SEMIGROUPS OF COMPOSITION OPERATORS ON SPACES OF ANALYTIC FUNCTIONS, A REVIEW.

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Abstract. If $\{\phi_t : t \geq 0\}$ is a semigroup under composition of analytic self maps of the unit disc $\mathbf{D}$ and $\mathbf{X}$ is a Banach space of analytic functions on $\mathbf{D}$ then the formula $T_t(f) = f \circ \phi_t$ defines an operator semigroup on $\mathbf{X}$. In this article we survey what we know about these semigroups.

1. Introduction

Let $\mathbf{D}$ be the unit disc in the complex plane $\mathbf{C}$ and $\mathbf{X}$ a Banach space of analytic functions on $\mathbf{D}$. Typical choices are the Hardy space H^p, the Bergman space A^p or the Dirichlet space $\mathcal{D}$. For $\phi : \mathbf{D} \to \mathbf{D}$ analytic consider the composition operator

$$C_\phi(f) = f \circ \phi.$$

The operator powers C_ϕ^n, $n \geq 0$, are composition operators C_{ϕ_n} induced by the discrete iterates of ϕ,

$$\phi_n = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n}.$$

Assume now ϕ has *fractional iterates.* This means that there is a family $\mathbf{\Phi} = \{\phi_t : t \geq 0\}$ of analytic self-maps of $\mathbf{D}$ such that $\phi_1 = \phi$ and $\mathbf{\Phi}$ satisfies

(1) $\phi_0(z) \equiv z$, the identity map of $\mathbf{D}$.
(2) $\phi_t \circ \phi_s = \phi_{t+s}$ for $t, s \geq 0$.
(3) The map $(t, z) \to \phi_t(z)$ is jointly continuous on $[0, \infty) \times \mathbf{D}$.

We can then define fractional powers of C_ϕ by setting

$$T_t(f) = f \circ \phi_t, \quad t \geq 0.$$

Clearly $T_n = C_{\phi_n}$ for integers n and each T_t is a bounded operator on $\mathbf{X} = H^p$ or A^p (we will see this is also true on $\mathcal{D}$). The family $\mathbf{T} = \{T_t : t \geq 0\}$ satisfies:

(1) $T_0 = I$, the identity operator on $\mathbf{X}$.
(2) $T_t \circ T_s = T_{t+s}$ for $t, s \geq 0$,

and is therefore a *one parameter semigroup* of bounded operators on $\mathbf{X}$.

We now have at our disposal the tools of the general theory of one-parameter semigroups in Banach spaces. We may study the composition semigroups $\mathbf{T} = \{T_t\}$ for their own sake or we may study $\mathbf{T}$ with the aim to obtain information about individual operators participating in the semigroup.

Date: February 12, 1997.
1991 *Mathematics Subject Classification.* Primary 47D03, 47B38; Secondary 47B37, 30C45.
I would like to express my appreciation to the organizers for a very rewarding conference. No version of this paper will be published elsewhere.

Semigroups of composition operators and their weighted versions make up a large class of explicit examples in the general theory of operator semigroups. Their nonanalytic counterparts, sometimes called substitution semigroups or semiflows, have been studied for a long time on Lebesgue spaces of integrable functions, on spaces of continuous functions etc. In this setting the requirements on the inducing functions $\mathbf{\Phi} = \{\phi_t\}$ are that ϕ_t is a measurable or measure preserving or continuous transformation of some base set which has a measure or topological structure.

On spaces of analytic functions the study of composition semigroups was started by E. Berkson and H. Porta [25]. In that paper the structure of semigroups of functions $\mathbf{\Phi}$ was determined and the basic properties of $\mathbf{T}$ on Hardy spaces were obtained. It turns out that there are close connections between the function theoretic properties of $\mathbf{\Phi}$ and the operator theoretic properties of $\mathbf{T}$. Thus the interplay between function theory and operator theory that is always present in single composition operators is also present in semigroups, with added the element of iteration.

Among the analytic self maps of the disc only the univalent ones can have fractional iterates and univalency is not a sufficient condition for fractional iterates to exist. In case ϕ has fractional iterates, properties of the composition operator C_ϕ are in close relation with properties of the semigroup $\mathbf{T}$ of which it is a member. Many times a question about C_ϕ can be translated to a question about $\mathbf{T}$ and vice versa. For example to determine the point spectrum of C_ϕ one has to solve a functional equation in $\mathbf{X}$. Using the spectral theorem for semigroups the problem is equivalent to finding the point spectrum of the infinitesimal generator of $\mathbf{T}$, and this involves solving a first order differential equation. The hard part is to determine which of the solutions belong to $\mathbf{X}$. An application of semigroups to finding the full spectrum of some composition operators can be found in [39, Th. 7.41].

It has been observed long ago that many properties of C_ϕ depend heavily on the dynamical behavior of the iterates $\{\phi_n\}$. When fractional iterates exist, all the information about the C_ϕ or C_{ϕ_n} is encapsulated in a single object: the infinitesimal generator of $\mathbf{\Phi}$ or equivalently the infinitesimal generator of $\mathbf{T}$.

The resolvent operators of composition semigroups are averaging integration operators as for example the Cesàro operator.

2. Semigroups of operators on Banach spaces

To make the paper self contained we recall some basic facts from the general theory of semigroups. More details can be found in [44] [57] [84].

Let $\mathbf{X}$ be a Banach space and $\mathbf{T} = \{T_t\}$ a semigroup of bounded operators on $\mathbf{X}$. $\mathbf{T}$ is called *strongly continuous* (or *c_0-semigroup*) if $\lim_{t\to 0} \|T_t(x) - x\| = 0$ for each $x \in \mathbf{X}$. If the stronger property $\lim_{t\to 0} \|T_t - I\| = 0$ holds then $\mathbf{T}$ is *uniformly continuous*. The *infinitesimal generator* of a strongly continuous semigroup is the (unbounded in general) operator defined by

$$A(x) = \lim_{t\to 0} \frac{T_t(x) - x}{t} = \frac{\partial T_t(x)}{\partial t}\Big|_{t=0}.$$

The generator A is defined only on those $x \in \mathbf{X}$ for which the limit exists. This set of x's is the domain $D(A)$ of A. $D(A)$ is a linear subset of $\mathbf{X}$ and is always dense in $\mathbf{X}$. It coincides with $\mathbf{X}$ if and only if A is bounded and this is equivalent to that $\mathbf{T}$ is uniformly continuous. The generator A is always a closed operator, i.e. its graph is closed in $\mathbf{X}\times\mathbf{X}$.

The collection of complex numbers λ for which $\lambda I - A$ has a bounded inverse on $\mathbf{X}$ is the resolvent set $\rho(A)$. For $\lambda \in \rho(A)$ the *resolvent operator* is $R(\lambda, A) = (\lambda I - A)^{-1}$. The spectrum of A is $\sigma(A) = \mathbf{C} \setminus \rho(A)$ and the point spectrum $\sigma_\pi(A)$ is defined as in the case of bounded operators. Since A is a closed operator its spectrum is a closed set in the plane. In contrast with bounded operators $\sigma(A)$ can vary in size from empty to a whole left half plane.

The *growth bound* (or *type*) of a strongly continuous semigroup is

$$\omega = \lim_{t\to\infty} \frac{\log \|T_t\|}{t}.$$

We have $-\infty \leq \omega < \infty$. For each $\tau > \omega$ there is a $M = M(\tau) < \infty$ such that $\|T_t\| \leq Me^{\tau t}$, $t \geq 0$. The spectral radius of T_t is $r(T_t) = e^{\omega t}$. If $\Re(\lambda) > \omega$ then $R(\lambda, A)$ is bounded and the Laplace formula holds

$$R(\lambda, A)(x) = \int_0^\infty e^{-\lambda t} T_t(x)\, dt \quad \text{for each } x \in \mathbf{X}. \tag{2.1}$$

A corollary to the Hille-Yosida-Phillips theorem says that if $\|T_t\| \leq e^{\omega t}$ for each t then

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega} \quad \text{for each } \lambda > \omega. \tag{2.2}$$

Roughly speaking operator semigroups are the operator analogue of the exponential function. When the semigroup is uniformly continuous (i.e. A is bounded) then $T_t = e^{tA}$. With suitable interpretation this formula remains true for all strongly continuous semigroups. The spectral theorem for semigroups says

$$e^{t\sigma(A)} \subseteq \sigma(T_t) \quad \text{for } t \geq 0. \tag{2.3}$$

Equality holds (modulo the point 0) for some special classes such as uniformly continuous or eventually compact semigroups, but in general the containment is strict. For the point spectrum however there is equality

$$e^{t\sigma_\pi(A)} = \sigma_\pi(T_t) \setminus \{0\} \quad \text{for } t \geq 0. \tag{2.4}$$

If $\{T_t\}$ contains a compact operator then it is *eventually compact* i.e. its tail consists of compact operators. If T_t is compact for every $t > 0$ then the semigroup is called *compact*. A theorem [84] states that a semigroup is compact if and only if $R(\lambda, A)$ is compact for $\lambda \in \rho(A)$ and $\lim_{t\to s} \|T_t - T_s\| = 0$ for each $s > 0$. It is easy to construct composition semigroups which are eventually compact but not compact. The resolvent equation

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \quad \lambda, \mu \in \rho(A),$$

implies that either $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$ or not compact for any λ.

