HAUSDORFF MATRICES AND COMPOSITION OPERATORS

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Abstract. We consider Hausdorff matrices as operators on Hardy spaces of analytic functions. When the generating sequence of the matrix is the moment sequence of a measure μ, we find conditions on μ such that the matrix represents a bounded operator. The results unify and extend some known special cases of operators on Hardy spaces such as the Cesàro and generalized Cesàro operators.

1. Introduction

1.1. **Hausdorff matrices.** Let Δ be the forward difference operator defined on scalar sequences $\{\mu_n\}_0^\infty$ by $\Delta\mu_n = \mu_n - \mu_{n+1}$ and

$$\Delta^k\mu_n = \Delta(\Delta^{k-1}\mu_n) \qquad \text{for} \quad k = 1, 2, \cdots, \qquad \Delta^0\mu_n = \mu_n.$$

A *Hausdorff matrix* $H = H(\mu_n)$ with generating sequence $\{\mu_n\}$ is the lower triangular matrix

$$H = \begin{pmatrix} c_{0,0} & 0 & 0 & \cdots \\ c_{1,0} & c_{1,1} & 0 & \cdots \\ c_{2,0} & c_{2,1} & c_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with entries

$$c_{n,k} = \binom{n}{k}\Delta^{n-k}\mu_k, \quad k \le n.$$

These matrices have been studied for a long time, originally in connection with summability of series and later as operators on sequence spaces. Their basic properties can be found in [H1] or [GA]. An important special case occurs when μ_n is the moment sequence of a measure. That is,

$$\mu_n = \int_0^1 t^n \, d\mu(t), \qquad n = 0, 1, \cdots,$$

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where μ is a finite (positive) Borel measure on $(0,1]$. These matrices are denoted by H_μ and their entries are found to be

$$c_{n,k} = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k}\, d\mu(t), \qquad k \le n.$$

When μ is probability measure the Hausdorff matrix H_μ is called *totally regular.*

It follows from the work of Hardy [H2] that if μ is a measure satisfying

$$\int_0^1 \frac{1}{t^{1/p}}\, d\mu(t) < \infty$$

then H_μ determines a bounded linear operator

$$H_\mu : \{a_n\} \longrightarrow \{A_n\}, \qquad A_n = \sum_{k=0}^{n} c_{n,k} a_k \qquad n = 0, 1, \cdots, \tag{1.1}$$

on the sequence space l^p, $1 < p < \infty$, whose norm is given by

$$\|H_\mu\|_{l^p \to l^p} = \int_0^1 \frac{d\mu(t)}{t^{1/p}}.$$

In recent years Hausdorff matrices, their generalizations and their continuous analogues have been studied as operators on sequence spaces or on spaces of functions by various authors, see for example [RH], [DE], [LE], [LM].

The purpose of this article is to consider Hausdorff matrices as operators on spaces of analytic functions and in particular on Hardy spaces. Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$ and let X denote a Banach space consisting of analytic functions on $\mathbb{D}$. Let $H_\mu = (c_{n,k})$ be a Hausdorff matrix arising from a Borel measure μ. If $f(z) = \sum_{n\ge 0} a_n z^n \in X$ we consider the transformed power series

$$H_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_{n,k} a_k \right) z^n, \tag{1.2}$$

which is obtained by letting the matrix H_μ multiply the Taylor coefficients of f. Putting aside for the moment the question of convergence we may ask whether the linear transformation

$$f \to H_\mu(f)$$

is bounded when considered as a transformation on X.

We also consider the transpose matrix $A_\mu = H_\mu^*$, to act on Taylor coefficient of a function $f \in X$. Formally,

$$A_\mu(f)(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} c_{n,k} a_n \right) z^k. \tag{1.3}$$

The matrices A_μ are called *quasi-Hausdorff matrices*. The convergence of the power series $A_\mu(f)$ is more delicate in this case. Nevertheless it is clear that if f is a polynomial then $A_\mu(f)$ is also a polynomial and, assuming that X contains the polynomials, we may ask if A_μ extends as a bounded operator on X.

Various choices of the measure μ give rise to some well known classical matrices. For example when μ is the Lebesgue measure one obtains the Cesàro matrix which is known to be bounded on Hardy and Bergman spaces. A weighted Lebesgue measure gives rise to generalized Cesàro operators which are also known to be bounded on Hardy spaces (see remark at the end of article). Other special cases of μ and the corresponding matrices can be found in [RH].

