WEIGHTED INTEGRALS OF ANALYTIC FUNCTIONS

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This paper is dedicated to Professor Ferenc Móricz on his 60th birthday.

ABSTRACT. We derive a formula of a weight v in terms of a given weight w such that the estimate

$$\int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z) \sim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p v(z)\, dm(z),$$

is valid for all analytic functions f on the unit disc.

1. INTRODUNCTION

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and $dm(z) = rdr\frac{d\theta}{\pi}$ the normalized Lebesgue area measure on $\mathbb{D}$. Our starting point is the estimate

$$\int_{\mathbb{D}} |f(z)|^p\, dm(z) \sim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p\, dm(z),$$

which is valid when $1 \le p < \infty$ for all analytic functions on the disc. The notation means that there are finite positive constants C and C' independent of f (but possibly depending on p) such that the left and right hand sides $L(f)$ and $R(f)$ satisfy

$$CR(f) \le L(f) \le C'R(f)$$

for all analytic f. In particular the two sides are either both infinite or both finite and in the latter case they are comparable. The weighting factor $(1-|z|)^p$ in the second integral compensates for the extra growth of the derivative as z approaches the boundary.

It is well known that a similar formula holds when the integrals are taken with respect to more general measures $d\mu(z) = (1-|z|)^\alpha dm(z)$, $\alpha > -1$:

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha\, dm(z) \sim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p+\alpha}\, dm(z),$$

and one can find in the literature other cases of analogous estimates. For example it was shown in Proposition 5 of [AS] that if $w(r)$, $0 < r < 1$, is a positive weight function which is integrable on $(0,1)$ and satisfies the conditions:

$$w(r) \ge \frac{C}{1-r}\int_r^1 w(u)\, du, \qquad \text{for} \quad 0 < r < 1. \tag{1.1}$$

for some positive constant C, and

$$w(sr+1-s) \ge C'w(r), \qquad \text{for} \quad 0 < r < 1, \tag{1.2}$$

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for some $s \in (0,1)$ and a positive constant C', then

$$\int_{\mathbb{D}} |f(z)|^p w(|z|)\, dm(z) \sim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p w(|z|)\, dm(z),$$

for all analytic functions on $\mathbb{D}$.

These kind of estimates or just the one sided inequalities are often useful in the study of spaces of analytic functions. One sided inequalities involving integrals of functions and their derivatives with respect to general measures on the disc have been studied by various authors, see for example [L].

The purpose of this article is to obtain a unifying statement of such estimates under some general conditions on w. And secondly to point out that these conditions are met for each of the most common weights, as well as for some less common ones, an instance of which is the doubly exponential weights of Example 3.3. To state our result we need some preliminaries which we now give.

We consider only radial weights. These arise from functions $w : [0,1) \to (0,\infty)$ that are Lebesgue integrable on $[0,1)$, and we put $w(z) = w(|z|)$ for each $z \in \mathbb{D}$. We further assume that our weights are sufficiently smooth on $[0,1)$. We will observe later that this requirement can be relaxed so that it will be sufficient for the weights to be sufficiently smooth near 1.

Given such a weight w we define the function

$$\psi(r) = \psi_w(r) \overset{\text{def}}{=} \frac{1}{w(r)} \int_r^1 w(u)\, du, \qquad 0 \le r < 1,$$

and we call it the **distortion function** of w. Some properties of ψ will be pointed out later. We put $\psi(z) = \psi(|z|)$ for $z \in \mathbb{D}$. A weight w is called **admissible** if it satisfies the following conditions:

Condition I1. There is a positive constant $A = A(w)$ such that

$$w(r) \ge \frac{A}{1-r} \int_r^1 w(u)\, du, \qquad \text{for} \quad 0 \le r < 1.$$

Condition I2. There is a positive constant $B = B(w)$ such that

$$w'(r) \le \frac{B}{1-r} w(r), \qquad \text{for} \quad 0 \le r < 1.$$

Condition D. For each sufficiently small positive δ there is a positive constant $C = C(\delta, w)$ such that

$$\sup_{0 \le r < 1} \frac{w(r)}{w(r + \delta\psi(r))} \le C.$$

Observe that (I1) implies $A\psi(r) \le 1 - r$ so that for each sufficiently small positive δ we have $r + \delta\psi(r) < 1$ and the quantity in the denominator of the fraction in (D) is well defined. Condition (I1) is the same as (1.1). Conditions (D) and (I2) will be discussed later. We can now state