3. Semigroups of analytic functions

For an analytic self map ϕ of the disc the sequence of iterates $\{\phi_n\}$ behaves in a rather predictable manner. According to the Denjoy-Wolff theorem, *unless ϕ is an elliptic Möbius automorphism of $\mathbf{D}$, there is a point b in the closed disc such that $\phi_n \to b$ uniformly on compact subsets of $\mathbf{D}$.* If b is in the interior then it is a fixed point of ϕ, while if b is on the boundary then it behaves as a fixed point in the sense that $\lim_{r\to 1} \phi(rb) = b$. This distinguished point is called the *Denjoy-Wolff point* (DW point) of ϕ. In the exceptional case of elliptic automorphisms the sequence of iterates moves around an interior fixed point without converging to it. We call also this point DW point and keep the notation b for the DW point in all cases.

Embedding the discrete iterates $\{\phi_n\}$ into a continuous parameter semigroup is not always possible even for univalent ϕ. The problem is related to the existence of "iterative roots" and leads to a question on functional equations. Complete conditions on ϕ that characterize the embeddability are not known. The interested reader can find more information in [112] and [113].

Nevertheless all semigroups of analytic self maps of $\mathbf{D}$ can be described as we will see below. First we give some simple examples:

1. Rotation + shrinking. For $\Re(c) \geq 0$ let

$$\phi_t(z) = e^{-ct}z.$$

If $\Re(c) = 0$ this is a group of rotations. It reduces to the trivial semigroup if $c = 0$. In all other cases ϕ_t maps the disc properly into itself. The point 0 is a common fixed point.

2. Shrinking the disc to a point. Let

$$\phi_t(z) = e^{-t}z + 1 - e^{-t}.$$

The typical image $\phi_t(\mathbf{D})$ is a small disc tangent to the unit circle at 1, whose diameter goes to 0 as $t \to \infty$. The point $b = 1$ is a common DW point for all ϕ_t.

3. Shrinking the disc to a segment. Let

$$\phi_t(z) = 1 - (1 - z)^{e^{-t}}.$$

There are two fixed points 0 and 1 common to all ϕ_t of which $b = 0$ is the DW point. The typical image $\phi_t(\mathbf{D})$ is an angular region inside $\mathbf{D}$ whose angle vertex is at 1. The size of the angle shrinks to 0 as $t \to \infty$.

4. Shrinking the disc to a smaller disc. Let

$$\phi_t(z) = \frac{e^{-t}z}{(e^{-t} - 1)z + 1}.$$

The points 0 and 1 are common fixed points and the DW point is $b = 0$. The typical image $\phi_t(\mathbf{D})$ is a disc tangent to the unit circle at 1 whose diameter shrinks to $1/2$ as $t \to \infty$.

5. Group of hyperbolic automorphisms. Let

$$\phi_t(z) = \frac{(1 + e^t)z - 1 + e^t}{(-1 + e^t)z + 1 + e^t}.$$

There are two common fixed points -1 and 1 of which $b = 1$ is the DW point. Every ϕ_t maps the disc onto itself.

6. Splitting the disc into two halves. Let $k(z) = \frac{z}{(1-z)^2}$ be the Koebe function and k^{-1} its inverse. For $t \geq 0$ let

$$\phi_t(z) = k^{-1}(k(z) + t).$$

The typical image $\phi_t(\mathbf{D})$ is a slit disc $\mathbf{D} \setminus (-1, r_t]$ with $r_t \nearrow 1$ as $t \to \infty$.

One can make more examples like these but there is a general construction that gives an unlimited list of semigroups. It amounts to conjugating one of the two basic semigroups of the complex plane, $z \longmapsto e^{-ct}z$ and $z \longmapsto z + ct$, by an analytic 1-1 map whose range is invariant under that semigroup. More precisely we have,

Semigroups of class $\mathbf{\Phi_0}$. Let $h : \mathbf{D} \to \mathbf{C}$ be an analytic univalent function with $h(0) = 0$. Suppose that $\Omega = h(\mathbf{D})$ satisfies the following property:

> There is a c with $\Re(c) \geq 0$ such that for each $w \in \Omega$ the entire spiral $\{we^{-ct} : t \geq 0\}$ is contained in Ω.

These are the *spirallike* (*starlike* if c is real) functions [46] and the above geometric condition is equivalent to

$$\Re\Big(\frac{1}{c}\frac{zh'(z)}{h(z)}\Big) \geq 0. \tag{3.1}$$

Given such an h we can write the functions

$$\phi_t(z) = h^{-1}(e^{-ct}h(z)), \quad z \in \mathbf{D},\, t \geq 0,$$

and it is easily seen that $\{\phi_t\}$ is a semigroup with common DW point $b = 0$.

Semigroups of class $\mathbf{\Phi_1}$. Let $h : \mathbf{D} \to \mathbf{C}$ be analytic and univalent with $h(0) = 0$. Suppose that $\Omega = h(\mathbf{D})$ satisfies the following property:

> There is a direction c with $\Re(c) \geq 0$ such that for each $w \in \Omega$ the entire half line $\{w + ct : t \geq 0\}$ is contained in Ω.

Such functions form a subclass of the *close-to-convex* univalent functions [46] and the geometric condition on Ω is equivalent to

$$\Re\Big(\frac{1}{c}(1-z)^2h'(z)\Big) \geq 0. \tag{3.2}$$

Given such an h the functions

$$\phi_t(z) = h^{-1}(h(z) + ct), \quad z \in \mathbf{D},\, t \geq 0,$$

form a semigroup with DW point $b = 1$.

It turns out that under a normalization, **every** semigroup of analytic self maps of $\mathbf{D}$ can be written in one of the preceding two ways. Before we see this we recall the basic structure of semigroups from [25]. Assume $\mathbf{\Phi} = \{\phi_t\}$ is a semigroup, then:

- Each ϕ_t is univalent.
- The limit
$$G(z) = \lim_{t\to 0}\frac{\partial\phi_t(z)}{\partial t}$$
exists uniformly on compact subsets of $\mathbf{D}$, and satisfies
$$G(\phi_t(z)) = \frac{\partial\phi_t(z)}{\partial t} = G(z)\frac{\partial\phi_t(z)}{\partial z}, \quad z \in \mathbf{D},\, t \geq 0. \tag{3.3}$$
The analytic function G(z) is the *infinitesimal generator* of $\mathbf{\Phi}$ and characterizes $\mathbf{\Phi}$ uniquely.
- The functions ϕ_t, $t > 0$, share a common DW point b, and G(z) has the unique representation
$$G(z) = F(z)(\bar{b}z - 1)(z - b)$$
where $F : \mathbf{D} \to \mathbf{C}$ is analytic with $\Re\,(F(z)) \geq 0$ for $z \in \mathbf{D}$.

Let now $\mathbf{\Phi}$ be an arbitrary semigroup with DW point b. Conjugating ϕ_t with Möbius automorphisms of the disc produces another semigroup but leaves the essential properties unchanged. If b is in the interior (on the circle) then the new semigroup will have its DW point in the interior (resp. on the circle). We can choose a suitable automorphism to conjugate and hence assume without loss of generality that $b = 0$ (when $b \in \mathbf{D}$) or $b = 1$ (when $b \in \partial\mathbf{D}$).

Case 1. If $b = 0$ then the generator is $G(z) = -zF(z)$. Let h be the solution on $\mathbf{D}$ of the following initial value problem

$$\frac{1}{F(0)}\frac{zh'(z)}{h(z)} = \frac{1}{F(z)}, \quad h(0) = 0.$$

Because $\Re(F) \geq 0$ the solution

$$h(z) = z\exp\Big(\int_0^z \frac{1}{\zeta}\Big(\frac{F(0)}{F(\zeta)} - 1\Big)\,d\zeta\Big). \tag{3.4}$$

is a univalent spirallike function [46] and Schröder's functional equation holds

$$h(\phi_t(z)) = e^{-F(0)t}h(z), \quad z \in \mathbf{D},\ t \geq 0.$$

It follows that $\mathbf{\Phi}$ belongs to the class $\mathbf{\Phi_0}$ of semigroups.