In this article we will examine the matrices H_μ and A_μ as operators on the Hardy space H^p, $1 \le p < \infty$. We find sufficient conditions on the measure μ that ensure boundedness. The main results are:

Theorem 1.1. *Let μ be a finite Borel measure on $(0,1]$ and H_μ the corresponding Hausdorff matrix.*
(i) Suppose $1 < p < \infty$ and $\int_0^1 t^{\frac{1}{p}-1}\, d\mu(t) < \infty$. Then H_μ is bounded on H^p and

$$\|H_\mu\|_{H^p \to H^p} \le C \int_0^1 t^{\frac{1}{p}-1}\, d\mu(t),$$

and the constant C can be taken $C = 1$ when $p \ge 2$.
(ii) For $p = 1$, H_μ is bounded on H^1 if and only if $\int_0^1 \log\frac{1}{t}\, d\mu(t) < \infty$. In this case

$$\|H_\mu\|_{H^1 \to H^1} \le C' \left(\mu((0,1]) + \int_0^1 \log\frac{1}{t}\, d\mu(t) \right).$$

for some constant C'.

Theorem 1.2. *Let μ be a finite Borel measure on $(0,1]$ and A_μ the corresponding quasi-Hausdorff matrix. If $1 \le p < \infty$ and $\int_0^1 t^{-1/p}\, d\mu(t) < \infty$ then A_μ is bounded on H^p and*

$$\|A_\mu\|_{H^p \to H^p} = \int_0^1 \frac{d\mu(t)}{t^{1/p}}.$$

The proof of these theorems will make essential use of a relation between Hausdorff matrices and certain composition operators. We point out this relation in section 2. The proofs of the theorems are given in section 3. In the rest of this section we present some background and fix the notation on Hardy spaces.

1.2. **Hardy spaces.** For $1 \le p < \infty$ the Hardy space H^p is the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_p = \sup_{r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

H^p is a Banach space with this norm and a Hilbert space for $p = 2$. If $1 \le p \le q < \infty$ then $H^1 \supset H^p \supset H^q$. Functions $f \in H^p$ possess boundary values $f(e^{i\theta})$ and these boundary functions are p-integrable on $\partial\mathbb{D}$. Identifying f with its boundary function provides an isometric embedding of H^p into $L^p(\partial\mathbb{D})$. If $f \in H^p$ then

$$|f(z)| \le \frac{c_p \|f\|_p}{(1-|z|)^{1/p}}, \qquad z \in \mathbb{D} \tag{1.4}$$

the constant c_p depending only on p, see [DU, p. 36].

If $1 < p < \infty$ the dual space $(H^p)^*$ is H^q, $\frac{1}{p} + \frac{1}{q} = 1$, in the sense that the continuous linear functionals on H^p are of the form $\Lambda_g(f) = \langle f,\ g\rangle$, where g ranges over H^q, and the pairing is given by

$$\langle f,\ g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta \qquad f \in H^p, \qquad g \in H^q \tag{1.5}$$

The duality $(H^p)^* \simeq H^q$ is only an isomorphism of Banach spaces and not an isometry unless $p = 2$. In general for $\Lambda_g \in (H^p)^*$ we have

$$\|\Lambda_g\| \le \|g\|_q \le C_q \|\Lambda_g\|$$

where C_q is a constant depending only on q.

Every analytic function $a(z) : \mathbb{D} \to \mathbb{D}$ induces a bounded composition operator

$$C_a(f)(z) = f(a(z))$$

on the Hardy space H^p, see [DU, p. 29]. In addition if $b(z)$ is a bounded analytic function on $\mathbb{D}$ then the weighted composition operator

$$C_{a,b}(f)(z) = b(z) f(a(z))$$

is bounded on H^p.

Additional properties of Hardy spaces and composition operators can be found in [DU], [CM], [SH].

2. Hausdorff matrices and composition operators

For each $t \in (0, 1]$ the function ϕ_t given by

$$\phi_t(z) = \frac{tz}{(t-1)z + 1}, \qquad z \in \mathbb{D} \tag{2.1}$$

maps the disc into itself. At the same time the weight functions

$$w_t(z) = \frac{1}{(t-1)z + 1} \tag{2.2}$$

are bounded on $\mathbb{D}$ for each $t \in (0,1]$ thus the weighted composition operators

$$T_t(f)(z) = w_t(z)f(\phi_t(z)), \qquad 0 < t \le 1, \tag{2.3}$$

are bounded on H^p.

Also for each $t \in (0,1]$ the functions

$$\psi_t(z) = tz + 1 - t, \qquad z \in \mathbb{D}, \tag{2.4}$$

map the disc into itself thus the induced composition operators

$$U_t(f)(z) = f(\psi_t(z)), \qquad 0 < t \le 1, \tag{2.5}$$

are bounded on H^p.

Lemma 2.1. *Let μ be a finite Borel measure on $(0,1]$ and $H_\mu = (c_{n,k})$ the corresponding Hausdorff matrix. Suppose $1 \le p < \infty$ and $f(z) = \sum_{n\ge 0} a_n z^n \in H^p$. Then*
(i) The power series $H_\mu(f)(z)$ in (1.2) represents an analytic function on $\mathbb{D}$.
(ii) $H_\mu(f)$ can be written in terms of the weighted composition operators (2.3) as

$$H_\mu(f)(z) = \int_0^1 w_t(z)f(\phi_t(z))\,d\mu(t) \tag{2.6}$$

for each $z \in \mathbb{D}$.