Theorem 1.1. *Suppose $1 \le p < \infty$ and w is an admissible weight with distortion function ψ. Then*

$$\int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z) \sim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \psi(z)^p w(z)\, dm(z), \tag{1.3}$$

for all amalytic functions f on the disc.

The proof of the Theorem is given in section 2. In section 3 we we calculate explicitly the distortion function ψ for some families of weights. We do not aim at generality but rather at concrete formulas that show how ψ depends on w for various classes of weights.

We need some further background. For f analytic on $\mathbb{D}$ and $0 \leq p < \infty$ we denote

$$M_p^p(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^p \,\frac{d\theta}{2\pi}, \qquad 0 \leq r < 1,$$

the p-mean of f on $|z| = r$. $M_p^p(r,f)$ is a monotone increasing function of r and satisfies

$$\frac{d}{dr} M_p^p(r,f) \leq p M_p^{p-1}(r,f) M_p(r,f'), \qquad 0 < r < 1. \tag{1.4}$$

for $1 \leq p < \infty$, see [AS].

To facilitate the notation but also to show the connections with spaces of analytic functions we consider the weighted Bergman space A^p_w of all analytic functions on the disc with norm

$$\|f\|^p_{w,p} = \int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z) = \int_0^1 M_p^p(r,f) w(r) r\, dr < \infty$$

The spaces A^p_w are Banach spaces when $1 \leq p < \infty$ and the point evaluations $f \to f(\lambda)$ are bounded linear functionals for each $\lambda \in \mathbb{D}$ (see Remark 1 of [AS])

Weighted Bergman spaces with weights other than the standard $w(r) = (1-r)^\alpha$ have been studied for example in [KM], [LR], focusing mainly in the Hilbert space case $p = 2$.

It is easy to see that if w and ω are two weights and there is a $\sigma \in (0,1)$ such that $w \sim \omega$ on $(\sigma, 1)$ then $A^p_w = A^p_\omega$ with equivalence in norms. In particular weights may be modified on intervals $[0,\sigma]$, with $\sigma < 1$ without changing the Bergman space.

Finally we list some properties of the distortion function. Suppose w is a weight and let ψ be its distortion function, then it is easy to see that

(i) If $w(r)$ is decreasing then

$$\int_r^1 w(u)\, du \leq w(r) \int_r^1 du = (1-r) w(r), \tag{1.5}$$

hence $\psi(r) \leq 1 - r$ and in particular $\lim_{r\to 1} \psi(r) = 0$.

(ii) The derivative of ψ is

$$\psi'(r) = -1 - \frac{w'(r)}{w(r)} \psi(r), \qquad 0 \leq r < 1.$$

Thus if $w(r)$ is increasing then $\psi(r)$ is decreasing and clearly $\lim_{r\to 1} \psi(r) = 0$. If w is increasing and admissible then condition (I1) gives $\psi(r) \leq (1/A)(1-r)$. On the other hand

$$\int_r^1 w(u)\, du \geq w(r)(1-r)$$

so $\psi(r) \geq (1-r)$ and we conclude that for increasing admissible weights we have

$$\psi(r) \sim 1 - r.$$

(iii) For the standard weights $w(r) = (1-r)^\alpha$, $\alpha > -1$, we have

$$\psi(r) = \frac{1}{\alpha+1}(1-r).$$

(iv) We can recover w from its distortion function by solving the differential equation $(w(r)\psi(r))' = -w(r)$. We get

$$w(r) = \frac{1}{\psi(r)} \exp\left(-\int_0^r \frac{1}{\psi(u)}\, du\right).$$

We use the letters C, C', C_1, ..., to denote generic constants whose value may change at the next step. The constant may depend on parameters p, σ, ..., and we write in that case $C(p,\sigma)$.