Case 2. If $b = 1$ then the generator is $G(z) = (1-z)^2F(z)$. Let h be defined by

$$h(z) = \int_0^z \frac{F(0)}{(1-\zeta)^2F(\zeta)}\,d\zeta.$$

Because $\Re(F) \geq 0$ the function h satisfies (3.2) and Abel's functional equation holds

$$h(\phi_t(z)) = h(z) + F(0)t, \quad z \in \mathbf{D},\ t \geq 0.$$

It follows that in this case $\mathbf{\Phi}$ is in class $\mathbf{\Phi_1}$.

The unique univalent function h corresponding to $\mathbf{\Phi}$ in either case is called the *associated univalent function*. A semigroup $\mathbf{\Phi}$ is characterized uniquely by the pair $\{b, F(z)\}$ or by the triple $\{b, c, h(z)\}$. The notation G=the generator, F=the function of positive real part in G, h=the associated univalent function and b=DW point, will be used exclusively with this assigned meaning. We also write c=$F(0)$=$-G'(0)$ if $b = 0$ and c=$F(0)$=$G(0)$ if $b = 1$.

Appropriate choices of h produce examples of $\mathbf{\Phi}$ or $\mathbf{T}$ with various desired properties. For example we can arrange for ϕ_t to have any number of fixed points on the boundary or to leave an arc on the boundary invariant. We can also arrange for the orbits $\{\phi_t(a) : t \geq 0\}$ of interior points a to approach the DW point on the boundary in a tangential or nontangential way, or for the limiting set $\cap_{t\geq 0}\phi_t(\mathbf{D})$ to consist of any number of connected components. The following is a construction of a semigroup $\mathbf{T}$ such that each composition operator T_t is not compact but the resolvent operator is compact. Let h be the Riemann map from $\mathbf{D}$ onto the starlike region $\Omega = \mathbf{D} \cup \{z : 0 < \Re(z) < \infty,\ 0 < \Im(z) < 1\}$, with $h(0) = 0$. Let $\phi_t(z) = h^{-1}(e^{-t}h(z))$. It is easily seen that $\phi_t(\partial\mathbf{D})$ intersects $\partial\mathbf{D}$ in a set of positive measure so none of the induced composition operators is compact. By results in section 6 the resolvent operator is compact on H^p. If in the definition of Ω we change the condition on $\Re(z)$ to $0 < \Re(z) < 1$ then we obtain a semigroup that is eventually compact but not compact.

4. Strong continuity of composition semigroups

Let $\mathbf{T} = \{T_t\}$ be a composition semigroup. Strong continuity requires that $\lim_{t\to 0}\|f\circ\phi_t - f\| = 0$ for each $f \in \mathbf{X}$. Assume $\mathbf{X}$ contains the polynomials then with P a polynomial we can write

$$\begin{aligned}\|f\circ\phi_t - f\| &\leq \|f\circ\phi_t - P\circ\phi_t\| + \|P\circ\phi_t - P\| + \|P - f\| \\ &\leq (\|T_t\| + 1)\|P - f\| + \|P\circ\phi_t - P\|.\end{aligned}$$

Table 1. Examples of semigroups

	$G(z)$	$h(z)$	$\phi_t(z)$
$b=0$	$-zc$, $\Re c \geq 0$	z	$e^{-ct}z$
	$-z(1-z)$	$\frac{z}{1-z}$	$\frac{e^{-t}z}{(e^{-t}-1)z+1}$
	$-(1-z)\log\frac{1}{1-z}$	$\log\frac{1}{1-z}$	$1-(1-z)^{e^{-t}}$
	$-z(1-z^n)$	$\frac{z}{(1-z^n)^{\frac{1}{n}}}$	$\frac{e^{-t}z}{((e^{-nt}-1)z^n+1)^{\frac{1}{n}}}$
	$-\frac{1-z^2}{2}\log\frac{1+z}{1-z}$	$\log\frac{1+z}{1-z}$	$\frac{(1+z)^{e^{-t}}-(1-z)^{e^{-t}}}{(1+z)^{e^{-t}}+(1-z)^{e^{-t}}}$
$b=1$	$1-z$	$\log\frac{1}{1-z}$	$e^{-t}z+1-e^{-t}$
	$c(1-z)^2$, $\Re c \geq 0$	$\frac{z}{1-z}$	$\frac{(1-ct)z-1+ct}{-ctz+1+ct}$
	$\frac{1}{2}(1-z^2)$	$\frac{1}{2}\log\frac{1+z}{1-z}$	$\frac{(1+e^t)z-1+e^t}{(-1+e^t)z+1+e^t}$
	$\frac{(1-z)^\alpha}{1+\alpha}$, $\alpha\in(-1,1]$	$\frac{1}{1+\alpha}\left((1-z)^{-\alpha-1}-1\right)$	$1-\left((1-z)^{-\alpha-1}+t\right)^{-\frac{1}{\alpha+1}}$

Assume further that polynomials are dense in $\mathbf{X}$ and that there is a $\delta > 0$ such that $\sup_{0\leq t<\delta}\|T_t\| < \infty$. Then to obtain strong continuity we only need to show $\lim_{t\to 0}\|P\circ\phi_t - P\| = 0$ for each polynomial and this will follow if we can show $\lim_{t\to 0}\|\phi_t(z)-z\| = 0$. The latter condition holds in many classical spaces as a result of the dominated convergence theorem.

Using this kind of argument one can show strong continuity of $\mathbf{T}$ on several function spaces. We recall the results from [25] on Hardy spaces. Suppose $1 \leq p < \infty$, then

- Each composition semigroup is strongly continuous on H^p.
- The infinitesimal generator Γ of $\mathbf{T}$ is given by

$$\Gamma(f)(z) = \left.\frac{\partial f(\phi_t(z))}{\partial t}\right|_{t=0} = G(z)f'(z),$$

  where $G(z)$ is the generator of $\mathbf{\Phi}$.
- $\mathbf{T}$ is not uniformly continuous unless it is trivial, i.e. $G\equiv 0$.

Similar statements hold on Bergman spaces A^p (in fact on weighted Bergman spaces [105]) and on the Dirichlet space $\mathcal{D}$ [108]. On all these spaces the generator Γ is a differential operator given by the same formal expression.

There are spaces of analytic functions where strong continuity can fail for some or all composition semigroups. For example on the disc algebra $A(\mathbf{D})$ strong continuity is equivalent to $\lim_{t\to 0}\|\phi_t(z)-z\|_\infty = 0$. Many semigroups of functions satisfy this condition (example $\phi_t(z) = e^{-t}z + 1 - e^{-t}$) but some others do not (example $\phi_t(z) = 1-(1-z)^{e^{-t}}$).

On H^∞ no composition semigroup is strongly continuous unless it is trivial. The easiest way to see this is to use a dichotomy result from the general theory of semigroups [75], which says that on a class of spaces which includes H^∞, every strongly continuous semigroup is automatically uniformly continuous. This means that the generator is bounded and for composition semigroups this implies the semigroup is trivial.

We discuss also the case of $BMOA$, the space of analytic functions whose boundary values have bounded mean oscillation. Recall the norm

$$\|f\|_* = |f(0)| + \sup_{a\in\mathbf{D}} \|f\circ\phi_a - f(a)\|_{H^2},$$

where $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$. The subspace $VMOA$ consists of those $f \in BMOA$ such that $\lim_{|a|\to 1}\sup_{a\in\mathbf{D}}\|f\circ\phi_a - f(a)\|_{H^2} = 0$. Alternatively $VMOA$ is the closure in $BMOA$ of the analytic polynomials. Using the norm inequality

$$\|f\circ\phi\|_* \le \Big(1 + \frac{1}{2}\log\frac{1+|\phi(0)|}{1-|\phi(0)|}\Big)\|f\|_*,$$

which holds for any analytic self map of the disc, and the fact that polynomials are dense in $VMOA$, one can show that every composition semigroup is strongly continuous on $VMOA$.

There is another characterization of $VMOA$ as follows. Suppose $f \in BMOA$ then the following conditions are equivalent

(1) $f \in VMOA$.
(2) $\lim_{t\to 0}\|f(e^{it}z) - f(z)\|_* = 0$.
(3) $\lim_{t\to 0}\|f(e^{-t}z) - f(z)\|_* = 0$,

see [93] or [117]. Thus functions in $VMOA$ are characterized by their strong continuity behavior under the rotation group $\phi_t(z) = e^{it}z$ or under the semigroup $\phi_t(z) = e^{-t}z$. The question arises whether there are other semigroups that can be used to test $VMOA$ functions. Examples show that in general there are functions, depending on $\mathbf{\Phi}$, that are not in $VMOA$ and which pass the test of strong continuity. For example take

$$\phi_t(z) = e^{-t}z + 1 - e^{-t} \quad \text{and} \quad f(z) = \log\frac{1}{1-z},$$

then $\lim_{t\to 0}\|f\circ\phi_t - f\|_* = \lim_{t\to 0} t = 0$ but $f \notin VMOA$. More generally let $h \in BMOA \setminus VMOA$ be a starlike univalent function with $h(0) = 0$, let $\phi_t(z) = h^{-1}(e^{-t}h(z))$ and take $f = h$. Then

$$\lim_{t\to 0}\|f\circ\phi_t - f\|_* = \|f\|_* \lim_{t\to 0}|e^{-t}-1| = 0,$$

but $f = h \notin VMOA$.