Proof. (i). Since $f \in H^p$ the sequence of Taylor coefficients of f is bounded say by $M < \infty$. Write $A_n = \sum_{k=0}^n c_{n,k} a_k$ and use and the identity $1 = (t+1-t)^n = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k}$ to obtain

$$\begin{aligned}
|A_n| \le& \sum_{k=0}^n \binom{n}{k} \int_0^1 t^k(1-t)^{n-k}\,d\mu(t)|a_k| \\
\le& M\int_0^1 \sum_{k=0}^n \binom{n}{k} t^k(1-t)^{n-k}\,d\mu(t) \\
=& M\mu\{(0,1]\}.
\end{aligned}$$

Thus the coefficients of the series (1.2) are bounded and it follows that its radius of convergence is ≥ 1.

(ii). Consider the composing functions ϕ_t defined in (2.1). Each ϕ_t fixes the origin so from Schwarz's lemma, for a fixed $z \in \mathbb{D}$ we have $|\phi_t(z)| \le |z|$ for each $t \in (0,1]$. Hence

$$\sup_{0<t\le 1} |f(\phi_t(z))| \le \sup_{|\zeta|\le|z|} |f(\zeta)| < \frac{c_p\|f\|_p}{(1-|z|)^{1/p}}.$$

Also the weight functions w_t satisfy $\sup_{0<t\le 1}|w_t(z)| \le 1/(1-|z|)$. Thus the integral

$$F(z) = \int_0^1 w_t(z)f(\phi_t(z))\,d\mu(t)$$

is finite and defines F as an analytic function on $\mathbb{D}$. Keeping $z \in \mathbb{D}$ fixed we have

$$\begin{aligned} F(z) &= \int_0^1 \frac{1}{(t-1)z+1} f(\frac{tz}{(t-1)z+1})\, d\mu(t) \\ &= \int_0^1 \sum_{k=0}^{\infty} a_k \frac{t^k z^k}{((t-1)z+1)^{k+1}}\, d\mu(t) \\ &= \sum_{k=0}^{\infty} a_k z^k \int_0^1 \frac{t^k}{((t-1)z+1)^{k+1}}\, d\mu(t) \end{aligned}$$

where the interchange between the sum and the integral is justified by the uniform convergence on t. Next

$$\begin{aligned} \frac{t^k}{((t-1)z+1)^{k+1}} &= \sum_{j=0}^{\infty} \binom{j+k}{k} t^k (1-t)^j z^j \\ &= \sum_{n=k}^{\infty} \binom{n}{k} t^k (1-t)^{n-k} z^{n-k}, \end{aligned}$$

see [ZY, p. 77]. This series converges uniformly on $t \in (0,1]$ hence we may interchange sum and integral to obtain

$$\begin{aligned} \int_0^1 \frac{t^k}{((t-1)z+1)^{k+1}}\, d\mu(t) &= \sum_{n=k}^{\infty} \binom{n}{k} \int_0^1 t^k (1-t)^{n-k}\, d\mu(t) z^{n-k} \\ &= \sum_{n=k}^{\infty} c_{n,k} z^{n-k}. \end{aligned}$$

Putting these together we find

$$F(z) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} c_{n,k} a_k z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_{n,k} a_k \right) z^n = H_\mu(f)(z),$$

where the change in the order of summation is valid because, as a consequence of the first part of the proof, the last sum converges absolutely. This finishes the proof.

□

We now turn to A_μ. Unlike the case of H_μ, the sums $B_k = \sum_{n=k}^{\infty} c_{n,k} a_n$ defining the coefficients of the power series (1.3) may not be finite. These sums are surely finite if f is a polynomial and we show that in this case $A_\mu(f)$ can be represented by an integral involving composition operators. We then use this integral to define $A_\mu(f)$ for other Hardy space functions.

Lemma 2.2. *Let μ be a finite Borel measure on $(0,1]$ and A_μ the corresponding quasi-Hausdorff matrix. Then* *(i) For each polynomial f, $A_\mu(f)$ can be written in terms of the composition operators (2.5) as*

$$A_\mu(f)(z) = \int_0^1 f(\psi_t(z))\, d\mu(t). \tag{2.7}$$

(ii) Suppose $1 \leq p < \infty$ and $\int_0^1 t^{-1/p}\, d\mu(t) < \infty$. Then for every $f \in H^p$ the above integral is finite and defines an analytic function on $\mathbb{D}$.