2. Proof of the theorem

In this section we prove Theorem (1.3) and and discuss the conditions on w. Of the two inequalities involved, one is valid with no extra hypothesis on w.

Lemma 2.1. *Suppose $1 \le p < \infty$ and w is a weight with associated distortion function ψ. Then there is a constant $C = C(p,w)$ such that*

$$\int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z) \le C\left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \psi(z)^p w(z)\, dm(z)\right)$$

for all analytic f on the disc.

Proof. The proof follows the lines of Lemma 2 from [AS]. For f constant the claim is clear. Assume f is not constant and both integrals are finite. We are going to prove first the case $f(0) = 0$. We have

$$\begin{aligned}
\|f\|_{w,p}^p &= \int_0^1 M_p^p(r,f) w(r) r\, dr \\
&= \int_0^1 r w(r) \left(\int_0^r \frac{d}{ds} M_p^p(s,f)\, ds\right) dr \\
&\le p \int_0^1 r w(r) \int_0^r M_p^{p-1}(s,f) M_p(s,f')\, ds dr && \text{(by (1.4))} \\
&= p \int_0^1 M_p^{p-1}(s,f) M_p(s,f') \int_s^1 w(r) r\, dr ds && \text{(Fubini's)} \\
&\le p \int_0^1 M_p^{p-1}(s,f) M_p(s,f') w(s)\psi(s)\, ds
\end{aligned}$$

If $p = 1$ then the proof is finished for the case $f(0) = 0$. If $p > 1$ apply in the last integral Hölder's inequality with exponents $p/(p-1)$ and p,

$$\begin{aligned}
&\le p \left(\int_0^1 M_p^p(s,f) w(s)\, ds\right)^{\frac{p-1}{p}} \left(\int_0^1 M_p^p(s,f')\psi(s)^p w(s)\, ds\right)^{\frac{1}{p}} \\
&\sim p \|f\|_{w,p}^{p-1} \left(\int_{\mathbb{D}} |f'(z)|^p \psi(z)^p w(z)\, dm(z)\right)^{\frac{1}{p}},
\end{aligned}$$

and this gives the desired inequality, with a constant $C = C(w,p)$, upon dividing by $\|f\|_{w,p}^{p-1}$. To remove the restriction $f(0) = 0$ write $f(z) = f(0)+g(z)$ with $g' = f'$ and $g(0) = 0$. We have

$$\begin{aligned}\|f\|_{w,p}^p &= \|f(0) + g\|_{w,p}^p \le (\|f(0)\|_{w,p} + \|g\|_{w,p})^p = (C|f(0)| + \|g\|_{w,p})^p \\ &\le \max(C^p, 1)\,(|f(0)| + \|g\|_{w,p})^p \le 2^p \max(C^p, 1)(|f(0)|^p + \|g\|_{w,p}^p) \\ &\le C(w,p)\left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \psi(z)^p w(z)\, dm(z)\right)\end{aligned}$$

as desired. To remove the restriction of the finiteness of the integrals, for f analytic in the disc let $f_\rho(z) = f(\rho z)$ for each $\rho < 1$, and apply the Monotone Convergence Theorem as $\rho \to 1$. □

The previous lemma gives the one inequality for (1.3). For the reverse inequality we need the conditions imposed on w.

As observed already, condition (I1) implies that for sufficiently small δ we have

$$0 < r + \delta\psi(r) < 1, \qquad 0 \le r < 1, \tag{2.1}$$

hence the quantity $w(r + \delta\psi(r))$ which appears in (D) is well defined for each $r \in [0, 1)$. For monotone decreasing weights we see from (1.5) that condition (I1) is automatically valid with $A = 1$, therefore (2.1) holds for any $\delta \in (0, 1)$. Condition (I2) is also automatic, with any $B > 0$, for decreasing w since in this case $w' \le 0$. Thus a decreasing w is admissible whenever it satisfies (D).

On the other hand condition (D) is automatic for admissible increasing weights. For such weights condition (I1) is used to insured the validity of (2.1) for some δ, and also together with (I2), to control the derivative of the quantity $r + \delta\psi(r)$.