Thus for a given $\mathbf{\Phi}$ there is a largest subspace $Y = Y_{\mathbf{\Phi}} \subseteq BMOA$ such that $\mathbf{\Phi}$ is strongly continuous on Y. Clearly $VMOA \subseteq Y$ and $Y = VMOA$ for the rotation group. We do not know if it is possible to have $Y = BMOA$ for nontrivial $\mathbf{\Phi}$. Similar phenomena arise on the Bloch space $\mathcal{B}$ and also on a whole chain of subspaces of $BMOA$, called Q_p-spaces. See [14], [15] for definitions and properties of these spaces.

5. Spectrum of the infinitesimal generator

The spectral theorem for semigroups (2.3) says that if we know the spectrum of the generator then we have information about spectra of single operators participating in the semigroup and conversely. Point spectra can be determined completely this way using (2.4).

To determine the point spectrum $\sigma_\pi(\Gamma)$ one has to solve $\Gamma(f) = \lambda f$ for f and λ. This is the differential equation $G(z)f'(z) = \lambda f(z)$. Recall the form of the generator. If $b = 0$ then

$$G(z) = -c\frac{h(z)}{h'(z)},$$

while if $b = 1$,

$$G(z) = \frac{c}{h'(z)}.$$

A straightforward solution of the differential equation in each case gives the point spectrum on H^p. If $b = 0$ then

$$\sigma_\pi(\Gamma) = \{-ck : h(z)^k \in H^p, k = 0, 1, 2, \dots\}$$

while if $b = 1$ then

$$\sigma_\pi(\Gamma) = \{c\lambda : e^{\lambda h(z)} \in H^p\}.$$

Thus in case of an interior DW point the point spectrum consists of a finite or infinite number of eigenvalues, depending on the "Hardy space size" of the associated univalent function. These eigenvalues are simple and they form a discrete set in the left half plane. In contrast, the point spectrum is usually large when the DW point is on the boundary. For example if the associated univalent function h maps the disc inside a strip then by subordination $h \in BMOA$ and then $e^{h(z)} \in H^q$ for some positive q [88] so $\sigma_\pi(\Gamma)$ contains a small disc around 0.

The available examples show that to find the full spectrum of Γ requires more elaborate work. The following fact from the operational calculus for unbounded closed operators relates $\sigma(\Gamma)$ to the spectrum $\sigma(R(\lambda, \Gamma))$ of the resolvent operator. For $\lambda \in \rho(\Gamma)$ the function $\Theta_\lambda(z) = \frac{1}{\lambda - z}$ is analytic in a neighborhood of $\sigma(\Gamma)$ and $\Theta_\lambda(\sigma(\Gamma) \cup \{\infty\}) = \sigma(R(\lambda, \Gamma))$. Hence there is a 1-1 map between $\sigma(\Gamma)$ and $\sigma(R(\lambda, \Gamma))$.

Hence when $b = 0$ and if $R(\lambda, \Gamma)$ is a compact operator then $\sigma(\Gamma)$ is a pure point spectrum. Compactness of the resolvent will be discussed in the next section. If $R(\lambda, \Gamma)$ is not compact then the general shape of $\sigma(\Gamma)$ is roughly as follows (assuming for simplicity that c is real): In addition to some eigenvalues, it contains a large portion of, and sometimes a whole, left half plane $\{z : \Re(z) \le s\}$ where $s = s(h)$ depends on the Hardy space size of the associated univalent function h.

When $b = 0$ the resolvent operator $R(\lambda, \Gamma)$ can be computed at the point λ=c=$F(0)$ and has a convenient form. To see that $c \in \rho(\Gamma)$ observe that $\|T_t\|_{H^p} = 1$ for each t when $b = 0$. Thus the growth bound of $\mathbf{T}$ is $\omega = 0$ hence $c \in \rho(\Gamma)$ if $\Re(c) > 0$. A straightforward calculation confirms this also when $\Re c = 0$. Inverting the operator $cI - \Gamma$ we obtain

$$R(c, \Gamma)(f)(z) = \frac{1}{ch(z)} \int_0^z f(\zeta)h'(\zeta)\, d\zeta. \tag{5.1}$$

We give below some examples of spectra of Γ on H^p.

Semigroups with DW point $\mathbf{b = 0}$.

Example 5.1. Let $\phi_t(z) = e^{-ct}z$, with associated univalent function $h(z) \equiv z$. The resolvent reduces to the integration operator

$$R(c,\Gamma)(f)(z) = \frac{1}{cz}\int_0^z f(\zeta)\,d\zeta$$

which is compact on H^p. Thus

$$\sigma(\Gamma) = \sigma_\pi(\Gamma) = \{-kc : k = 0,1,2,\dots\}.$$

Example 5.2. Let $\phi_t(z) = 1-(1-z)^{e^{-t}}$, with $h(z) = \log(1/(1-z))$. Using the angular derivative criterion [39, Cor. 3.21] we see that each composition operator T_t, $t > 0$, is compact, and it follows that $R(\lambda,\Gamma)$ is compact. Thus

$$\sigma(\Gamma) = \sigma_\pi(\Gamma) = \{-k : k = 0,1,2,\dots\}.$$

Example 5.3. Let

$$\phi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}.$$

Here $h(z) = z/(1-z)$ and we find $\sigma_\pi(\Gamma) = \{0\}$. For each $\lambda \in \mathbf{C}$ with $\Re(\lambda) \le -1/p$ and each positive integer n define

$$P_n(z) = 1 + \sum_{k=1}^{n} \frac{\lambda(-1)^k}{k}\binom{n}{k} z^k,$$

and let $f_{n,\lambda}(z) = (1-z)^\lambda e^{P_n(z)}$. A calculation shows that the differential equation $(\lambda I - \Gamma)(y) = \lambda f_{n,n+\lambda+1}$, equivalently,

$$z(1-z)y'(z) + \lambda y(z) = \lambda(1-z)^{\lambda+n+1}e^{P_n(z)}$$

has the unique analytic solution $y(z) = f_{n,\lambda}(z)$ on the unit disc. If we choose n such that $\Re(n+\lambda+1) > -1/p$ then $f_{n,n+\lambda+1} \in H^p$. If $\lambda I - \Gamma$ were invertible then we would have $f_{n,\lambda} = R(\lambda,\Gamma)(f_{n,n+\lambda+1}) \in H^p$ which is impossible because $\Re(\lambda) \le -1/p$. It follows that the half plane $\{z : \Re(z) \le -1/p\}$ is contained in $\sigma(\Gamma)$.

We can proceed to determine $\sigma(\Gamma)$ along the lines of [103] by considering a weighted composition semigroup, but it is easier here to use the spectra of single composition operators in the semigroup. For each t, ϕ_t has the interior DW point 0 and a single fixed point 1 on the boundary, with angular derivative $\phi_t'(1) = e^t$. Further each ϕ_t is analytic in a neighborhood of the disc, hence we have the spectrum on H^p, $\sigma(T_t) = \{z : |z| \le e^{-t/p}\} \cup \{1\}$. Using the spectral theorem (2.3) we conclude

$$\sigma(\Gamma) = \{z : \Re(z) \le -1/p\} \cup \{0\}.$$

Similar arguments applied to the semigroup $\phi_t(z) = k^{-1}(e^{-t}k(z))$ where $k(z) = z/(1-z)^2$, give the H^p spectrum [106],

$$\sigma(\Gamma) = \{z : \Re(z) \le -1/2p\} \cup \{0\}.$$

Example 5.4. Let $\phi_t(z) = h^{-1}(e^{-t}h(z))$ where $h : \mathbf{D} \to \Omega$ is a starlike map with $h(0) = 0$. Suppose that $\Omega = h(\mathbf{D})$ has the following property,

$$\sup\{r : D(z,r) \subset \Omega\} < \infty,$$

that is, the radii of schlicht discs $D(z,r)$ that can be inscribed in Ω are bounded. Thus Ω can be a strip of finite width or a Y-shaped union of half strips extending to infinity in each direction. Such an h is in the Bloch space $\mathcal{B}$ and since it is univalent it is also in $BMOA$ [88]. Since $BMOA \subset H^p$ for all finite p we conclude that $h \in \cap_{p<\infty} H^p$. In the next section we will see that this implies compactness of $R(\lambda, \Gamma)$ hence

$$\sigma(\Gamma) = \sigma_\pi(\Gamma) = \{0, -1, -2, \dots\}.$$

Semigroups with DW point $\mathbf{b = 1}$.