Proof. (i). Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a polynomial so that $a_n = 0$ for $n > N$. It is clear that in this case $A_\mu(f)$ is a polynomial of degree at most N. Let ψ_t is given by (2.4). Then clearly the integral (2.7) is finite for each $z \in \mathbb{D}$ and we have

$$\begin{aligned}
\int_0^1 f(\psi_t(z))\, d\mu(t) &= \sum_{n=0}^{\infty} a_n \int_0^1 (tz + 1 - t)^n\, d\mu(t) \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{n} \binom{n}{k} \int_0^1 t^k (1-t)^{n-k}\, d\mu(t) z^k \right) \\
&= \sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{n} c_{n,k} z^k \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} c_{n,k} a_n \right) z^k \\
&= A_\mu(f)(z),
\end{aligned}$$

where the interchange of sums and integrals is justified because all sums are finite.

(ii). Let $f \in H^p$. Applying (1.4) we find

$$|f(\psi_t(z))| \leq \frac{c_p \|f\|_p}{(1 - |tz + 1 - t|)^{1/p}} \leq \frac{c_p \|f\|_p}{t^{1/p}(1 - |z|)^{1/p}},$$

for each $z \in \mathbb{D}$ and $0 < t \leq 1$. From the hypothesis it follows that the integral

$$G(z) = \int_0^1 f(\psi_t(z))\, d\mu(t)$$

is finite for each $z \in \mathbb{D}$ and defines G as an analytic function on $\mathbb{D}$. □

Using the above Lemma we can define A_μ on analytic functions $f \in H^p$ by the integral, whenever μ satisfies the hypothesis of the Lemma. The resulting function $A_\mu(f)$ is analytic on the disc and it makes sense examine if A_μ is a bounded operator on H^p.

Remarks
(**1**) Suppose μ is a measure satisfying

$$\int_0^1 \frac{d\mu(t)}{t} < \infty. \tag{2.8}$$

Then the series (1.3) does define an analytic function on $\mathbb{D}$ for every $f \in H^1$. This is because

$$\begin{aligned} |B_k| \le \sum_{n=k}^{\infty} c_{n,k}|a_n| \le M \sum_{n=k}^{\infty} c_{n,k} &= M \sum_{n=k}^{\infty} \binom{n}{k} \int_0^1 t^k (1-t)^{n-k}\, d\mu(t) \\ =& M \int_0^1 t^k \sum_{n=k}^{\infty} \binom{n}{k} (1-t)^{n-k}\, d\mu(t) \\ =& M \int_0^1 t^k \frac{1}{(1-(1-t))^{k+1}}\, d\mu(t) \\ =& M \int_0^1 \frac{d\mu(t)}{t}, \end{aligned}$$

so that the coefficients B_k are finite and they form a bounded sequence.
(**2**) If $d\mu(t) = dt$ the Lebesgue measure, then (2.8) is not fulfilled. Nevertheless the series (1.3) converges and defines for each $f \in H^1$ an analytic function on $\mathbb{D}$. Indeed the elements $c_{n,k}$ are found to be $c_{n,k} = \frac{1}{n+1}$, $k = 0, 1, \cdots n$, (A_μ is the transpose of the Cesàro matrix). The coefficients B_k of (1.3) are given by

$$B_k = \sum_{n=k}^{\infty} \frac{a_n}{n+1}, \qquad k = 0, 1, \cdots.$$

Now Hardy's inequality for functions $f(z) = \sum_{n\ge 0} a_n z^n \in H^1$ says

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le \pi \|f\|_1, \tag{2.9}$$

see [DU, p. 48]. It follows that for each $f \in H^1$ the coefficients B_k are finite and they form a bounded sequence so that the series (1.3) converges on the disc.

Lemma 2.3. *Suppose* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q}$. *Then under the pairing (1.5) the following duality*

$$\langle w_t f \circ \phi_t,\ h \rangle = \langle f,\ h \circ \psi_t \rangle \tag{2.10}$$

holds for all $f \in H^p$ *and* $h \in H^q$.

Proof. Suppose $f \in H^p$. We can write f as a Cauchy integral

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - e^{-i\theta} z}\, d\theta.$$

Write $g(z) = T_t(f)(z)$ then

$$\begin{aligned} g(z) =& \frac{1}{(t-1)z+1} f\left(\frac{tz}{(t-1)z+1}\right) \\ =& \frac{1}{(t-1)z+1} \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \frac{e^{-i\theta}tz}{(t-1)z+1}}\, d\theta \\ =& \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1-(e^{-i\theta}t+1-t)z}\, d\theta \\ =& \sum_{n=0}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} (e^{-i\theta}t+1-t)^n f(e^{i\theta})\, d\theta \right) z^n, \end{aligned}$$

so that

$$\int_0^{2\pi} e^{-in\theta} g(e^{i\theta})\, d\theta = \int_0^{2\pi} (e^{-i\theta}t+1-t)^n f(e^{i\theta})\, d\theta, \qquad n = 0,1,2,... \tag{2.11}$$

or equivalently,

$$\langle g,\ e^{in\theta} \rangle = \langle f,\ (e^{i\theta}t+1-t)^n \rangle \qquad n = 0,1,2,...$$

