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On the 2/1 resonant planetary dynamics - Periodic orbits and dynamical stability

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ABSTRACT

The 2/1 resonant dynamics of a two-planet planar system is studied within the framework of the three-body problem by computing families of periodic orbits and their linear stability. The continuation of resonant periodic orbits from the restricted to the general problem is studied in a systematic way. Starting from the Keplerian unperturbed system we obtain the resonant families of the circular restricted problem. Then we find all the families of the resonant elliptic restricted three body problem, which bifurcate from the circular model. All these families are continued to the general three body problem, and in this way we can obtain a global picture of all the families of periodic orbits of a two-planet resonant system. The parametric continuation, within the framework of the general problem, takes place by varying the planetary mass ratio ρ. We obtain bifurcations which are caused either due to collisions of the families in the space of initial conditions or due to the vanishing of bifurcation points. Our study refers to the whole range of planetary mass ratio values ($\rho \in (0, \infty)$) and, therefore we include the passage from external to internal resonances. Thus we can obtain all possible stable configurations in a systematic way. Finally, we study whether the dynamics of the four known planetary systems, whose currently observed periods show a 2/1 resonance, are associated with a stable periodic orbit.

Key words: celestial mechanics – planetary systems.

1 INTRODUCTION

A good model to study the motion of three celestial bodies considered as point masses is the famous *three-body problem* (TBP), whose study goes back to Poincaré. In the present study we consider a planetary system with two planets, namely the case where only one of the three bodies is the more massive one, and the other two bodies have much smaller masses.

The simplest model is the *circular restricted TBP*. Although much work has been carried out for periodic orbits of this model, (see e.g. Bruno, 1994; Hénon, 1997), new interesting results continue to appear in the literature (Maciejewski and Rybicki 2004; Papadakis and Goudas, 2006; Bruno and Varin 2006,2007). In this model we consider two bodies with non zero mass, called *primaries*, which are called Sun (S) and Jupiter (J) for convinience, moving in circular orbits around their common center of mass, and a third body with negligible mass, for example an asteroid, moving under the gravitational attraction of the two primaries. A more realistic model is the *elliptic restricted TBP*, where the two primaries move in elliptic orbits. However, the gravitational interaction between the small body and the two primaries is not taken into account in the restricted models. When we introduce this gravitational interaction, we have a more realistic model, the *general TBP*. Within this framework we can study a system consisting of a Sun and two small bodies (*planets*).

In the study of a dynamical system, the topology of its phase space plays a crucial role. The topology is determined by the position and the stability properties of the periodic orbits, or equivalently, of the fixed points of the Poincaré map on a surface of section. This makes clear the importance of the knowledge of the families of periodic orbits in a dynamical system. Particularly, in a planetary system, many families of periodic orbits are associated with resonances, which are mean motion resonances between the two planets. Since in our study of planetary systems only one body is the more massive one (the sun), a good method is to start from the simplest model, which is the circular restricted problem and find all the basic families of periodic orbits. Then we extend the model to the elliptic restricted model, and find all the families of resonant periodic orbits that bifurcate from the circular to the elliptic model. Finally, we give mass to the massless body and continue all these families to the model of the general problem. This is

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the method that we shall use in the present study in order to describe the dynamics of a resonant planetary system. The existence of periodic orbits of the planetary type in the general TBP, as a continuation from the restricted problem, has been studied by Hadjidemetriou (1975, 1976) and recently this method found a fruitful field of applicability in the dynamics of resonant extrasolar systems (e.g. Rivera and Lissauer,2001; Ji et al, 2003; Haghighipour et al., 2003; Ferraz-Mello et al., 2003; Psychoyos and Hadjidemetriou, 2005; Voyatzis and Hadjidemetriou, 2005; Voyatzis, 2008; Michtchenko et al, 2008). An alternative method for approximating resonant periodic motion is the computation of stationary solutions (corotations) of the averaged model (Beauge et al. 2003; Michtchenko at al. 2006).