Example 5.5. Let $\phi_t(z) = e^{-t}z + 1 - e^{-t}$. This is obtained as

$$\phi_t(z) = h^{-1}(h(z) + t), \quad \text{with} \quad h(z) = \log \frac{1}{1-z}.$$

We have $e^{\lambda h(z)} = (1-z)^{-\lambda} \in H^p$ if and only if $\Re(\lambda) < 1/p$, hence

$$\sigma_\pi(\Gamma) = \{z : \Re(z) < 1/p\}.$$

Using the norm estimate $\|T_t\|_{H^p} \leq 2e^{t/p}$ we find for the growth bound $\omega \leq 1/p$, thus $\sigma(\Gamma) \subseteq \{z : \Re(z) \leq 1/p\}$. Since $\sigma(\Gamma)$ is a closed set we obtain

$$\sigma(\Gamma) = \{z : \Re(z) \leq 1/p\}.$$

Example 5.6. Let $h(z) = \frac{1}{2} \log \frac{1+z}{1-z}$ and

$$\phi_t(z) = h^{-1}(h(z) + \frac{1}{2}t) = \frac{(1+e^t)z - 1 + e^t}{(-1+e^t)z + 1 + e^t}.$$

This is a group of hyperbolic automorphisms. We have $e^{\lambda h(z)} = (\frac{1+z}{1-z})^{\lambda/2} \in H^p$ if and only if $-1/p < \Re(\lambda/2) < 1/p$. It follows that

$$\sigma_\pi(\Gamma) = \{z : -1/p < \Re(z) < 1/p\}.$$

On the other hand the angular derivative at the DW point is $\phi_t'(1) = e^{-t}$ thus the spectrum of T_t on H^p is $\sigma(T_t) = \{z : e^{-t/p} \leq |z| \leq e^{t/p}\}$. As in the previous example we conclude

$$\sigma(\Gamma) = \{z : -1/p \leq \Re(z) \leq 1/p\}.$$

Example 5.7. Let $h(z) = z/(1-z)$ and

$$\phi_t(z) = h^{-1}(h(z) + it) = \frac{(1-it)z + it}{-itz + 1 + it} \tag{5.2}$$

This is a group of parabolic automorphisms with DW point $b = 1$. We find $e^{\lambda h(z)} = e^{\frac{\lambda z}{1-z}} \in H^p$ if and only if $\lambda \leq 0$. It follows that

$$\sigma_\pi(\Gamma) = \{i\lambda : \lambda \leq 0\}.$$

The growth bound in this case is $\omega = 0$. Recalling the spectra of T_t [39] we find $\sigma(\Gamma) = \sigma_\pi(\Gamma) = \{i\lambda : \lambda \leq 0\}$.

6. Compactness of the resolvent operator

In the previous section we saw that in the case of an interior DW point, $\sigma(\Gamma)$ is a pure point spectrum if the resolvent $R(\lambda, \Gamma)$ is compact. In this section we discuss compactness of $R(\lambda, \Gamma)$. Compactness is characterized on Hardy and Bergman spaces by a number of equivalent conditions in terms of the associated univalent function h. Characterizations are also given for membership of $R(\lambda, \Gamma)$ in the Schatten classes $\mathcal{S}^p(H^2)$ of the Hilbert space H^2 and similarly for A^2.

Recall that for compactness we may calculate $R(\lambda, \Gamma)$ at the convenient point $c \in \rho(\Gamma)$, and $R(c, \Gamma)$ is a constant multiple of the following averaging operator,

$$R_h(f)(z) = \frac{1}{h(z)} \int_0^z f(\zeta)h'(\zeta)\, d\zeta.$$

Define two auxiliary operators P_h and Q_h as follows

$$P_h(f)(z) = \frac{1}{zh(z)} \int_0^z f(\zeta)\zeta h'(\zeta)\, d\zeta,$$

and

$$Q_h(f)(z) = \frac{1}{z} \int_0^z f(\zeta)\frac{\zeta h'(\zeta)}{h(\zeta)}\, d\zeta,$$

and denote by M_z the operator of multiplication by the independent variable z. Obviously P_h is bounded on H^p since R_h is, and we will see in a moment that Q_h is also bounded. First by direct computation the following identities hold

$$M_z P_h = R_h M_z, \qquad Q_h = P_h + Q_h P_h. \tag{6.1}$$

Next using the identity $zh'(z)/h(z) = 1 + z(\log \frac{h(z)}{z})'$ we can write

$$\begin{aligned} Q_h(f)(z) &= \frac{1}{z}\int_0^z f(\zeta)\, d\zeta + \frac{1}{z}\int_0^z f(\zeta)\zeta\Big(\log\frac{h(\zeta)}{\zeta}\Big)'\, d\zeta \\ &= J(f)(z) + L_h M_z(f)(z), \end{aligned}$$

where J is the integration operator and L_h is

$$L_h(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\Big(\log\frac{h(\zeta)}{\zeta}\Big)'\, d\zeta.$$

We will use a result about certain operators T_g studied in [8],

$$T_g(f)(z) = \frac{1}{z}\int_0^z f(\zeta)g'(\zeta)\, d\zeta \tag{6.2}$$

of which L_h is a special case. To state this result we need the analytic Besov spaces B_p. For $1 < p < \infty$, B_p consists of analytic f such that

$$\|f\|_{B_p}^p = \int_{\mathbf{D}} |f'(z)|^p (1 - |z|^2)^{p-2}\, dm(z) < \infty,$$

where $dm(z)$ is the Lebesgue area measure on $\mathbf{D}$. These are small spaces, all contained in $VMOA$. More details for B_p can be found in [117].

Let T_g be as above with g analytic on the disc and $1 \le p < \infty$. Then the following hold [8].

- T_g is bounded on H^p if and only if $g \in BMOA$.
- T_g is compact on H^p if and only if $g \in VMOA$.
- T_g is in the Schatten class $\mathcal{S}^q(H^2)$ if and only if $g \in B_q$, $1 < q < \infty$.

Now for any univalent function h with $h(0) = 0$ the function $\log(h(z)/z)$ is in $BMOA$ [16]. The operator L_h above is identical to T_g with $g = \log(h(z)/z)$. It follows that L_h is bounded on H^p and since J is compact, Q_h is also bounded. Further (6.1) implies that R_h is compact if and only if Q_h is compact if and only if L_h is compact and this is equivalent to $\log(h(z)/z) \in VMOA$.

For the Schatten classes of R_h we can use the same reasoning together with the fact that $J \in \mathcal{S}^q(H^2)$ for all $q > 1$. We find $R_h \in \mathcal{S}^q(H^2)$ if and only if $\log(h(z)/z) \in B_q$.

The conditions above have other equivalent forms due to the fact that the associated univalent functions h are of special form, i.e. spirallike. First, a result of function theory says that if g is analytic in $VMOA$ then $e^g \in \cap_{p<\infty} H^p$ [88]. It follows that if R_h is compact then $h \in \cap_{p<\infty} H^p$. The converse of this also holds, that is if $h \in \cap_{p<\infty} H^p$ then R_h is compact [6]. As a byproduct of these we find that for h univalent spirallike,

$$h \in \cap_{p<\infty} H^p \quad \text{if and only if} \quad \log \frac{h(z)}{z} \in VMOA.$$

Recall now the form of the generator $G(z) = -zF(z)$, $\Re(F) \geq 0$. Then $\Re(1/F) \geq 0$ and assuming for simplicity $F(0) = 1$ we can write the Herglotz representation of $1/F$,

$$\frac{1}{F(z)} = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta),$$

where μ is a probability measure on the unit circle. Integrating (3.4) with a change of the order of integration we find

$$h(z) = z \exp\left(2 \int_0^{2\pi} \log \frac{1}{1 - e^{-i\theta} z} \, d\mu(\theta) \right).$$

From [25, Th. 4.10] we see that such a function h is in H^p for all finite p if and only if μ has no point masses on the unit circle, so this condition on μ is also equivalent to compactness of R_h.