Now the set $\{e^{in\theta} : n = 0,1,2,..\}$ spans H^q and clearly the same is true for the set $\{(e^{i\theta}t+1-t)^n : n = 0,1,2,...\}$. Taking linear combinations of elements of each set and then limits of such combinations in the H^q norm we conclude that for each $h \in H^q$ we have

$$\langle g,\ h \rangle = \langle f,\ h \circ \psi_t \rangle$$

where $\psi_t(z) = tz + 1 - t$, as desired. □

3. Hausdorff matrices on Hardy spaces

We start by finding estimates for the Hardy space norms of the composition operators.

Lemma 3.1. *For each $1 \le p < \infty$ the Hardy space norms of the composition operators $U_t(f)(z) = f(tz+1-t)$ are given by*

$$\|U_t\|_{H^p \to H^p} = \frac{1}{t^{1/p}}, \qquad 0 < t \le 1. \tag{3.1}$$

Proof. Suppose $1 \le p < \infty$ and let $\lambda \in \mathbb{C}$ with $\Re(\lambda) < 1/p$. The functions

$$f_\lambda(z) = \frac{1}{(1-z)^\lambda}$$

belong to H^p and we have

$$f_\lambda(\psi_t(z)) = \frac{1}{t^\lambda} f_\lambda(z),$$

so $t^{-\lambda}$ is an eigenvalue of U_t. It follows that $\|U_t\|_{H^p \to H^p} \ge t^{-1/p}$.

The opposite inequality can be seen as follows. Theorem 9.4 of [CM] implies that $\|U_t\|_{H^2\to H^2} = 1/\sqrt{t}$ so that the assertion of the lemma is valid for $p = 2$. Now for $f \in H^p$ with Blaschke factor $B(z)$ write $f(z) = B(z)F(z)$. Then

$$\begin{aligned}
\|U_t(f)\|_p^p &= \int_{\partial\mathbb{D}} |B(\psi_t(z))|^p |F(\psi_t(z))|^p \,|dz| \\
&\le \int_{\partial\mathbb{D}} |F(\psi_t(z))|^p \,|dz| \\
&= \int_{\partial\mathbb{D}} |F^{p/2}(\psi_t(z))|^2 \,|dz| \\
&\le \|U_t\|^2_{H^2\to H^2} \|F^{p/2}\|_2^2 \\
&= (1/t)\|F\|_p^p \\
&= (1/t)\|f\|_p^p,
\end{aligned}$$

which gives the opposite inequality and the proof is complete.

□

Lemma 3.2. *For the Hardy space norms of the weighted composition operators*

$$T_t(f)(z) = \frac{1}{(t-1)z+1} f\Big(\frac{tz}{(t-1)z+1}\Big)$$

we have:
(i) If $2 < p < \infty$ then

$$\|T_t\|_{H^p\to H^p} \le t^{-1+1/p}, \qquad 0 < t \le 1.$$

(ii) If $1 < p < 2$ then there is a constant C_p depending only on p such that

$$\|T_t\|_{H^p\to H^p} \le C_p t^{-1+1/p}, \qquad 0 < t \le 1.$$

(ii) If $p = 1$ then there is a constant C' such that

$$\|T_t\|_{H^1\to H^1} \le C'(1 + \log \frac{1}{t}), \qquad 0 < t \le 1.$$

Proof. For $1 \le p < \infty$ let $H^p(\mathbb{P})$ be the Hardy space of the right half plane $\mathbb{P} = \{z : \Re(z) > 0\}$, consisting of analytic functions $f : \mathbb{P} \to \mathbb{C}$ such that

$$\|f\|^p_{H^p(\mathbb{P})} = \sup_{0<x<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^p \,dy < \infty.$$

These are Banach spaces, isometric to the corresponding Hardy spaces of the disc through the linear map $V_p : H^p(\mathbb{P}) \to H^p$,

$$V_p(f)(z) = \frac{(4\pi)^{1/p}}{(1-z)^{2/p}} f(\mu(z)), \quad \text{where} \quad \mu(z) = \frac{1+z}{1-z}.$$

A calculation shows that the inverse of V_p is

$$V_p^{-1}(g)(z) = \frac{1}{\pi^{1/p}(1+z)^{2/p}} g(\mu^{-1}(z)), \quad g \in H^p.$$