Condition (D) contains the convoluted term $w(r + \delta\psi(r))$ and would seem hard to check. However it is seen to hold in all examples of section 3. In fact I have not been able to find a reasonable decreasing weight where it fails.

Lemma 2.2. *Suppose $1 \le p < \infty$ and w is an admissible weight with associated distortion function ψ. Then there is a constant $C = C(p,w)$ such that*

$$|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \psi(z)^p w(z)\, dm(z) \le C \int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z)$$

for all analytic f on the disc.

Proof. Let f analytic and $z = re^{i\theta} \in \mathbb{D}$. Let $\rho \in (r, 1)$ and apply Cauchy's theorem to obtain

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = \frac{\rho}{2\pi} \int_0^{2\pi} \frac{f(\rho e^{i(t+\theta)}) e^{i(t-\theta)}}{(\rho e^{it} - r)^2}\, dt.$$

Integrate with respect to $\theta \in [0, 2\pi]$ and apply the continuous form of Minkowski's inequality,

$$M_p(r, f') \le \frac{M_p(\rho, f)}{2\pi} \int_0^{2\pi} \frac{1}{|\rho e^{it} - r|^2}\, dt = \frac{M_p(\rho, f)}{\rho^2 - r^2}.$$

Thus whenever $r < \rho < 1$ we have

$$(\rho^2 - r^2) M_p(r, f') \le M_p(\rho, f). \tag{2.2}$$

Now put

$$\rho = \rho(r) = r + \delta\psi(r), \qquad 0 \le r < 1,$$

where δ is chosen in the following two steps. First from condition (I1) on w, any choice of $\delta \in (0, A)$ gives

$$r < \rho(r) < 1, \qquad 0 \le r < 1.$$

Second the derivative of $\rho(r)$ is

$$\rho'(r) = 1 - \delta - \delta \frac{w'(r)}{w(r)} \psi(r)$$

and conditions (I1) and (I2) together imply that

$$\frac{w'(r)}{w(r)} \psi(r) \le \frac{B}{A}, \qquad 0 \le r < 1,$$

so that

$$\rho'(r) = 1 - \delta - \delta \frac{w'(r)}{w(r)} \psi(r) \ge 1 - \delta(1 + \frac{B}{A}).$$

We can now pick a $\delta \in (0, A)$ small enough to have $\rho'(r) > 1/2$ and at the same time condition (D) to hold for that value of δ. With this choice $\rho(r)$ is strictly increasing in $[0, 1)$, hence $\rho(r) + r \ge \rho(r) \ge \rho(0) =: C_0$. From (2.2) then we have

$$C_0 \delta \psi(r) M_p(r, f') \le M_p(\rho(r), f).$$

Raise the two sides to the pth power and multiply the left by $w(r)r$ and the right by $w(r)\rho(r)$ to obtain

$$\begin{aligned} M_p^p(r, f')\psi(r)^p w(r) r &\le C M_p^p(\rho(r), f) w(r) \rho(r) \\ &= C M_p^p(\rho(r), f) \frac{w(r)}{w(\rho(r))\rho'(r)} w(\rho(r))\rho(r)\rho'(r) \\ &\le 2CC(\delta, w) M_p^p(\rho(r), f) w(\rho(r))\rho(r)\rho'(r), \end{aligned}$$

where condition (D) and inequality $\rho'(r) > 1/2$ were used in the last step. Next integrate and make the change of variable $u = \rho(r)$ to obtain

$$\begin{aligned} \int_0^1 M_p^p(r, f')\psi(r)^p w(r) r \, dr &\le C' \int_0^1 M_p^p(\rho(r), f) w(\rho(r))\rho(r)\rho'(r) \, dr \\ &= C' \int_{\rho(0)}^1 M_p^p(u, f) w(u) u \, du \\ &\le C' \int_0^1 M_p^p(u, f) w(u) u \, du \\ &= C' \|f\|_{w,p}^p. \end{aligned}$$

Also, the linear functional $f \to f(0)$ is bounded thus $|f(0)|^p \le C'' \|f\|_{w,p}^p$ for some constant C''. The assertion follows by addition with a constant $C = C' + C''$ depending only on p and w.

