CAUCHY TRANSFORMS AND CESÀRO AVERAGING OPERATORS

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Abstract. We show that the Cesàro operator is bounded on the space of Cauchy transforms.

1. Introduction

Cauchy transforms. An analytic function f on the unit disc $\mathbf{D}$ is a Cauchy transform if it admits a representation

$$f(z) = \int_0^{2\pi} \frac{1}{1 - e^{i\theta} z}\, d\mu(\theta), \qquad z \in \mathbf{D}, \tag{1.1}$$

where $\mu \in \mathbb{M}$, the space of all finite complex valued Borel measures on the unit circle $\mathbf{T} \equiv [0, 2\pi]$. $\mathbb{M}$ is a Banach space under the total variation norm $\|\mu\|$. The space K of all Cauchy transforms is a Banach space under the norm

$$\|f\|_K = \inf\{\|\mu\| : \mu \in \mathbb{M} \text{ and } (1.1) \text{ holds}\}. \tag{1.2}$$

The representation (1.1) is unique up to measures of vanishing Cauchy transform; If $\nu \in \mathbb{M}$ is such that

$$\int_0^{2\pi} e^{in\theta}\, d\nu(\theta) = 0, \qquad n = 0, 1, \cdots, \tag{1.3}$$

then μ and $\mu + \nu$ represent the same function. By the F. and M. Riesz theorem these measures ν have the form $d\nu = g \frac{d\theta}{2\pi}$ for some $g \in H^1_0$, the subspace of the Hardy space H^1 consisting of functions that vanish at 0. Thus K is isometrically isomorphic to the quotient space $\mathbb{M}/H^1_0$.

Each function in H^1 is the Cauchy integral of its boundary values [D], thus $H^1 \subset K$. On the other hand for each θ the kernels $\frac{1}{1-e^{i\theta}z}$ are in the Hardy space H^p for all $p < 1$ and it is clear that $K \subset \bigcap_{p<1} H^p$.

Next K is a dual space. Let $C(\mathbf{T})$ be the space of continuous functions on $\mathbf{T}$ with the sup norm. By the Riesz representation theorem the pairing

$$\langle \mu, f \rangle = \int_0^{2\pi} f(e^{i\theta})\, d\mu(\theta), \quad f \in C(\mathbf{T}),\ \mu \in \mathbb{M},$$

establishes an isometric isomorphism $\mathbb{M} = C(\mathbf{T})^*$, and in particular the norm of a measure μ considered as a linear functional on $C(\mathbf{T})$ is equal to its total variation $\|\mu\|$. Let $\mathbf{A}$ be the disc algebra of functions continuous on the closed disc and analytic in the interior. $\mathbf{A}$ is identified with the subspace of those functions

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$f \in C(\mathbf{T})$ for which the negative Fourier coefficients vanish: $\int_0^{2\pi} f(e^{i\theta})e^{in\theta}\,d\theta = 0$ for $n = 1, 2, \cdots$. A linear functional $\nu \in C(\mathbf{T})^*$ annihilates all functions in $\mathbf{A}$ if and only if (1.3) holds. It follows that the dual space $\mathbf{A}^*$ is isometrically isomorphic to the space K of Cauchy transforms.

For $f \in K$ the integral (1.1) may be written

$$f(z) = \langle \mu, \kappa_z \rangle, \qquad z \in \mathbf{D},$$

where κ_z are the kernels $\kappa_z(e^{i\theta}) = \frac{1}{1-e^{i\theta}z}$. Further

$$\begin{aligned} \|f\|_K &= \inf\{\|\mu + \nu\| : \nu = g\frac{d\theta}{2\pi},\ g \in H_0^1\} \\ &= \sup\{|\langle \mu, h\rangle| : h \in \mathbf{A},\ \|h\|_\infty \le 1\} \qquad \text{(by duality)} \\ &= \|\mu|_{\mathbf{A}}\| \end{aligned}$$

where $\mu|_{\mathbf{A}}$ is restriction of μ on $\mathbf{A}$.

