COMPOSITION OPERATORS AND THE HILBERT MATRIX

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Abstract. The Hilbert matrix acts on Hardy spaces by multiplication with Taylor coefficients. We find an upper bound for the norm of the induced operator.

1. Introduction

The classical Hilbert inequality

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{1/p} \leq \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{n=0}^{\infty}|a_k|^p\right)^{1/p}, \tag{1.1}$$

is valid for sequences $a = \{a_n\}$ in the sequence spaces l^p for $1 < p < \infty$, and the constant $\pi/\sin(\frac{\pi}{p})$ is best possible [HLP]. Thus the Hilbert matrix

$$H = (\frac{1}{i+j+1}) \qquad i, j = 0, 1, 2, \cdots$$

acting by multiplication on sequences, induces a bounded linear operator

$$Ha = b, \qquad b_n = \sum_{k=0}^{\infty}\frac{a_k}{n+k+1},$$

on the l^p spaces with norm $\|H\|_{l^p \to l^p} = \pi/\sin(\frac{\pi}{p})$ for $1 < p < \infty$.

The Hilbert matrix also induces an operator $\mathcal{H}$ on Hardy spaces H^p, as explained below, by its action on Taylor coefficients. In this article we prove an analogue of the inequality (1.1) on Hardy spaces. More precisely we show

Theorem 1.1. *The following inequalities are valid*

(1) *If* $2 \leq p < \infty$ *then*

$$\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_{H^p}$$

for each $f \in H^p$.

(2) *If* $1 < p < 2$ *then*

$$\|\mathcal{H}(f)\|_{H^p} \leq \frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_{H^p}$$

for each $f \in H^p$ *with* $f(0) = 0$

Date: January 21 2000.
1991 *Mathematics Subject Classification.* Primary 30D55, 47B38; Secondary 47A30, 47B35.
The first author was supported with a postgraduate state fellowship from I.K.Y.

The proof will be given in section 3 and involves an expression of $\mathcal{H}$ in terms of weighted composition operators of which we can estimate the Hardy space norms.

Recall that the Hardy space H^p, $1 \leq p \leq \infty$, of the unit disc $\mathbb{D}$ is the Banach space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\|f\|_{H^p} = \sup_{r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty, \tag{1.2}$$

for finite p, and $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. For $1 \leq p \leq q \leq \infty$ we have $H^1 \supset H^p \supset H^q \supset H^\infty$ and H^p is embedded as a closed subspace in $L^p(\mathbb{T})$, the Lebesgue space on the unit circle, by identifying H^p with the closure of analytic polynomials in $L^p(\mathbb{T})$. Additional properties of Hardy spaces can be found in [DU].

To study the effect of Hilbert matrix on Hardy spaces let $f(z) = \sum_{n=0}^\infty a_n z^n$ belong to H^1. Hardy's inequality [DU, p. 48] says

$$\sum_{n\geq 0} \frac{|a_n|}{n+1} \leq \pi \|f\|_{H^1},$$

and it follows that the power series

$$F(z) = \sum_{n=0}^\infty \left(\sum_{k=0}^\infty \frac{a_k}{n+k+1} \right) z^n,$$

has bounded coefficients, hence its radius of convergence is ≥ 1. In this way we obtain a well defined analytic function $F = \mathcal{H}(f)$ on the disc for each $f \in H^1$. A calculation shows that we can write

$$\mathcal{H}(f)(z) = \int_0^1 f(t) \frac{1}{1-tz} \, dt, \tag{1.3}$$

where the convergence of the integral is guaranteed by the Fejer-Riesz inequality [DU, p. 46] and the fact that $1/(1-tz)$ is bounded in t for each $z \in \mathbb{D}$.