□

Putting together the two lemmas we have the assertion of Theorem 1.1.

Remark 1. Because the behavior of a weight w on an interval $[0, \sigma)$ with $\sigma < 1$ does not affect the Bergman space A_w^p or its topology, it is clear that any of the conditions above may be assumed to hold only on some interval of the form $(\sigma, 1)$ without affecting the conlusions. Thus we may include such weights in our class of admissible weights.

Remark 2. If we want to relate the weighted p-integral of an analytic function to that of its antiderivative we need to solve, with respect to w, an equation of the form $\psi(r)^p w(r) = v(r)$ where v is the given weight. This of course is possible only when v belongs to the class of weights that are of the form $\psi^p w$ for an admissible w. If such is the case then this problem leads, when $p > 1$, to a Bernoulli diferential equation. Indeed the equation above is $\left(\int_r^1 w(u)\,du\right)^p = v(r)w(r)^{p-1}$ or equivalently

$$\left(\int_r^1 w(u)\,du\right)^{\frac{p}{p-1}} = v(r)^{\frac{1}{p-1}} w(r). \tag{2.3}$$

Now let $y(r) = \int_r^1 w(u)\,du$ then $y' = -w$ and (2.3) becomes

$$y'(r) = -\frac{1}{v(r)^{\frac{1}{p-1}}} y(r)^{\frac{p}{p-1}},$$

which is a Bernoulli type differential equation. It is a Riccati equation when $p = 2$.

3. Some examples

In this section we give examples of weights that satisfy our conditions and we compute the distortion function in each case.

Example 3.1. Each of the following weights is admissible,

$$w(r) = (1-r)^\alpha \left(\log \frac{e}{1-r}\right)^\beta, \qquad \alpha > -1 \quad \text{and} \quad \beta \in \mathbb{R}$$

$$w(r) = \left(\log\log \frac{e}{1-r}\right)^\alpha, \qquad \alpha > 0$$

$$w(r) = \exp\left(-\beta \left(\log \frac{e}{1-r}\right)^\alpha\right), \qquad \beta > 0 \quad \text{and} \quad 0 < \alpha \le 1$$

and the distortion function is

$$\psi(r) \sim 1 - r$$

in each case. We omit the straightforward computations.

Example 3.2. For $\alpha > 0$, $\gamma > 0$ and $\beta \in \mathbb{R}$ the weight

$$w(r) = (1-r)^\beta \exp\left(\frac{-\gamma}{(1-r)^\alpha}\right),$$

is admissible and $\psi(r) \sim (1-r)^{\alpha+1}$.

Proof. First we find ψ. With a change of variables $t = t(u) = (1-u)^{-\alpha} - (1-r)^{-\alpha}$ we have

$$\begin{aligned}
\int_r^1 w(u)\,du &= \int_r^1 (1-u)^\beta \exp\left(\frac{-\gamma}{(1-u)^\alpha}\right) du \\
&= \exp\left(\frac{-\gamma}{(1-r)^\alpha}\right) \frac{1}{\alpha} \int_0^\infty \left(\frac{1}{t + (1-r)^{-\alpha}}\right)^{\frac{\alpha+\beta+1}{\alpha}} e^{-\gamma t}\,dt \\
&= (1-r)^{\alpha+\beta+1} \exp\left(\frac{-\gamma}{(1-r)^\alpha}\right) \frac{1}{\alpha} \int_0^\infty \left(\frac{(1-r)^{-\alpha}}{t + (1-r)^{-\alpha}}\right)^{\frac{\alpha+\beta+1}{\alpha}} e^{-\gamma t}\,dt \\
&= w(r)(1-r)^{\alpha+1} I(r),
\end{aligned}$$

where $I(r)$ represents the integral. Write $\eta = \frac{\alpha+\beta+1}{\alpha}$. Since $(1-r)^{-\alpha} \geq 1$ we have

$$\left(\frac{1}{1+t}\right)^{\eta} \leq \left(\frac{(1-r)^{-\alpha}}{t+(1-r)^{-\alpha}}\right)^{\eta} \leq 1 \qquad \text{for} \quad 0 < t < \infty \quad \text{if} \quad \eta \geq 0,$$

and

$$1 \leq \left(\frac{(1-r)^{-\alpha}}{t+(1-r)^{-\alpha}}\right)^{\eta} \leq \left(\frac{1}{1+t}\right)^{\eta} \quad \text{for} \quad 0 < t < \infty \quad \text{if} \quad \eta < 0$$