Next we derive a geometric condition on $F(z)$. Let $w \in \partial\mathbf{D}$ then we can write $\mu = \mu\{w\}\delta_w + \mu_1$ where δ_w is the Dirac measure at w and μ_1 is the rest of μ. From the Herglotz formula

$$\frac{1}{F(z)} = \mu\{w\} \frac{w + z}{w - z} + \int_{\partial\mathbf{D}} \frac{\zeta + z}{\zeta - z} \, d\mu_1(\zeta),$$

so that

$$\frac{w - z}{F(z)} = \mu\{w\}(w + z) + \int_{\partial\mathbf{D}} (\zeta + z) \frac{w - z}{\zeta - z} \, d\mu_1(\zeta).$$

Letting $z \to w$ nontangentially from inside the disc, the integral above goes to zero by the bounded convergence theorem. Taking reciprocals we may then write (with the convention $\frac{1}{0} = \infty$)

$$\lim_{z \to w} \frac{F(z)}{z - w} = -\frac{1}{2w\mu\{w\}}.$$

This limit can be finite only when $\mu\{w\} \neq 0$ and then only if the nontangential limit $F(w) \stackrel{\text{def}}{=} \lim_{z \to w} F(z)$ is equal to zero. This means that $F(\mathbf{D})$ touches the imaginary axis at 0. We interpret the limit

$$F'(w) \stackrel{\text{def}}{=} \lim_{z \to w} \frac{F(z) - F(w)}{z - w}$$

as the *angular derivative* of F at w. We see that when $F'(w)$ is finite then $F(\mathbf{D})$ touches the imaginary axis in a substantial way. And $F'(w)$ is finite if and only if $\mu\{w\} \neq 0$.

The preceding arguments can be applied with small changes to Bergman spaces A^p. Here the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$ replace $BMOA$ and $VMOA$. For the operators T_g analogous statements hold [9] for $1 \leq p < \infty$.

- T_g is bounded on A^p if and only if $g \in \mathcal{B}$.

- T_g is compact on A^p if and only if $g \in \mathcal{B}_0$.
- $T_g \in \mathcal{S}^q(A^2)$ if and only if $g \in B_q$, $1 < q < \infty$.

Using this we can obtain characterizations of compactness of R_h on A^p in terms of h or F. The conditions turn out to be the same as for Hardy spaces because of the following result from function theory [88]. For any h univalent on the disc with $h(0) = 0$,

$$\log \frac{h(z)}{z} \in \mathcal{B}_0 \quad \text{if and only if} \quad \log \frac{h(z)}{z} \in VMOA.$$

We collect all these conditions for the resolvent in the following

Theorem 6.1. *Suppose $1 \leq p < \infty$. Let $\mathbf{T}$ be a semigroup of composition operators induced by the semigroup of functions $\mathbf{\Phi}$ on H^p or A^p. Let $G(z) = -zF(z)$ be the generator of $\mathbf{\Phi}$, μ the measure in the Herglotz representation of $1/F$ and h the associated univalent function. Then*

a. *The following are equivalent:*

(1) R_h *is compact on* H^p.
(2) R_h *is compact on* A^p.
(3) $\log \frac{h(z)}{z} \in VMOA$.
(4) $\log \frac{h(z)}{z} \in \mathcal{B}_0$.
(5) $h \in \cap_{p<\infty} H^p$.
(6) $h \in \cap_{p<\infty} A^p$.
(7) $\mu\{w\} = 0$ *for all* $w \in \partial\mathbf{D}$.
(8) F *has no finite angular derivative on* $\partial\mathbf{D}$.

b. *For $q > 1$ the following are equivalent:*

(1) $R_h \in \mathcal{S}^q(H^2)$.
(2) $R_h \in \mathcal{S}^q(A^2)$.
(3) $\log \frac{h(z)}{z} \in B_q$.

To see that (6)$\Rightarrow$(5) in (a) assume for simplicity h is starlike and consider growth. If $h \in \cap_{p<\infty} A^p$ then $|h(z)| = O((1-|z|)^{-2/p})$ for all positive p. If there is a q such that $h \notin H^q$, then there is a $w \in \partial\mathbf{D}$ such that $\mu\{w\} = a > 0$. From [91, Prop. 3.19] it follows that $|h(rw)| > d(1-r)^{-a/\pi}$ for some $d > 0$ and all $0 < r < 1$ and this a contradiction.

The above conditions for compactness suggest that on Hardy and Bergman spaces, the essential norm $\|R_h\|_e = \inf\{\|R_h - K\| : K \text{ a compact operator}\}$ or the essential spectral radius $r_e(R_h)$ may be comparable to one or more of the following quantities

$$(i)\ \ \mathrm{dist}\Big(\log \frac{h(z)}{z}, VMOA\Big), \quad (ii)\ \sum_{w \in \partial\mathbf{D}} \mu\{w\}, \quad (iii)\ \frac{1}{s(h)},$$

where $s(h) = \sup\{p > 0 : h \in H^p\}$ is the Hardy space size of the associated univalent function [85].

We next discuss the case of the Dirichlet space. As a consequence of the norm estimate

$$\|f \circ \phi\|_{\mathcal{D}} \leq \Big(1 + \Big(\log \frac{1+|\phi(0)|}{1-|\phi(0)|}\Big)^{1/2}\Big)\|f\|_{\mathcal{D}},$$

which is valid for any univalent analytic self map ϕ of the disc, the growth bound of every composition semigroup on $\mathcal{D}$ is $\omega \leq 0$ [108]. To discuss compactness of the resolvent operator in the case of an interior DW point we may employ the arguments of the Hardy space case, but this time we cannot go as far.The reason is that we do not have good analogues of the results for operators T_g on $\mathcal{D}$. In particular the operator Q_h is bounded on $\mathcal{D}$ if and only if the measure

$$d\nu(z) = \Big|\Big(\log \frac{h(z)}{z}\Big)'\Big|^2 dm(z)$$

is a *Carleson measure* for $\mathcal{D}$ in the sense of [110] and Q_h is compact if and only if $d\nu(z)$ is a *vanishing Carleson measure*. The identity (6.1) shows that the latter condition on $d\nu(z)$ is equivalent to compactness of R_h. However this Carleson measure characterization of compactness is not as transparent as the Hardy space conditions.

For the Schatten classes of R_h on $\mathcal{D}$ the situation is even less clear. An easy calculation shows that $R_h \in \mathcal{S}^2(\mathcal{D})$ (the Hilbert-Schmidt class) if and only if

$$\int_{\mathbf{D}} \Big|\Big(\log \frac{h(z)}{z}\Big)'\Big|^2 \log \frac{1}{1-|z|^2}\, dm(z) < \infty.$$

For other Schatten classes $\mathcal{S}^q(\mathcal{D})$ one needs to know the Schatten classes of the operators T_g on $\mathcal{D}$. Partial results can be obtained when q is an even integer (see also [10, Th. 25], [92, Th. 1]). For example $R_h \in \mathcal{S}^4(\mathcal{D})$ if and only if

$$\iint_{\mathbf{D}\times\mathbf{D}} \Big|\Big(\log \frac{h(z)}{z}\Big)'\Big|^2 \Big|\Big(\log \frac{h(w)}{w}\Big)'\Big|^2 \log \frac{1}{1-\bar{z}w} \log \frac{1}{1-\bar{w}z}\, dm(z)dm(w) < \infty.$$

Analogous conditions can be written for $q = 6, 8, \ldots$, but these do not suggest anything for noninteger values of q. The analogy in behavior of R_h to Hankel operators with symbol $\lambda(z) = \log(h(z)/z)$ on Hardy and Bergman spaces (see [11], [117]), suggests that R_h may also behave like a Hankel operator on $\mathcal{D}$ with symbol $\lambda(z)$.

7. Weighted composition semigroups

Let $\mathbf{X}$ be a Banach space of analytic functions and $\mathbf{\Phi}$ be a semigroup of analytic self maps of the disc. If $w : \mathbf{D} \to \mathbf{C}$ is analytic, the formula

$$S_t(f)(z) = \frac{w(\phi_t(z))}{w(z)} f(\phi_t(z)), \quad f \in \mathbf{X}, \tag{7.1}$$

defines, for suitable w, bounded operators on $\mathbf{X}$. The family $\mathbf{S} = \{S_t : t \geq 0\}$ is an operator semigroup. If $w \equiv 1$ it reduces to an unweighted semigroup.

An obvious choice of w that makes S_t bounded operators is to take w an invertible pointwise multiplier of $\mathbf{X}$ (i.e. such that for each $f \in \mathbf{X}$ both wf and f/w are in $\mathbf{X}$). There are however many other possible choices. For example if the DW point is $b = 0$ then $w(z) = z$ is a good choice and in fact $w(z) = z^r$, r real, which are not even analytic on $\mathbf{D}$, give semigroups of bounded operators on H^p

$$S_t(f)(z) = \Big(\frac{\phi_t(z)}{z}\Big)^r f(\phi_t(z)).$$

As another example let $\phi_t(z) = 1-(1-z)^{e^{-t}}$. The function $w(z) = 1/(1-z)$ is far from being a multiplier of H^p but the resulting semigroup

$$S_t(f)(z) = \frac{w(\phi_t(z))}{w(z)} f(\phi_t(z)) = (1-z)^{1-e^{-t}} f(\phi_t(z)),$$

consists of bounded operators on H^p.