Although the planetary TBP can show various modes of stable motion for small or moderate eccentricities, the stability domains that are associated with stable periodic orbits are of great importance in planetary dynamics because i) they offer a phase protection mechanism and long term stability even for large eccentricities and ii) stable periodic orbits may be traps for planetary systems after a migration process (Lee, 2004; Beauge et al, 2003, 2006). Since the planetary TBP shows large chaotic domains in phase space, planetary evolution depends significantly on the initial conditions. Therefore in order to reveal the dynamics of a particular observed planetary system we require reliable orbit determination and, inversely, the orbital fits of the observations should be valid only if they provide dynamically stable evolution (see e.g. Beauge, 2008).

The central target of the present paper is the description of a systematic way for obtaining the resonant periodic orbits of a planetary pair and its application to the 2/1 resonance. In the next section we discuss briefly known aspects on the families of periodic orbits in the circular unperturbed and in the restricted problem. These issues are fundamental for continuing our study in the elliptic restricted and in the general TBP problem. In section 3 we present the continuation of resonant periodic orbits from the circular to the elliptic model and then, in section 4, we consider the continuation in the general problem. In section 5 we study the bifurcation of families of periodic orbits within the framework of the general problem, as the mass ratio of the planets varies. In section 6 we present the chart of the 2:1 stable resonant periodic orbits and we discuss the possible association of the the dynamics of observed planetary systems with periodic orbits.

2 THE CIRCULAR RESTRICTED PROBLEM

Consider a body S (Sun) with mass m_0 and a second body J (Jupiter) with mass m_1, which describe circular orbits around their common center of mass. We define a rotating frame of reference xOy, whose x-axis is the line SJ, the origin is at their center of mass and the xy plane is the orbital plane of the circular motion of the these two bodies. The circular restricted problem describes the motion of a massless body A in the rotating frame, which moves under the gravitational attraction of S and J. In our computations we consider the normalization of units $m_0+m_1=1$, $G=1$ and $n'=1$, where G is the gravity constant and n' the mean motion of J, implying that the orbital radius of J is equal to 1.

2.1 The unperturbed case ($m_1=0$)

We assume that the body J has zero mass, $m_1=0$, constant radius $a'=1$ and rotates with constant angular velocity $n'=1$. Actually is used only to define the rotating frame xOy, whose origin is the body S. Evidently, the motion of the body A is a Keplerian orbit, presented in the rotating frame.

The Hamiltonian function H that describes the unperturbed motion of A, in polar coordinates, r, ϕ (in the rotating frame), is

$$H_0=\frac{p_r^2}{2}+\frac{p_\phi^2}{2r^2}-n'p_\phi-\frac{Gm_0}{r}.$$

The momenta are $p_r=\dot r$ and $p_\phi=r^2(\dot\phi+n')$. Note that the angle ϕ is an ignorable coordinate and consequently, in addition to the energy integral $H_0=h$ =constant, we also have the angular momentum integral p_ϕ =constant. In terms of the elements of the orbit, the Hamiltonian and the angular momentum are expressed as

$$H_0=-\frac{Gm_0}{2a}-n'p_\phi,\quad p_\phi=\sqrt{Gm_0a(1-e^2)}.$$

We shall consider two types of orbits of the body A: circular orbits and elliptic orbits. All the circular orbits are symmetric periodic in the rotating frame. The elliptic orbits are periodic in the rotating frame only if they are resonant, and may be symmetric or asymmetric. The initial conditions of a symmetric periodic orbit are the initial position x_0 and the initial velocity $\dot y_0$, perpendicular to the x-axis (the other two initial conditions are $y_0=0$, $\dot x_0=0$). So, a family of symmetric periodic orbits is represented by a smooth curve, called *characteristic curve*, in the space $x_0\dot y_0$. The energy h can be used instead of the velocity $\dot y_0$.