In either case there are positive constants C_1 and C_2 independent of r such that $C_1 \leq I(r) \leq C_2$ for each $r \in [0,1)$. And

$$\psi(r) = \frac{1}{w(r)} \int_r^1 w(u)du = (1-r)^{\alpha+1} I(r)$$

so the assertion about $\psi(r)$ follows.

These weights are decreasing on an interval near 1. We verify condition (D) by showing that for each $\delta \in (0,1)$ there is a $\sigma = \sigma(\delta) < 1$ such that (D) holds for all $r \in (\sigma, 1)$. With $\rho(r) = r + \delta\psi(r)$ we have

$$\begin{aligned} \frac{w(r)}{w(\rho(r))} &= \left(\frac{1-r}{1-\rho(r)}\right)^{\beta} \exp\left(\frac{\gamma}{(1-\rho(r))^{\alpha}} - \frac{\gamma}{(1-r)^{\alpha}}\right) \\ &= \left(\frac{1}{h(r)}\right)^{\beta} \exp\left(\gamma \frac{1-h(r)^{\alpha}}{(1-r)^{\alpha} h(r)^{\alpha}}\right), \end{aligned}$$

where $h(r) = 1 - \delta(1-r)^{\alpha} I(r)$. Clearly $\lim_{r\to 1} h(r) = 1$ and using the binomial expansion we see that $|h(r)^{\alpha} - 1| \leq C(1-r)^{\alpha}$ with C independent of r. It follows that for each $\delta \in (0,1)$ there is a σ such that condition (D) holds for $r \in (\sigma, 1)$.

□

Example 3.3. For α, β, $\gamma > 0$ the weight

$$w(r) = \exp\left(-\gamma \exp\left(\frac{\beta}{(1-r)^{\alpha}}\right)\right),$$

is admissible and its distortion function is

$$\psi(r) \sim (1-r)^{\alpha+1} \exp\left(\frac{-\beta}{(1-r)^{\alpha}}\right).$$

Proof. Denote $T(r) = \exp\left(\frac{\beta}{(1-r)^{\alpha}}\right)$. With the change of variables $t = t(u) = \exp\left(\frac{\beta}{(1-u)^{\alpha}}\right) - T(r)$ we find

$$\begin{aligned} \int_r^1 w(u)\,du &= \int_r^1 \exp\left(-\gamma \exp\left(\frac{\beta}{(1-u)^{\alpha}}\right)\right) du \\ &= \alpha^{-1}\beta^{1/\alpha} \exp\left(-\gamma T(r)\right) \int_0^{\infty} \left(\frac{1}{\log(t+T(r))}\right)^{\frac{\alpha+1}{\alpha}} \frac{1}{t+T(r)} e^{-\gamma t}\,dt. \end{aligned}$$

Thus we have

$$\begin{aligned}
\psi(r) &= \alpha^{-1}\beta^{1/\alpha}\int_0^\infty \left(\frac{1}{\log(t+T(r))}\right)^{\frac{\alpha+1}{\alpha}} \frac{1}{t+T(r)}e^{-\gamma t}\,dt\\
&= \frac{\alpha^{-1}\beta^{1/\alpha}}{T(r)}\left(\frac{1}{\log T(r)}\right)^{\frac{\alpha+1}{\alpha}}\int_0^\infty \left(\frac{\log T(r)}{\log(t+T(r))}\right)^{\frac{\alpha+1}{\alpha}} \frac{T(r)}{t+T(r)}e^{-\gamma t}\,dt\\
&= (1-r)^{\alpha+1}\exp\left(\frac{-\beta}{(1-r)^\alpha}\right)\frac{1}{\alpha\beta}\int_0^\infty \left(\frac{\log T(r)}{\log(t+T(r))}\right)^{\frac{\alpha+1}{\alpha}} \frac{T(r)}{t+T(r)}e^{-\gamma t}\,dt\\
&\sim (1-r)^{\alpha+1}\exp\left(\frac{-\beta}{(1-r)^\alpha}\right),
\end{aligned}$$