Now let $\widetilde{T}_t : H^p(\mathbb{P}) \to H^p(\mathbb{P})$ be the operators defined by

$$\widetilde{T}_t = V_p^{-1} T_t V_p.$$

Because V_p and V_p^{-1} are isometries we have $\|T_t\|_{H^p \to H^p} = \|\widetilde{T}_t\|_{H^p(\mathbb{P}) \to H^p(\mathbb{P})}$. A calculation shows that for $f \in H^p(\mathbb{P})$,

$$\widetilde{T}_t(f)(z) = \left(\frac{z+1}{tz+2-t}\right)^{1-\frac{2}{p}} f(tz+1-t).$$

Further it is easy to see that

$$\left|\frac{z+1}{tz+2-t}\right| \leq \frac{1}{t} \tag{3.2}$$

for each $z \in \mathbb{P}$ and $t \in (0, 1]$.

(i) Suppose first $p \geq 2$ then $p - 2 \geq 0$ and using the last inequality we find

$$\begin{aligned}
\|\widetilde{T}_t(f)\|_{H^p(\mathbb{P})} &= \sup_{0<x<\infty} \left(\int_{-\infty}^{\infty} \left|\frac{z+1}{tz+2-t}\right|^{p-2} |f(t(x+iy)+1-t)|^p \, dy \right)^{1/p} \\
&\leq t^{-1+\frac{2}{p}} \sup_{0<x<\infty} \left(\int_{-\infty}^{\infty} |f(t(x+iy)+1-t)|^p \, dy \right)^{1/p} \\
&= t^{-1+\frac{2}{p}} \sup_{1-t<u<\infty} \left(\int_{-\infty}^{\infty} |f(u+iv)|^p \, \frac{dv}{t} \right)^{1/p},
\end{aligned}$$

where we have made the change of variables $u = tx + 1 - t$ and $v = ty$,

$$\begin{aligned}
&\leq t^{-1+\frac{1}{p}} \sup_{0<u<\infty} \left(\int_{-\infty}^{\infty} |f(u+iv)|^p \, dv \right)^{1/p} \\
&= t^{-1+\frac{1}{p}} \|f\|_{H^p(\mathbb{P})}
\end{aligned}$$

and the conclusion follows for $p \geq 2$.

(ii) Suppose next $1 < p < 2$, then the above estimates fail because $p - 2$ is negative and there is no suitable lower inequality to replace (3.2). We can however use duality. Let $f \in H^p$ and let q be the conjugate index. Recalling the representation of bounded linear functionals on H^p we have

$$\begin{aligned}
\|T_t(f)\|_p &= \sup\{|\Lambda(T_t(f))| : \Lambda \in (H^p)^*, \, \|\Lambda\| \leq 1\} \\
&= \sup\{|\langle T_t(f), \, g\rangle| : g \in H^q, \, \|\Lambda_g\| \leq 1\}.
\end{aligned}$$

Using (2.10) we find

$$|\langle T_t(f), \, g\rangle| = |\langle f, \, U_t(g)\rangle| \leq \|f\|_p \|U_t(g)\|_q \leq \|f\|_p \|g\|_q t^{1/q},$$

therefore,

$$\|T_t(f)\|_p \leq (\sup\{\|g\|_q : \|\Lambda_g\| \leq 1\}) \|f\|_p t^{1/q} = C_q t^{1-\frac{1}{p}} \|f\|_p$$

and this is the desired conclusion.

(iii) Suppose now $p = 1$. In this case we first show that if $f \in H^1$ with $f(0) = 0$ then $\|T_t(f)\|_1 \leq \|f\|_1$. Indeed we can write $f(z) = zg(z)$ with $g \in H^1$ and $\|f\|_1 = \|g\|_1$. Applying the operator T_t we find

$$T_t(f)(z) = \frac{tz}{((t-1)z+1)^2} g(\frac{tz}{(t-1)z+1}).$$

The weighted composition operators

$$S_t(g)(z) = \frac{1}{((t-1)z+1)^2} g(\frac{tz}{(t-1)z+1}),$$

are clearly bounded on H^1 and $T_t(f)(z) = tzS_t(g)(z)$. Thus

$$\|T_t(f)\|_1 = t\|S_t(g)\|_1 \leq t\|S_t\|_{H^1 \to H^1} \|g\|_1 = t\|S_t\|_{H^1 \to H^1} \|f\|_1.$$

We can now repeat the method of part (i) of the proof to estimate $\|S_t\|_{H^1 \to H^1}$. Thus let $\widetilde{S}_t = V_1^{-1} S_t V_1$, then $\|S_t\|_{H^1 \to H^1} = \|\widetilde{S}_t\|_{H^1(\mathbb{P}) \to H^1(\mathbb{P})}$. A calculation shows that if $h \in H^1(\mathbb{P})$ then

$$\widetilde{S}_t(h)(z) = h(tz + 1 - t).$$

As in the case (i) we integrate for the norm to obtain

$$\|\widetilde{S}_t(h)\|_{H^1(\mathbb{P})} \leq \frac{1}{t} \|h\|_{H^1(\mathbb{P})}.$$

It follows that $\|S_t\|_{H^1 \to H^1} \leq 1/t$ and we conclude $\|T_t(f)\|_1 \leq \|f\|_1$ whenever $f \in H^1$ and $f(0) = 0$.