Next we see the inf in (1.2) is attained, that is for each $f \in K$ there is a μ representing f such that $\|f\|_K = \|\mu|_{\mathbf{A}}\| = \|\mu\|$. Indeed let μ_n, $n = 1, 2, \cdots$ be such that each μ_n represents f and $\|\mu_n\| \to \|f\|_K$. Because the sequence is bounded it has a weak* accumulation point $\mu = \lim_{k\to\infty} \mu_{n_k}$. Then

$$\langle \mu, \kappa_z \rangle = \lim_{k\to\infty} \langle \mu_{n_k}, \kappa_z \rangle = f(z), \qquad z \in \mathbf{D},$$

so μ represents f and for any $h \in C(\mathbf{T})$ we have

$$|\langle \mu, h\rangle| = \lim_{k\to\infty} |\langle \mu_{n_k}, h\rangle| \le \limsup_{k\to\infty} \|\mu_{n_k}\|\|h\|_\infty = \|f\|_K \|h\|_\infty$$

thus $\|\mu\| = \|f\|_K$.

More general spaces of Cauchy transforms have been studied in the literature [MG], see also [HN]. For each $\gamma > 0$ let K_γ be the space of all analytic functions f on the disc that are representable as

$$f(z) = \int_0^{2\pi} \frac{1}{(1 - e^{i\theta}z)^\gamma}\, d\mu(\theta), \qquad z \in \mathbf{D}, \tag{1.4}$$

for some $\mu \in \mathbb{M}$. These are Banach spaces with norm defined in analogy with (1.2) and have properties similar to those of $K = K_1$. They are nested by the inclusions $K_\beta \subset K_\gamma$ for $\beta < \gamma$ and are connected among themselves by the following two properties, see [MG] for more details and proofs.

(P1) If $f \in K_\beta$ and $g \in K_\gamma$ then $fg \in K_{\beta+\gamma}$, for all β, $\gamma > 0$.
(P2) $f \in K_\gamma$ if and only if $f' \in K_{\gamma+1}$, for all $\gamma > 0$.

We will not use these spaces except in the concluding remarks and the above two properties are the only ones we need.

The Cesàro operator. For a function $f(z) = \sum_{n\ge 0} a_n z^n$ analytic on $\mathbf{D}$ the Cesàro transformation of f is

$$\mathcal{C}(f)(z) = \sum_{n\ge 0} \left(\frac{a_0 + a_1 \cdots a_n}{n+1} \right) z^n.$$

It is well known that $\mathcal{C}$ acts as a bounded linear operator on various spaces of analytic functions (see [A], [M], [S1], [S2], [X]) including the Hardy and Bergman spaces. In particular $\mathcal{C}(H^p) \subset H^p$ for each $p \in (0, \infty)$. Because $H^1 \subset K \subset \cap_{p<1} H^p$ the question arises whether $\mathcal{C}$ is also bounded on K. The purpose of this note is to show that $\mathcal{C}$, and its generalized versions $\mathcal{C}^\alpha$ defined below, are bounded on K.

It is possible to give a quick proof of this fact for $\mathcal{C}$ by appealing to the generalized spaces K_γ and their properties. The argument is presented in the remarks. That proof however does not apply to the generalized Cesàro operators $\mathcal{C}^\alpha$. Our proof below relies on the duality $K = \mathbf{A}^*$

A simple calculation shows that we can write

$$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\,\frac{1}{1-\zeta}\,d\zeta,$$

and if we choose the integration path to be the curve

$$\zeta(t) = \zeta(t,z) = \frac{tz}{(t-1)z+1}, \qquad 0 \le t \le 1,$$

we find

$$\mathcal{C}(f)(z) = \int_0^1 \frac{1}{(t-1)z+1} f\left(\frac{tz}{(t-1)z+1}\right) dt. \tag{1.5}$$

This representation of $\mathcal{C}$ is valid for all functions f analytic on $\mathbf{D}$. Next observe that for each $t \in [0,1]$ the functions

$$\phi_t(z) = \frac{tz}{(t-1)z+1}$$

map the disc into itself and we can consider the weighted composition operators

$$T_t(f)(z) = \frac{1}{(t-1)z+1}\; f(\phi_t(z)). \tag{1.6}$$

With this notation (1.5) says that $\mathcal{C}$ is the average of the weighted composition operators T_t:

$$\mathcal{C}(f)(z) = \int_0^1 T_t(f)(z)\,dt.$$

2. Proof of boundedness

Lemma 2.1. *For each $t \in [0,1]$ the weighted composition operators T_t are bounded on K and $\|T_t\|_{K\to K} = 1$.*

