Addison-Wesley, 1969).

> collision and so on.[5]

John von Neumann and Stanislas Ulam suggested that the problem be solved by modeling the experiment by chance devices on a computer. Their work being secret, it was necessary to give it a code name. Von Neumann chose the name "Monte Carlo." Since that time, this method of simulation has been called the *Monte Carlo Method.*

William Feller indicated the possibilities of using computer simulations to illustrate basic concepts in probability in his book *An Introduction to Probability Theory and Its Applications.* In discussing the problem about the number of times in the lead in the game of "heads or tails" Feller writes:

> The results concerning fluctuations in coin tossing show that widely held beliefs about the law of large numbers are fallacious. These results are so amazing and so at variance with common intuition that even sophisticated colleagues doubted that coins actually misbehave as theory predicts. The record of a simulated experiment is therefore included.[6]

Feller provides a plot showing the result of 10,000 plays of *heads or tails* similar to that in Figure 1.5.

The martingale betting system described in Exercise 10 has a long and interesting history. Russell Barnhart pointed out to the authors that its use can be traced back at least to 1754, when Casanova, writing in his memoirs, *History of My Life,* writes

> She [Casanova's mistress] made me promise to go to the casino [the Ridotto in Venice] for money to play in partnership with her. I went there and took all the gold I found, and, determinedly doubling my stakes according to the system known as the martingale, I won three or four times a day during the rest of the Carnival. I never lost the sixth card. If I had lost it, I should have been out of funds, which amounted to two thousand zecchini.[7]

Even if there were no zeros on the roulette wheel so the game was perfectly fair, the martingale system, or any other system for that matter, cannot make the game into a favorable game. The idea that a fair game remains fair and unfair games remain unfair under gambling systems has been exploited by mathematicians to obtain important results in the study of probability. We will introduce the general concept of a martingale in Chapter 6.

The word *martingale* itself also has an interesting history. The origin of the word is obscure. The *Oxford English Dictionary* gives examples of its use in the

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[5] D. D. McCracken, "The Monte Carlo Method," *Scientific American,* vol. 192 (May 1955), p. 90.

[6] W. Feller, *Introduction to Probability Theory and its Applications,* vol. 1, 3rd ed. (New York: John Wiley & Sons, 1968), p. xi.

[7] G. Casanova, *History of My Life,* vol. IV, Chap. 7, trans. W. R. Trask (New York: Harcourt-Brace, 1968), p. 124.

early 1600s and says that its probable origin is the reference in Rabelais's Book One, Chapter 19:

> Everything was done as planned, the only thing being that Gargantua doubted if they would be able to find, right away, breeches suitable to the old fellow's legs; he was doubtful, also, as to what cut would be most becoming to the orator—the martingale, which has a draw-bridge effect in the seat, to permit doing one's business more easily; the sailor-style, which affords more comfort for the kidneys; the Swiss, which is warmer on the belly; or the codfish-tail, which is cooler on the loins.[8]

In modern uses martingale has several different meanings, all related to *holding down,* in addition to the gambling use. For example, it is a strap on a horse's harness used to hold down the horse's head, and also part of a sailing rig used to hold down the bowsprit.

The Labouchere system described in Exercise 9 is named after Henry du Pre Labouchere (1831–1912), an English journalist and member of Parliament. Labouchere attributed the system to Condorcet. Condorcet (1743–1794) was a political leader during the time of the French revolution who was interested in applying probability theory to economics and politics. For example, he calculated the probability that a jury using majority vote will give a correct decision if each juror has the same probability of deciding correctly. His writings provided a wealth of ideas on how probability might be applied to human affairs.[9]

Exercises

1 Modify the program **CoinTosses** to toss a coin n times and print out after every 100 tosses the proportion of heads minus 1/2. Do these numbers appear to approach 0 as n increases? Modify the program again to print out, every 100 times, both of the following quantities: the proportion of heads minus 1/2, and the number of heads minus half the number of tosses. Do these numbers appear to approach 0 as n increases?

2 Modify the program **CoinTosses** so that it tosses a coin n times and records whether or not the proportion of heads is within .1 of .5 (i.e., between .4 and .6). Have your program repeat this experiment 100 times. About how large must n be so that approximately 95 out of 100 times the proportion of heads is between .4 and .6?

3 In the early 1600s, Galileo was asked to explain the fact that, although the number of triples of integers from 1 to 6 with sum 9 is the same as the number of such triples with sum 10, when three dice are rolled, a 9 seemed to come up less often than a 10—supposedly in the experience of gamblers.

[8] Quoted in the *Portable Rabelais,* ed. S. Putnam (New York: Viking, 1946), p. 113.

[9] Le Marquise de Condorcet, *Essai sur l'Application de l'Analyse à la Probabilité dès Décisions Rendues a la Pluralité des Voix* (Paris: Imprimerie Royale, 1785).