In a more general setting let the semigroup $\mathbf{\Phi}$ be given. A *cocycle* for $\mathbf{\Phi}$ is a family $\mathbf{m} = \{m_t : t \geq 0\}$ of analytic functions $m_t : \mathbf{D} \to \mathbf{C}$ satisfying

(1) $m_0 \equiv 1$
(2) $m_{t+s}(z) = m_t(z) m_s(\phi_t(z)) \quad$ for each $z \in \mathbf{D}$ and $t, s \geq 0$.
(3) The map $t \to m_t(z)$ is continuous for each $z \in \mathbf{D}$.

Given $\mathbf{\Phi}$ and a suitable w, the family of functions

$$m_t(z) = \frac{w(\phi_t(z))}{w(z)}, \quad z \in \mathbf{D},\ t \geq 0,$$

is a cocycle for $\mathbf{\Phi}$. A *coboundary* for $\mathbf{\Phi}$ is a cocycle that can be written in this form for some w. Not all cocycles are coboundaries [60].

A large class of cocycles is constructed in the following way. Let g be arbitrary analytic on $\mathbf{D}$ then the functions

$$m_t(z) = \exp\left(\int_0^t g(\phi_s(z))\, ds\right), \quad z \in \mathbf{D},\ t \geq 0, \tag{7.2}$$

define a cocycle for $\mathbf{\Phi}$. In particular if G is the generator of $\mathbf{\Phi}$ and w is analytic such that $w'(z)G(z)/w(z)$ is also analytic (this allows w to have zeros only at the DW point) then choosing $g = w'G/w$ we have

$$g \circ \phi_s = \frac{w' \circ \phi_s}{w \circ \phi_s} G \circ \phi_s = \frac{w' \circ \phi_s}{w \circ \phi_s} \frac{\partial}{\partial s}(\phi_s) = \frac{\frac{\partial (w \circ \phi_s)}{\partial s}}{w \circ \phi_s}$$

so the cocycle obtained for this g

$$m_t(z) = \exp\left(\int_0^t \frac{\frac{\partial (w \circ \phi_s)}{\partial s}}{w \circ \phi_s}\, ds\right) = \frac{w \circ \phi_t}{w},$$

is a coboundary.

If $\mathbf{m}$ is a cocycle for $\mathbf{\Phi}$ the formula

$$U_t(f)(z) = m_t(z) f(\phi_t(z)), \quad f \in \mathbf{X}, \tag{7.3}$$

defines a semigroup $\mathbf{U} = \{U_t\}$ of operators on $\mathbf{X}$ provided that each U_t is bounded. A condition sufficient to make each U_t bounded is that each m_t is a multiplier of $\mathbf{X}$. On Hardy and Bergman spaces this means $m_t \in H^\infty$ for each t and this is equivalent to $\limsup_{t\to 0} \|m_t\|_\infty < \infty$ [68]. If $\mathbf{\Phi}$ is a group of Möbius automorphisms, only cocycles consisting of bounded functions give bounded operators U_t on H^p and A^p. For other $\mathbf{\Phi}$ however m_t need not be bounded in order to obtain bounded U_t. This is due to the fact that a weighted composition operator $C_{m,\phi}(f) = m f \circ \phi$ can be bounded on a space $\mathbf{X}$ without m having to be a multiplier of $\mathbf{X}$. For example if ϕ maps $\mathbf{D}$ inside a smaller disc of radius $r < 1$ then every $m \in H^p$ produces a bounded $C_{m,\phi}$ on H^p. See [13] for some results on this.

The question of strong continuity of weighted composition semigroups $\mathbf{S}$ or $\mathbf{U}$ is more complex than in the unweighted case, and depends on the weight fuctions. For coboundaries $m_t = w \circ \phi_t / w$ then the following conditions imply strong continuity on H^p, $1 \leq p < \infty$, [102],

$$\limsup_{t\to 0} \|\frac{w \circ \phi_t}{w}\|_\infty \leq 1. \tag{C_1}$$

$$(C_2) \qquad w \in H^q \quad \text{for some } q > 0, \quad \text{and} \quad \limsup_{t\to 0} \|\frac{w \circ \phi_t}{w}\|_\infty < \infty.$$

Further if either of these conditions is satisfied then $\mathbf{S}$ is not uniformly continuous on H^p unless $\mathbf{\Phi}$ is the trivial semigroup. W. König [68] extended these results to arbitrary cocycles $\mathbf{m}$. He proved,

- If $1 \le p < \infty$ and $\mathbf{U}$ is strongly continuous on H^p then $g = \frac{\partial m_t}{\partial t}|_{t=0}$ exists, it is analytic on $\mathbf{D}$ and $\mathbf{m}$ is given by (7.2).
- If $\mathbf{m}$ is of the form (7.2) and $\sup_{z\in\mathbf{D}} \Re(g(z)) < \infty$ then $\mathbf{U}$ is strongly continuous on H^p and the infinitesimal generator is given by
$$\Delta(f)(z) = G(z)f'(z) + g(z)f(z),$$
where G is the generator of $\mathbf{\Phi}$.
- If $\mathbf{U}$ is uniformly continuous on H^p then $\mathbf{\Phi}$ is trivial and g is bounded.

Thus the generator Δ is a *perturbation*, by the multiplication operator $M_g(f) = gf$ which may be unbounded, of the generator Γ of the unweighted semigroup. In particular if $m_t = w \circ \phi_t/w$ is a coboundary then the generator is $\Delta(f) = Gf' + (w'G/w)f = (G/w)(wf)'$ and we have the following formal identity

$$M_w \circ \Delta = \Gamma \circ M_w, \tag{7.4}$$

where M_w is the operator of multiplication by w. This is an intertwining property for generators inherited from a corresponding property of the semigroups. Indeed comparing the unweighted semigroup $\{T_t\}$ with $\{S_t\}$ we see that

$$M_w \circ S_t = T_t \circ M_w \qquad \text{for all } t \ge 0.$$

In case M_w is bounded on $\mathbf{X}$ this relation is precise and can be used to study the unweighted and weighted semigroup simultaneously. Further, this intertwining relation is inherited by the resolvent operators.

Spectra of generators of weighted composition semigroups can be discussed in much the same way as for the unweighted case. In particular the point spectra are found by solving a first order differential equation [68], [102]. Because of the above intertwining relation $\sigma(\Gamma)$ and $\sigma(\Delta)$ are related and sometimes, modulo eigenvalues, they coincide. The constant functions are no longer eigenvectors for S_t and examples show that, in the case of an interior DW point, if we replace w in (7.1) by a larger power w^n then some additional eigenvalues of T_t are removed. And there is a critical value of n, related to the Hardy space size of the associated univalent function such that when n exceeds this critical value all eigenvalues have been removed. For Hardy and Bergman spaces compactness of resolvent in case of an interior DW point can be discussed as for unweighted semigroups. If w is a bounded function in which case M_w is a bounded operator on H^p the conditions for compactness are those for unweighted semigroups. This can be seen from the form of the resolvent at c or at 0 assuming the latter is in $\rho(\Delta)$,

$$R(c,\Delta)(f)(z) = \frac{1}{w(z)h(z)} \int_0^z f(\zeta)w(\zeta)h'(\zeta)\, d\zeta,$$
$$R(0,\Delta)(f)(z) = \frac{1}{w(z)} \int_0^z f(\zeta)w(\zeta)\frac{h'(\zeta)}{h(\zeta)}\, d\zeta.$$

For general w we do not have information on compactness of these resolvents.

Examples and applications of weighted composition semigroups

Example 7.1. Let $\phi_t(z) = e^{-t}z + 1 - e^{-t}$, $w(z) = 1 - z$ and

$$S_t(f)(z) = \frac{w(\phi_t(z))}{w(z)} f(\phi_t(z)) = e^{-t} f(\phi_t(z)).$$

The generator of $\{S_t\}$ is

$$\Delta(f)(z) = (1-z)f'(z) - f(z) = ((1-z)f(z))'.$$

Using (7.4) and the spectrum of the generator of the unweighted semigroup from example (5.5) we find $\sigma_p(\Delta) = \{z : \Re(z) \leq -1 + \frac{1}{p}\}$, [$\sigma_p(\Delta)$ denotes the spectrum of Δ as an operator on H^p]. Thus $0 \in \rho(\Delta)$ if and only if $p > 1$. Denoting by $\mathcal{A}$ the resolvent $R(0, \Delta)$ we have

$$\mathcal{A}(f)(z) = \frac{1}{z-1} \int_1^z f(\zeta)\, d\zeta$$

If $p = 1$ then $0 \in \sigma(\Delta)$ and this is an immediate proof that $\mathcal{A}$ is not bounded on H^1. With the standard basis for H^p the matrix for $\mathcal{A}$ is

$$\begin{pmatrix} 1, & 1/2, & 1/3, & \cdots \\ 0, & 1/2, & 1/3, & \cdots \\ 0, & 0, & 1/3, & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

This is the transpose of the matrix for $\mathcal{C}$, the Cesàro operator. On H^2 therefore $\mathcal{A}$ and $\mathcal{C}$ are Hilbert space adjoints. This relation between the Cesàro operator and the semigroup $\{S_t\}$ was exploited in [37] to show that $\mathcal{C}$ is subnormal on H^2.