The correspondence $f \to \mathcal{H}(f)$ is clearly linear and we consider the restriction of this mapping on the spaces H^p for $p \geq 1$. For $p = 2$, the isometric identification of H^2 with l^2 gives

$$\|\mathcal{H}\|_{H^2 \to H^2} = \pi.$$

On the other hand $\mathcal{H}$ is not bounded on the spaces H^1 and H^∞. For H^∞ this is because the constant function 1 is mapped to

$$\mathcal{H}(1)(z) = \frac{1}{z} \log \frac{1}{1-z},$$

which is not a bounded function. For H^1, let $\epsilon > 0$ and let

$$f_\epsilon(z) = \frac{1}{(1-z)\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{1+\epsilon}},$$

a function which belongs to H^1 [DU, p. 13] and is real and positive on $[0, 1]$. We assert that the analytic function $\mathcal{H}(f_\epsilon)$ does not belong to H^1 for small values of ϵ. Indeed using (1.3) we find

$$\mathcal{H}(f_\epsilon)(z) = \sum_{n=0}^\infty \left(\int_0^1 t^n f_\epsilon(t) \, dt \right) z^n$$

and if we assume $\mathcal{H}(f_\epsilon) \in H^1$ then Hardy's inequality implies that the quantity

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n+1} \int_0^1 t^n f_\epsilon(t)\, dt &= \int_0^1 f_\epsilon(t) \sum_{n=0}^{\infty} \frac{t^n}{n+1}\, dt \\ &= \int_0^1 f_\epsilon(t)(\frac{1}{t} \log \frac{1}{1-t})\, dt \\ &= \int_0^1 \frac{1}{(1-t)\left(\frac{1}{t}\log\frac{1}{1-t}\right)^\epsilon}\, dt\end{aligned}$$

is finite. With $\epsilon \leq 1$ this is a contradiction.

The operator $\mathcal{H}$ is however bounded on H^p for all $1 < p < \infty$. This is known and a quick way to see this is to view $\mathcal{H}$ as a Hankel operator. In fact $\mathcal{H}$ is a prototype for Hankel operators see [PA] for details. We will not pursue this aspect further except to note that a Hankel operator is bounded on H^2 if and only if it is bounded on each H^p for $1 < p < \infty$, see [CS]. The results of [CS] also imply that $\mathcal{H}$ is not bounded on H^1, a fact that we obtained by a direct argument above.

2. $\mathcal{H}$ in terms of composition operators

In this section we indicate how $\mathcal{H}$ can be written as an average of certain weighted composition operators.

Every analytic function $\phi : \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator

$$C_\phi : f \to f \circ \phi$$

on H^p for $1 \leq p \leq \infty$, see [DU, p. 29]. In addition if $w(z)$ is a bounded analytic function then the weighted composition operator

$$C_{w,\phi}(f)(z) = w(z) f(\phi(z))$$

is bounded on each H^p. More information about these operators can be found in [CM] or [SH]. We will not need here any of their properties except from the fact that they are bounded.

The connection of the Hilbert matrix with composition operators comes as follows. For $f \in H^1$ the Fejér-Riesz theorem, which guarantees convergence, along with analyticity shows that the integral in (1.3) is independent of the path of integration. For $z \in \mathbb{D}$ we can choose the path

$$\zeta(t) = \zeta_z(t) = \frac{t}{(t-1)z+1}, \qquad 0 \leq t \leq 1, \tag{2.1}$$

i.e. a circular arc in $\mathbb{D}$ joining 0 to 1. The change of variable in (1.3) gives

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{(t-1)z+1} f(\frac{t}{(t-1)z+1})\, dt. \tag{2.2}$$

This expression says that the transformation $\mathcal{H}$ is an average

$$\mathcal{H}(f)(z) = \int_0^1 T_t(f)(z)\, dt$$

of the weighted composition operators

$$T_t(f)(z) = w_t(z) f(\phi_t(z)), \tag{2.3}$$

where

$$w_t(z) = \frac{1}{(t-1)z+1} \qquad \text{and} \qquad \phi_t(z) = \frac{t}{(t-1)z+1}.$$

It is easy to see that ϕ_t is a self map of the disc, hence $f \to f \circ \phi_t$ is bounded on H^p, and that for each $0 < t < 1$, $w_t(z)$ is a bounded analytic function. Thus $T_t : H^p \to H^p$, $1 \leq p \leq \infty$, is bounded for $0 < t < 1$.

3. Proof of the theorem

We first obtain estimates for the norms of the weighted composition T_t. The estimates are achieved by transferring T_t to operators $\widetilde{T}_t$ acting on Hardy spaces of the right half plane, which are isometric to Hardy spaces of the disc. The form of $\widetilde{T}_t$ permits estimates of its norm, thereby the estimate for the norm of T_t follows.