Next let $F \in H^1$ be arbitrary. Write $F(z) = F(0) + f(z)$ where $f(0) = 0$ and $\|f\|_1 = \|F - F(0)\|_1 \leq \|F\|_1 + |F(0)| \leq 2\|F\|_1$. We then have

$$T_t(F)(z) = T_t(F(0) + f(z)) = F(0)\frac{1}{(t-1)z+1} + T_t(f)(z)$$

so that

$$\begin{aligned}
\|T_t(F)\|_1 &\leq |F(0)| \|\frac{1}{(t-1)z+1}\|_1 + \|T_t(f)\|_1 \\
&\leq \|F\|_1 \|\frac{1}{(t-1)z+1}\|_1 + \|f\|_1 \\
&\leq \|F\|_1 \|\frac{1}{(t-1)z+1}\|_1 + 2\|F\|_1 \\
&= \left(2 + \|\frac{1}{(t-1)z+1}\|_1\right) \|F\|_1.
\end{aligned}$$

Now we need the well known inequality

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|} \, d\theta \leq C \log \frac{1}{1-r}, \qquad 0 < r < 1,$$

where C is a constant. This can be found for example in [PO] where it is shown that, as $r \to 1$, the integral is asymptotically equivalent to $\frac{1}{\pi} \log \frac{1}{1-r}$.

The inequality can also be derived by an elementary estimate of the integral upon making first the change of variables $s = \tan\theta$. We omit the details. Applying this inequality to the function $\frac{1}{1-(1-t)z}$ we find

$$\|\frac{1}{(t-1)z+1}\|_1 \le C \log\frac{1}{t}, \qquad 0 < t < 1,$$

and this gives

$$\begin{aligned}\|T_t(F)\|_1 &\le (2 + C\log\frac{1}{t})\|F\|_1 \\ &\le C'(1+\log\frac{1}{t})\|F\|_1,\end{aligned}$$

where $C' = \max(2, C)$. This finishes the proof.

□

Using the above norm estimates we can now prove Theorems 1.1 and 1.2.

3.1. **Proof of Theorem 1.1.** (i) Suppose $1 < p < \infty$ and let $f \in H^p$. From Lemma (2.1) the power series (1.2) represents an analytic function on $\mathbb{D}$ and for its Hardy space norm we have

$$\begin{aligned}\|H_\mu(f)\|_p =& \sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|H_\mu(f)(re^{i\theta})|^p\,d\theta\right)^{1/p} \\ =& \sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 T_t(f)(re^{i\theta})\,d\mu(t)\right|^p\,d\theta\right)^{1/p} \\ \le& \int_0^1\left(\sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}\left|T_t(f)(re^{i\theta})\right|^p\,d\theta\right)^{1/p}\,d\mu(t),\end{aligned}$$

where we have used Minkowski's integral inequality before putting the sup inside the integral. Continuing,

$$\begin{aligned}=& \int_0^1 \|T_t(f)\|_p\,d\mu(t) \\ \le& C\int_0^1 t^{1-\frac{1}{p}}\,d\mu(t)\|f\|_p,\end{aligned}$$

with the constant $C = 1$ when $p \ge 2$ and $C = C_p$ (the constant of Lemma 3.2 (ii)) when $1 < p < 2$.

(ii) Suppose $p = 1$ and $f \in H^1$. Lemma (2.1) says again that the power series $H_\mu(f)$ represents an analytic function on $\mathbb{D}$, and

$$\begin{aligned}
\|H_\mu(f)\|_1 =& \sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|H_\mu(f)(re^{i\theta})|\,d\theta\right)\\
=& \sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 T_t(f)(re^{i\theta})\,d\mu(t)\right|\,d\theta\right)\\
\leq& \int_0^1\left(\sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}\left|T_t(f)(re^{i\theta})\right|\,d\theta\right)\,d\mu(t),\\
=& \int_0^1 \|T_t(f)\|_1\,d\mu(t)\\
\leq& C'\int_0^1(1+\log\frac{1}{t})\,d\mu(t)\|f\|_1\\
=& C'\left(\mu((0,1])+\int_0^1\log\frac{1}{t}\,d\mu(t)\right)\|f\|_1.
\end{aligned}$$

Thus H_μ is bounded on H^1 whenever $\int_0^1 \log(\frac{1}{t})\,d\mu(t) < \infty$.