Norm estimates for S_t can be used in conjunction with (2.2) to obtain the norm and spectrum of $\mathcal{A}$,

$$\|\mathcal{A}\|_{H^p} = \frac{p}{p-1} \quad \text{and} \quad \sigma_p(\mathcal{A}) = \{z : |z - \frac{p}{2(p-1)}| \leq \frac{p}{2(p-1)}\},$$

for all $p > 1$. See [103] for details.

Example 7.2. Let $\phi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}$ and

$$S_t(f)(z) = \frac{\phi_t(z)}{z} f(\phi_t(z)).$$

The generator is given by

$$\Delta(f)(z) = -z(1-z)f'(z) - (1-z)f(z) = -(1-z)(zf(z))'.$$

For the spectrum of Δ we can use the spectrum of the generator of the unweighted semigroup in example (5.3), and (7.4) to find $\sigma(\Delta) = \{z : \Re(z) < -1/p\}$. Thus $0 \in \rho(\Delta)$ and $R(0, \Delta) = \mathcal{C}$ the Cesàro operator,

$$\mathcal{C}(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1-\zeta}\, d\zeta = \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} \sum_{k=0}^{n} a_k \Big) z^n.$$

It requires some work to obtain the norms $\|S_t\|_{H^p} = e^{-t/p}$ which are valid however only for $p \geq 2$. The growth bound therefore is $\omega = -1/p$ for $p \geq 2$ and by (2.2) we find $\|\mathcal{C}\|_{H^p} = p$ for $p \geq 2$. The case $1 \leq p < 2$ is discussed in [103] where some variation of the weighted semigroup produces only one sided estimates for the norm. In [106] it is shown that there are values of $p > 1$ such that $\|\mathcal{C}\|_{H^p} > p$. On the other hand we have

$$\sigma_p(\mathcal{C}) = \{z : |z - \frac{p}{2}| \leq \frac{p}{2}\},$$

for all $p \geq 1$. Using this semigroup in a similar manner we can study $\mathcal{C}$ on the Bergman space [107].

It is interesting to note that $\mathcal{C}$ can be also obtained from the semigroup $\phi_t(z) = 1-(1-z)^{e^{-t}} = h^{-1}(e^{-t}h(z))$ with $h(z) = \log(1/(1-z))$. Indeed from (5.1) we find

$$R(1,\Gamma)(f)(z) = \frac{1}{\log(1/(1-z))} \int_0^z f(\zeta) \frac{1}{1-\zeta}\, d\zeta.$$

If we write $R(1,\Gamma)$ as a Laplace transform (2.1) and then change variables we obtain

$$\mathcal{C}(f)(z) = \frac{\log(1/(1-z))}{z} \int_0^1 f(1-(1-z)^s)\, ds.$$

Example 7.3. For any semigroup $\mathbf{\Phi}$ with generator $G(z) = -zF(z)$ let

$$S_t(f)(z) = \frac{\phi_t(z)}{z} f(\phi_t(z)).$$

This is strongly continuous on H^p and the generator is

$$\Delta(f)(z) = -zF(z)f'(z) - F(z)f(z) = -F(z)(zf(z))'.$$

The special case $F(z) = \frac{1-z}{1+z}$ corresponds to $\phi_t(z) = k^{-1}(e^{-t}k(z))$ where $k(z) = z/(1-z)^2$ is the Koebe function. In some sense this is extremal among the semigroups with DW point 0 and every other such semigroup is "subordinate" to this. By studying first the extremal semigroup we find $0 \in \rho(\Delta)$ for all $\mathbf{\Phi}$ with DW point 0 and this implies that the resolvent $Q_F = R(0,\Delta)$,

$$Q_F(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{F(\zeta)}\, d\zeta,$$

is a bounded operator on H^p for all F with $\Re(F) \geq 0$. Writing F in terms of the associated univalent function we see that Q_F is identical to the operator Q_h of section 6. This is another proof (without use of operators T_g) that Q_h are bounded on H^p. In addition we see that Q_F is compact if and only if the measure in the Herglotz representation of $1/F$ has no point masses on the unit circle.

Example 7.4. Let $\mathbf{\Phi}$ be any semigroup of functions with generator G and choose $w(z) = G(z)$ to obtain

$$S_t(f)(z) = \frac{G(\phi_t(z))}{G(z)} = \phi_t'(z) f(\phi_t(z)),$$

This is an interesting and intriguing operator semigroup. In principle there should be conditions (for example on G) characterizing those $\mathbf{\Phi}$ for which S_t are bounded operators say on H^p. Additional conditions should imply strong continuity. We do not know any such conditions.

Example 7.5. The groups of isometries from H^p onto H^p, $1 \leq p < \infty$, $p \neq 2$, are weighted composition semigroups. This is a consequence of Forelli's theorem [51] which says that if $1 \leq p < \infty$, $p \neq 2$ and T is a linear isometry from H^p onto H^p then T is given by

$$T(f)(z) = e^{i\gamma}(\phi'(z))^{1/p} f(\phi(z)),$$

where γ is a real number and ϕ a Möbius automorphism of the disc. On H^2 this formula does give an onto isometry but there are many more isometries that cannot be described as weighted composition operators. The groups of Möbius automorphisms of $\mathbf{D}$ were determined explicitly in [21]. After normalization they are essentially the ones in example 5.1 with c pure imaginary (elliptic), example 5.6 (hyperbolic), and example 5.7 (parabolic). The one-parameter groups of isometries are given by

$$U_t(f)(z) = e^{i\gamma t}(\phi_t'(z))^{1/p} f(\phi_t(z)), \qquad -\infty < t < \infty,$$

where $\{\phi_t\}$ is a one-parameter group of automorphisms of $\mathbf{D}$. The generators and spectral properties of $\{U_t\}$ were studied in [21].

There are also linear isometries of H^p which are not onto. These are again weighted composition operators but the weights are more general. Semigroups of such isometries were studied in [19]. Further, groups of isometries on Hardy spaces of the torus or of the unit ball of $\mathbf{C}^n$ were studied in [23], [24], [26].

8. Final Remarks

There are several questions about the operator semigroups $\mathbf{T}$, $\mathbf{S}$, and $\mathbf{U}$ that remain to be studied. For example we have very little information in case the DW point of $\mathbf{\Phi}$ is on the boundary. Composition semigroups can (or should) be studied in parallel with single composition operators. Even though the inducing functions are strongly restricted by the requirement of participating in a continuous parameter semigroup, results from semigroups can give insight for properties of general single composition operators. In fact it may be possible to use semigroups more directly to study single composition operators. It is shown in [33] that under some mild conditions on ϕ, for each $z \in \mathbf{D}$ all sufficiently large fractional iterates of ϕ can be defined at z.

We have not discussed analytic semigroups, that is those $\mathbf{\Phi} = \{\phi_t\}$ for which there is a sector S in the plane containing the positive x-axis such that $\mathbf{\Phi}$ extends to a family $\{\phi_w\}$, $w \in S$, with the semigroup properties satisfied. These give rise to *analytic composition semigroups* $\{T_w\}$ which behave better as operator semigroups, for example there is equality in the spectral theorem (2.3).

Semigroups of functions in several complex variables are studied in [2]. I am not aware of any papers studying semigroups of composition operators on spaces of holomorphic functions in several variables.

Finally we would like to mention an observation which leads to some questions. The univalent function h associated to a semigroup of functions $\mathbf{\Phi}$, is in principle determined by the boundary of the region Ω onto which it maps the disc conformally. The size of h and its other properties depend heavily on the geometry of $\partial\Omega$. It follows that properties of the induced operator semigroup $\mathbf{T}$ may be related to the geometry of the boundary. For example how much different are the two operator semigroups, one induced by $\phi_t(z) = e^{-t}z$ (for which $h(z) \equiv z$) and the second by $\phi_t(z) = h^{-1}(e^{-t}h(z))$ where h maps the disc onto a bounded starlike region with a fractal boundary?

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