Lemma 3.1. *Suppose $p \geq 2$, then*

$$\|T_t(f)\|_{H^p} \leq \frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{p}}} \|f\|_{H^p}, \qquad 0 < t < 1. \tag{3.1}$$

for each $f \in H^p$.

Proof. The Hardy space $H^p(\Pi)$ of the right half plane $\Pi = \{z : \Re(z) > 0\}$, consists of analytic functions $f : \Pi \to \mathbb{C}$ such that

$$\|f\|^p_{H^p(\Pi)} = \sup_{0<x<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy < \infty \tag{3.2}$$

These are Banach spaces for $1 \leq p < \infty$.

Let $\mu(z) = \frac{1+z}{1-z}$ be the conformal map of $\mathbb{D}$ onto Π with inverse $\mu^{-1}(z) = \frac{z-1}{z+1}$, and let

$$V(f)(z) = \frac{(4\pi)^{1/p}}{(1-z)^{2/p}} f(\mu(z)), \qquad f \in H^p(\Pi).$$

It can be checked that this map is a linear isometry from $H^p(\Pi)$ onto H^p with inverse given by

$$V^{-1}(g)(z) = \frac{1}{\pi^{1/p}(1+z)^{2/p}} g(\mu^{-1}(z)), \qquad g \in H^p.$$

Let $\widetilde{T}_t : H^p(\Pi) \to H^p(\Pi)$ be the operators defined by

$$\widetilde{T}_t = V^{-1} T_t V$$

and suppose $h \in H^p(\Pi)$. A calculation shows that $\widetilde{T}_t$ are weighted composition operators given by

$$\widetilde{T}_t(h)(z) = \frac{1}{(1-t)^{\frac{2}{p}}} \left(\frac{1}{(t-1)\mu^{-1}(z)+1} \right)^{1-\frac{2}{p}} h(\Phi_t(z)), \qquad 0 < t < 1, \tag{3.3}$$

where

$$\Phi_t(z) = \mu \circ \phi_t \circ \mu^{-1}(z) = \frac{t}{1-t} z + \frac{1}{1-t}$$

is an analytic function mapping Π into itself. By an elementary argument we see that if $z \in \Pi$ then $|(t-1)\mu^{-1}(z) + 1| \geq t$ and since $1 - \frac{2}{p} \geq 0$ we have

$$|\widetilde{T}_t(h)(z)| \leq \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} |h(\Phi_t(z))|.$$

Integrating for the norm we have

$$\begin{aligned}\|\widetilde{T}_t(h)\|_{H^p(\Pi)} &= \sup_{0<x<\infty}\left(\int_{-\infty}^{\infty}|\widetilde{T}_t(h)(z)|^p\,dy\right)^{1/p}\\ &\le \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}}\sup_{0<x<\infty}\left(\int_{-\infty}^{\infty}|h(\frac{t}{1-t}(x+iy)+\frac{1}{1-t})|^p\,dy\right)^{1/p}\\ &= \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}}\sup_{1/(1-t)<X<\infty}\left(\int_{-\infty}^{\infty}|h(X+iY)|^p\frac{1-t}{t}\,dY\right)^{1/p}\end{aligned}$$

where we have changed the variables $X=\frac{t}{1-t}x+\frac{1}{1-t}$ and $Y=\frac{t}{1-t}y$, to obtain

$$\begin{aligned}&\le \frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{p}}}\sup_{0<X<\infty}\left(\int_{-\infty}^{\infty}|h(X+iY)|^p\,dY\right)^{1/p}\\ &= \frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{p}}}\|h\|_{H^p(\Pi)}.\end{aligned}$$

The conclusion follows. □

For the final step of the proof we will need some classical identities about the Gamma and Beta functions, see for example [WW]. The Beta function is defined by

$$B(s,t)=\int_0^1 x^{s-1}(1-x)^{t-1}\,dx$$

for each s,t with $\Re(s)>0$, $\Re(t)>0$. The value $B(s,t)$ can be expressed in terms of the Gamma function as $B(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$. We are going to use also the functional equation for the Gamma function

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}.$$

which is valid for non-integer complex z.