Proof. We first verify the assertion for the endpoints. If $t = 1$, T_1 is the identity operator. If $t = 0$ then $T_0(f)(z) = f(0)\frac{1}{1-z}$ for each $f \in K$. Choose a representing measure μ for f such that $\|f\|_K = \|\mu\|$, then

$$|f(0)| = \Big|\int_{\mathbf{T}} d\mu(\theta)\Big| \le \int_{\mathbf{T}} d|\mu|(\theta) = \|f\|_K$$

and because $\|\frac{1}{1-z}\|_K = 1$ we conclude $\|T_0\| \le 1$. To obtain the equality observe that the constant function 1 has $\|1\|_K = 1$ and $T_0(1)(z) = \frac{1}{1-z}$.

Now let $0 < t < 1$ and write

$$\begin{aligned} T_t(f)(z) &= \frac{1}{tz}\phi_t(z)f(\phi_t(z)) \\ &= \frac{1}{tz}(C_t \circ M_z)(f)(z) \end{aligned}$$

where C_t is the composition operator induced by ϕ_t and M_z is multiplication by z. Both are bounded on K (for the boundedness of composition operators on K see [BC]) and further, if a function $F \in K$ vanishes at 0 then $F(z)/z \in K$. It follows that T_t maps K into itself.

We next show that $\|T_t\| \leq 1$. Let $f \in K$ be the Cauchy integral of μ with $\|f\|_K = \|\mu\| = \|\mu|_{\mathbf{A}}\|$ and let $\nu_t \in \mathbb{M}$ be a representing measure for $T_t(f)$, then

$$\begin{aligned}\int_0^{2\pi} \frac{1}{1-e^{i\theta}z}\, d\nu_t(\theta) &= \frac{1}{(t-1)z+1} f(\phi_t(z)) \\ &= \int_0^{2\pi} \frac{1}{(t-1)z+1} \frac{1}{1-e^{i\theta}\phi_t(z)}\, d\mu(\theta) \\ &= \int_0^{2\pi} \frac{1}{1-(te^{i\theta}+1-t)z}\, d\mu(\theta).\end{aligned}$$

It follows that ν_t is a representing measure for $T_t(f)$ if and only if

$$\int_0^{2\pi} e^{in\theta}\, d\nu_t(\theta) = \int_0^{2\pi} (te^{i\theta}+1-t)^n\, d\mu(\theta) \qquad \text{for all } n = 0,1,2,\cdots.$$

Now the linear span of the set $\{e^{in\theta} : n = 0,1,2,\cdots\}$ coincides with the linear span of $\{(te^{i\theta}+1-t)^n : n = 0,1,2,\cdots\}$ and the closure of each in the sup norm is the disc algebra $\mathbf{A}$. Taking linear combinations and then limits of such linear combinations we see that for each $h \in \mathbf{A}$,

$$\int_0^{2\pi} h(e^{i\theta})\, d\nu_t(\theta) = \int_0^{2\pi} h(te^{i\theta}+1-t)\, d\mu(\theta). \tag{2.1}$$

Next let $S_t(h) = h \circ \psi_t$ be the composition operator induced by $\psi_t(z) = tz+1-t$ on $\mathbf{A}$. Clearly $\|S_t\|_{\mathbf{A}\to\mathbf{A}} = 1$ and (2.1) says that

$$\langle \nu_t, h\rangle = \langle \mu, S_t(h)\rangle, \qquad h \in \mathbf{A}.$$

We therefore have

$$\begin{aligned}\|T_t(f)\|_K &= \|\nu_t|_{\mathbf{A}}\| \\ &= \sup\{|\langle \nu_t, h\rangle| : h \in \mathbf{A},\ \|h\|_\infty \leq 1\} \\ &= \sup\{|\langle \mu, S_t(h)\rangle| : h \in \mathbf{A},\ \|h\|_\infty \leq 1\} \\ &\leq \|\mu|_{\mathbf{A}}\| \|S_t\|_{\mathbf{A}\to\mathbf{A}} \\ &= \|f\|_K,\end{aligned}$$

and we conclude $\|T_t\| \leq 1$. To obtain the equality notice that the function $f(z) = \frac{1}{1-z}$ is an eigenfunction of T_t corresponding to the eigenvalue 1. The proof is complete. □

Theorem 2.2. *The Cesàro operator is bounded on K and $\|\mathcal{C}\|_{K\to K} = 1$.*