provided that the integral $I(r)$ in the previous line is bounded below and above between two positive constants for $0 \le r < 1$. To see that this is the case observe that $T(r) \ge e^\beta$ for all r thus

$$\frac{e^\beta}{e^\beta+t} \le \frac{T(r)}{t+T(r)} \le 1,$$

for all $r \in [0,1)$ and $t \in (0,\infty)$. In addition since $\frac{\log x}{\log(t+x)}$ is an increasing function of x we have

$$\frac{\beta}{\log(e^\beta+t)} \le \frac{\log(T(r))}{\log(t+T(r))} \le 1, \qquad r\in[0,1), \quad 0<t<\infty.$$

It follows that there is a positive C such that $C \le I(r) \le 1/\gamma$ for all $r \in [0,1)$ and the assertion about $\psi(r)$ follows.

Next we show that this weight satisfies condition (D). Let $\rho(r) = r + \delta\psi(r)$ with $\delta \in (0,1)$. For the quotient $\frac{w(r)}{w(\rho(r))}$ to be bounded it suffices to show that the quantity

$$\exp\left(\frac{\beta}{(1-\rho(r))^\alpha}\right) - \exp\left(\frac{\beta}{(1-r)^\alpha}\right),$$

is bounded above by a constant as $r \to 1$, or equivalently,

$$\exp\left(\frac{\beta}{(1-\rho(r))^\alpha} - \frac{\beta}{(1-r)^\alpha}\right) - 1 \le C\exp\left(\frac{-\beta}{(1-r)^\alpha}\right).$$

But $r < \rho(r)$ thus the quantity inside the first exponential is nonnegative, and since $e^x \ge 1 + x$ for $x \ge 0$ it suffices to show

$$\frac{1}{(1-\rho(r))^\alpha} - \frac{1}{(1-r)^\alpha} \le C\exp\left(\frac{-\beta}{(1-r)^\alpha}\right) \tag{3.1}$$

for r sufficiently close to 1. Next

$$\rho(r) = r + \delta(1-r)^{\alpha+1}\exp\left(\frac{-\beta}{(1-r)^\alpha}\right)I(r) \le r + \frac{\delta}{\gamma}(1-r)^{\alpha+1}\exp\left(\frac{-\beta}{(1-r)^\alpha}\right),$$

and we find

$$\begin{aligned}
\frac{1}{(1-\rho(r))^\alpha} - \frac{1}{(1-r)^\alpha} &= \left(\left(\frac{1-r}{1-\rho(r)}\right)^\alpha - 1\right)\frac{1}{(1-r)^\alpha}\\
&\le \left(\left(\frac{1}{1-h(r)}\right)^\alpha - 1\right)\frac{1}{(1-r)^\alpha},
\end{aligned}$$

where $h(r) = (\delta/\gamma)(1-r)^\alpha \exp(\frac{-\beta}{(1-r)^\alpha})$. The rest follows by observing that if r is close to 1 then $h(r)$ is close to 0 and $1/(1-h(r)) \sim 1+h(r)$. One can use then the binomial theorem to obtain

$$\sim \Big(1+\alpha h(r)+\cdots-1\Big)\frac{1}{(1-r)^\alpha}$$
$$\sim \frac{\alpha h(r)}{(1-r)^\alpha},$$

and (3.1) follows.

□

Example 3.4. For $\alpha > 1$ and $\beta > 0$ let

$$w(r) = \exp\left(-\beta\left(\log\frac{e}{1-r}\right)^\alpha\right).$$

Working as in the previous examples we find that w is admissible and has distortion function

$$\psi(r) \sim \frac{1-r}{\left(\log\frac{e}{1-r}\right)^{\alpha-1}}.$$

We omit the calculations.

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