Conversely suppose H_μ is bounded on H^1, then the image $H_\mu(1)$ of the constant function 1 is in H^1. The power series of $H_\mu(1)(z)$ is

$$\begin{aligned}
H_\mu(1)(z) &= \int_0^1 \frac{1}{(t-1)z+1}\,d\mu(t)\\
&= \sum_{n=0}^{\infty}\left(\int_0^1(1-t)^n\,d\mu(t)\right)z^n.
\end{aligned}$$

Now apply Hardy's inequality (2.9) in the form $\sum_{n\geq 1}\frac{|a_n|}{n} \leq 2\pi\|f\|_1$ to obtain

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1(1-t)^n\,d\mu(t) \leq 2\pi\|H_\mu(1)\|_1 \leq 2\pi\|H_\mu\|_{H^1\to H^1}.$$

Putting the sum inside the integral we have

$$\int_0^1 \log\frac{1}{t}\,d\mu(t) \leq 2\pi\|H_\mu\|_{H^1\to H^1}$$

and the proof is complete.

3.2. **Proof of Theorem 1.2.** Suppose $1 \leq p < \infty$. If $\int_0^1 t^{-1/p}\,d\mu(t) < \infty$ then by Lemma 2.2, for each $f \in H^p$ the analytic function $A_\mu(f)$ is well defined in the disc and for its Hardy space norm we have

$$\begin{aligned}\|A_\mu(f)\|_p &= \sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|A_\mu(f)(re^{i\theta})|^p\,d\theta\right)^{1/p}\\ &= \sup_{r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 U_t(f)(re^{i\theta})\,d\mu(t)\right|^p\,d\theta\right)^{1/p}\\ &\le \int_0^1\left(\sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}\left|U_t(f)(re^{i\theta})\right|^p\,d\theta\right)^{1/p}\,d\mu(t)\\ &= \int_0^1 \|U_t(f)\|_p\,d\mu(t)\\ &\le \int_0^1 \frac{1}{t^{1/p}}\,d\mu(t)\|f\|_p.\end{aligned}$$

This shows that A_μ is bounded on H^p and gives the inequality $\|A_\mu\|_{H^p\to H^p} \le \int_0^1 t^{-1/p}\,d\mu(t)$ for the norm. To show the opposite inequality recall that for each $\lambda \in \mathbb{C}$ with $\Re(\lambda) < 1/p$ the functions $f_\lambda(z) = \frac{1}{(1-z)^\lambda}$ belong to H^p. Applying A_μ to these functions we find

$$\begin{aligned}A_\mu(f_\lambda)(z) &= \int_0^1 f_\lambda(tz+1-t)\,d\mu(t)\\ &= \int_0^1 \frac{1}{t^\lambda}\,d\mu(t)f_\lambda(z).\end{aligned}$$

This says f_λ is an eigenfunction of A_μ corresponding to the eigenvalue $\int_0^1 t^{-\lambda}\,d\mu(t)$, hence the point spectrum of A_μ contains the set

$$\left\{\int_0^1 t^{-\lambda}\,d\mu(t) : \Re(\lambda) < 1/p\right\}.$$

Thus $\|A_\mu\|_{H^p\to H^p} \ge \int_0^1 t^{-1/p}\,d\mu(t)$ and the proof is complete.

3.3. **Some remarks.** Theorem 1.1 covers the case of the generalized Cesàro operators $\mathcal{C}^\alpha$ which were studied on Hardy and other spaces by different methods in [ST], [AN] and in [XI]. In our approach these operators arise from the measures

$$d\mu(t) = (1-t)^\alpha dt, \qquad \Re(\alpha) > -1,$$

and Theorem 1.1 says that $\mathcal{C}^\alpha$ are bounded on H^p for $p \ge 1$. On the other hand, in the above mentioned works it is shown that $\mathcal{C}^\alpha$ are in fact bounded on H^p for all $0 < p < \infty$. This leads to the question of finding conditions on μ that imply the boundedness of H_μ on H^p for $0 < p < 1$. Because these spaces are not Banach spaces our method will not apply directly, since for example the proof of Theorem 1.1 is valid only when the norm is a Banach space norm.

After this paper was accepted we have learned of the paper [RU] of O. Rudolf, where he has obtained results which partly overlap with those of our Theorem 1.1 (i) and Theorem 1.2. More precisely he obtains part (i) of Theorem 1.1 for $2 \leq p < \infty$. However in the range $1 \leq p < 2$ he gives a sufficient condition for boundedness of H_μ which is neither natural nor optimal, and does not distinguish the case $p = 1$, in which as our Theorem 1.1 shows the integrability of $\log(1/t)$ characterizes boundedness. He also obtains Theorem 1.2 except for $p = 1$. His work includes further studies for Hausdorff matrices on Bergman spaces and examination of their spectra, topics that we have not considered here.

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