Proof. Pick $f \in K$ with $\|f\|_K \leq 1$. For each $z \in \mathbf{D}$ we can write the integral in (1.5) as a limit of Riemann sums

$$\mathcal{C}(f)(z) = \lim_{n\to\infty} R_n(f)(z),$$

where

$$R_n(f)(z) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(t_k-1)z+1} f(\phi_{t_k}(z)),$$

and $t_k = \frac{k}{n}$, $k = 1,2,\cdots,n$. Let $\mu_n \in \mathbb{M}$ be such that μ_n represents $R_n(f)$ and $\|\mu_n\| = \|R_n(f)\|_K$. From the lemma we have $\|\mu_n\| \leq 1$ and because the unit ball of

$\mathbb{M}$ is weak* compact there is a subsequence $\{\mu_{n_m}\}$ and a $\mu \in \mathbb{M}$ such that $\mu_{n_m} \to \mu$ in the weak* topology. Then

$$\begin{aligned}\mathcal{C}(f)(z) &= \lim_{m\to\infty} R_{n_m}(f)(z) = \lim_{m\to\infty} \langle \mu_{n_m}, \kappa_z\rangle = \langle \mu, \kappa_z\rangle \\ &= \int_0^{2\pi} \frac{1}{1-e^{i\theta}z}\, d\mu(\theta)\end{aligned}$$

thus $\mathcal{C}(f) \in K$. Further since $\|\mu_n\| \le 1$ and the set $\{k_z : z \in \mathbf{D}\}$ has dense linear span in $\mathbf{A}$ the uniform boundedness principle applies and gives $\|\mu\| = \|\mathcal{C}(f)\|_K \le 1$. It follows that $\|\mathcal{C}\| \le 1$. Finally $\frac{1}{1-z}$ is an eigenfunction of $\mathcal{C}$ corresponding to the eigenvalue 1 thus $\|\mathcal{C}\| = 1$.

□

3. Generalized Cesàro operators

For each complex α with $\Re(\alpha) > -1$ and k a nonnegative integer let A_k^α be defined as the kth coefficient in the expansion

$$\frac{1}{(1-x)^{\alpha+1}} = \sum_{k=0}^\infty A_k^\alpha x^k,$$

so that

$$A_k^\alpha = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\,\Gamma(\alpha+1)} = \frac{(\alpha+1)\cdots(\alpha+k)}{k!}.$$

The generalized Cesàro transformation $\mathcal{C}^\alpha$, defined on analytic functions $f(z) = \sum_{n\ge 0} a_n z^n$, is

$$\mathcal{C}^\alpha(f)(z) = \sum_{n=0}^\infty \left(\frac{1}{A_n^{\alpha+1}} \sum_{k=0}^n A_{n-k}^\alpha a_k \right) z^n. \tag{3.1}$$

These operators were introduced in [ST] on Hardy spaces and have been subsequently studied and proved bounded on all Hardy spaces in [A] and [X]. Because of the identity $\frac{1}{A_n^{\alpha+1}} \sum_{k=0}^n A_k^\alpha = 1$, we can view $\mathcal{C}^\alpha$ as a weighted versions of $\mathcal{C} = \mathcal{C}^0$ for the specific sequence of weights A_0^α, A_1^α, $A_2^\alpha, \cdots$.

Our proof above applies to show that all $\mathcal{C}^\alpha$, $\Re(\alpha) > -1$, are bounded operators on K. Indeed the integral form of $\mathcal{C}^\alpha$ is (see [ST])

$$\mathcal{C}^\alpha(f)(z) = \frac{\alpha+1}{z^{\alpha+1}} \int_0^z f(\zeta) \frac{(z-\zeta)^\alpha}{(1-\zeta)^{\alpha+1}}\, d\zeta,$$

and integrating on the same path $\zeta(t) = \frac{tz}{(t-1)z+1}$ we find

$$\mathcal{C}^\alpha(f)(z) = (\alpha+1) \int_0^1 \frac{1}{(t-1)z+1} f\left(\frac{tz}{(t-1)z+1}\right) (1-t)^\alpha\, dt,$$

which expresses $\mathcal{C}^\alpha$ as a weighted average of the same weighted composition operators T_t:

$$\mathcal{C}^\alpha(f)(z) = (\alpha+1) \int_0^1 T_t(f)(z)(1-t)^\alpha\, dt.$$

The argument in the proof of Theorem 2.2 applies with these changes: The Riemann sums that approximate $\mathcal{C}^\alpha(f)(z)$ now are

$$R_n^\alpha(f)(z) = (\alpha+1)\frac{1}{n} \sum_{k=1}^n \frac{1}{(t_k-1)z+1} f(\phi_{t_k}(z))(1-t_k)^\alpha,$$

and if $\varepsilon > 0$ is given we have

$$\begin{aligned}\|R_n^\alpha(f)\|_K &\le |\alpha+1|\frac{1}{n}\sum_{k=1}^{n}(1-t_k)^{\Re(\alpha)}\\ &\le |\alpha+1|\int_0^1(1-t)^{\Re(\alpha)}\,dt+\varepsilon\\ &= \frac{|\alpha+1|}{\Re(\alpha)+1}+\varepsilon\end{aligned}$$

for all large n. The rest applies without change and we obtain

Corollary 3.1. *For each complex number α with $\Re(\alpha) > -1$ the weighted Cesàro operator $\mathcal{C}^\alpha$ is bounded on K and $\|\mathcal{C}^\alpha\|_{K\to K} \le \frac{|\alpha+1|}{\Re(\alpha)+1}$.*

4. Concluding remarks

Remark 1. The following proof of boundedness is conceptually similar to the one given in Theorem (2.2). First identify preadjoint of $\mathcal{C}$ on the disc algebra. A computation shows that the duality $\mathbf{A}^* = K$ may be realized by the integral pairing

$$\langle f,g\rangle_0 = \lim_{r\to 1}\int_0^{2\pi} g(e^{i\theta})f(re^{-i\theta})\,\frac{d\theta}{2\pi}, \qquad g\in\mathbf{A},\quad f\in K.$$

and a further computation gives

$$\langle \mathcal{C}f,g\rangle_0 = \langle f,\mathcal{A}g\rangle_0,$$

where $\mathcal{A}$ is the operator acting on functions $g(z)=\sum_{n\ge 0}a_nz^n\in\mathbf{A}$ by

$$\mathcal{A}(g)(z)=\int_0^1 g(tz+1-t)\,dt=\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k+1}\right)z^n.$$

From the integral expression it is clear that $\mathcal{A}$ is a bounded operator on $\mathbf{A}$ and the duality $\mathcal{C}=\mathcal{A}^*$ implies that $\mathcal{C}$ is bounded on K.

Further it is easy to see that each $f_\lambda(z)=(1-z)^\lambda$, $\Re(\lambda)>0$, is an eigenfunction of $\mathcal{A}$ corresponding to the eigenvalue $\frac{1}{\lambda+1}$. It follows that the disc $\{z : |z-\frac{1}{2}|\le\frac{1}{2}\}$ is contained in the spectrum of $\mathcal{A}$ and each interior point is an eigenvalue. With some additional computation which we omit it is possible to show that in fact the spectrum of $\mathcal{A}$ on the disc algebra is this closed disc. By duality then the spectrum of $\mathcal{C}$ on K is this same disc.

Remark 2. We can obtain a short proof of the boundedness of $\mathcal{C}$ by using the properties (P1) and (P2) of the generalized spaces K_γ of Cauchy transforms defined in the introduction, as follows. Let $f\in K$. Then from (P1) we have $f(z)\frac{1}{1-z}\in K_2$ because $\frac{1}{1-z}\in K$ and from (P2) we find that the function

$$F(z)=\int_0^z f(\zeta)\frac{1}{1-\zeta}\,d\zeta,$$

is in K. Further $F(0)=0$ hence $F(z)/z\in K$. Thus $f\in K$ implies $\mathcal{C}(f)\in K$. The boundedness follows from the closed graph theorem. As remarked earlier this proof does not apply to $\mathcal{C}^\alpha$ for $\alpha\ne 0$. On the other hand this arguments shows that $\mathcal{C}$ is a bounded operator on K_γ for each $\gamma\ge 1$.

Remark 3. Speaking of the space K is equivalent to speaking about one-sided sequences of Fourier–Stieltjes coefficients. Averages of Fourier–Stieltjes cosine and sine coefficients of measures were studied in [G], where it is shown, using different techniques, that sequences of these averages are again Fourier–Stieltjes sequences.

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