Now suppose $p\ge 2$ and $f\in H^p$ with $\|f\|_{H^p}=1$. Then

$$\begin{aligned}\|\mathcal{H}(f)\|_{H^p} &= \sup_{r<1}\left(\int_0^{2\pi}|\mathcal{H}(f)(re^{i\theta})|^p\,\frac{d\theta}{2\pi}\right)^{1/p}\\ &= \sup_{r<1}\left(\int_0^{2\pi}\left|\int_0^1 T_t(f)(re^{i\theta})\,dt\right|^p\frac{d\theta}{2\pi}\right)^{1/p}\\ &\le \int_0^1\sup_{r<1}\left(\int_0^{2\pi}|T_t(f)(re^{i\theta})|^p\,\frac{d\theta}{2\pi}\right)^{1/p}dt\end{aligned}$$

(by the continuous version of Minkowski's inequality)

$$
\begin{aligned}
&= \int_0^1 \|T_t(f)\|_{H^p}\, dt \\
&\le \int_0^1 t^{1/p-1}(1-t)^{-1/p}\, dt \\
&= B(\frac{1}{p}, 1-\frac{1}{p}) \\
&= \Gamma(\frac{1}{p})\Gamma(1-\frac{1}{p}) \\
&= \frac{\pi}{\sin(\frac{\pi}{p})}.
\end{aligned}
$$

and this gives the assertion for $p \ge 2$.

Suppose now $1 < p < 2$ and $f \in H^p$ with $f(0) = 0$. Then $f(z) = zf_0(z)$ with $\|f\|_{H^p} = \|f_0\|_{H^p}$. Writing $\mathcal{H}$ in the integral form (2.2) we see that

$$\mathcal{H}(f)(z) = \int_0^1 \mathcal{T}_t(f_0)(z)\, dt$$

where $\mathcal{T}_t$ are the weighted composition operators

$$\mathcal{T}_t(g)(z) = \frac{t}{((t-1)z+1)^2} g(\frac{t}{(t-1)z+1}).$$

We now follow the proof (with same notation) of Lemma 3.1 to estimate the norms of $\mathcal{T}_t$. Letting $\widetilde{\mathcal{T}}_t = V^{-1}\mathcal{T}_t V : H^p(\Pi) \to H^p(\Pi)$ we find

$$\widetilde{\mathcal{T}}_t(h)(z) = \frac{t}{(1-t)^{\frac{2}{p}}} \left(\frac{1}{(t-1)\mu^{-1}(z)+1} \right)^{2-\frac{2}{p}} h(\Phi_t(z)), \qquad 0 < t < 1, \tag{3.4}$$

for each $h \in H^p(\Pi)$. Because $2 - \frac{2}{p} > 0$ for $p > 1$, the rest of the calculation in Lemma 3.1 goes through and we conclude

$$\|\mathcal{T}_t(g)\|_{H^p} \le \frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{p}}} \|g\|_{H^p}, \qquad 0 < t < 1.$$

for each $g \in H^p$. Using this norm estimate we can repeat the final step of the proof of the case $p \ge 2$ to obtain

$$
\begin{aligned}
\|\mathcal{H}(f)\|_{H^p} &\le \frac{\pi}{\sin(\frac{\pi}{p})} \|f_0\|_{H^p} \\
&= \frac{\pi}{\sin(\frac{\pi}{p})} \|f\|_{H^p},
\end{aligned}
$$

and this finishes the proof of the theorem.

3.1. **Remarks.** We do not know if the inequalities in the theorem are sharp but it is expected they are. Also we have not been able to remove the restriction $f(0) = 0$ in the case $1 < p < 2$. One can obtain an inequality which holds for all $f \in H^p$, $1 < p < 2$, as follows: Write $f(z) = f(0) + f_0(z)$ with $f_0(0) = 0$ then

$$
\begin{aligned}
\mathcal{H}(f)(z) &= f(0)\mathcal{H}(1)(z) + \mathcal{H}(f_0)(z) \\
&= f(0)\frac{1}{z}\log\frac{1}{1-z} + \mathcal{H}(f_0)(z).
\end{aligned}
$$

Using the second part of Theorem 1.1 and the fact that $\|f - f(0)\|_{H^p} \leq \|f\|_{H^p} + |f(0)| \leq 2\|f\|_{H^p}$ we obtain

$$\|\mathcal{H}(f)\|_{H^p} \leq \left(\frac{2\pi}{\sin(\frac{\pi}{p})} + \| \log \frac{1}{1-z} \|_{H^p} \right) \|f\|_{H^p},$$

an inequality which is certainly not the best.

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