Circular orbits

In the rotating frame there exist circular orbits of the body A with an arbitrary radius r_0, which correspond to the periodic solution $r=r_0$, $p_r=0$, $\dot\phi=n-n'$, $p_\phi=nr_0^2$, where $n=p_\phi/r_0^2$ is the angular velocity of the circular orbit (in the inertial frame). A circular orbit in xOy is a Keplerian orbit in the inertial frame, with semi major axis $a=r_0$. Consequently, a *family of circular periodic orbits* exists along which the radius r_0 or the frequency n varies. The characteristic curve of the family is obtained from the energy integral (Hamiltonian H_0) for $e=0$:

$$-\frac{Gm_0}{2a}-n'\sqrt{Gm_0a}=h.$$

It is presented in Fig. 1a in the space $x-h$. In this case, x denotes the intersection of the circular orbit with the Ox axis ($x>0$) and its value coincides with the radious or, equivelantly, the semimajor axis a. The circular family is divided into two parts at the point $x=1$, where the orbit of the small body A coincides with the orbit of J. For $x<1$ we have the inner orbits and for $x>1$ we have the outer orbits.

Along the family of circular orbits the semimajor axis and consequently the resonance n/n' varies.

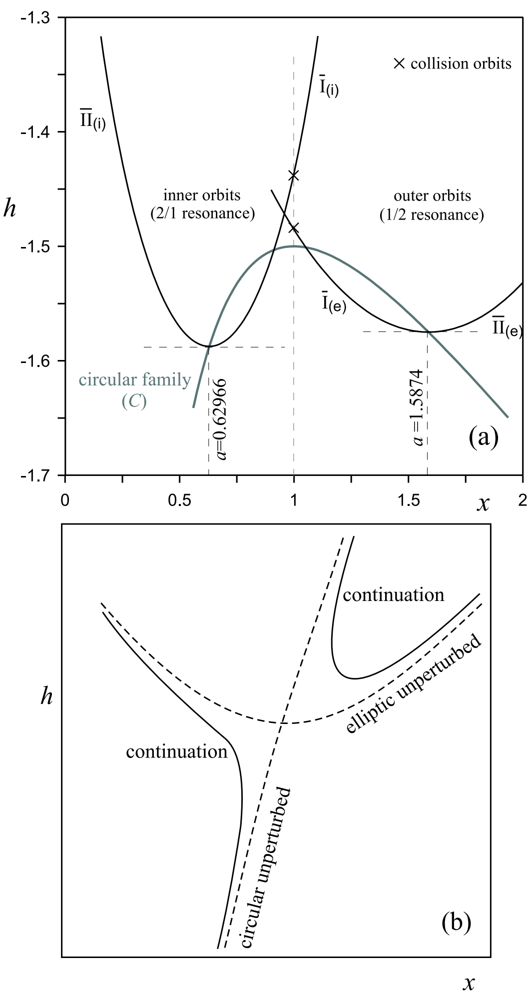

Figure 1. (a) The unperturbed case where Jupiter mass is zero ($m_1 = 0$). The family of circular periodic orbits and the the two resonant families at the 2/1 and 1/2 resonance that bifurcate from the circular family. (b) The bifurcation to the two 2/1 resonant elliptic families (inner resonance), when the mass of Jupiter is non zero. The same topology appears at the bifurcation point at the 1/2 resonance (outer resonance).

Elliptic orbits An elliptic orbit of the small body A in the inertial frame is periodic in the rotating frame only if it is resonant, i.e. $\frac{n}{n'} = \frac{p}{q}$ =rational. The corresponding semi major axis $a_{p/q}$ of the resonant orbit must satisfy the relation

$$\frac{(Gm_0)^{1/2} a_{p/q}^{-3/2}}{n'} = \frac{p}{q}.$$

The orbit is resonant periodic for *any* eccentricity e, so a *family of resonant elliptic periodic orbits* exists, with the eccentricity as a parameter along the family. This family bifurcates from the circular family at the point where the semi major axis corresponds to the resonance p/q. The resonance p/q is constant along the family. There is however another parameter, defining the *orientation* of the elliptic orbit, which is the angle ω of the line of apsides with a fixed direction (which we take to be the x-axis at $t = 0$). In general, an elliptic orbit is not symmetric with respect to the rotating x-axis, contrary to the circular orbits, which are all symmetric. Symmetry exists only when $\omega = 0$ or $\omega = \pi$.

Let us consider the symmetric elliptic periodic orbits at a fixed resonance. As in the circular case, an elliptic orbit intersects the Ox axis perpendicularly at a point $x > 0$. We distinguish two different cases when, at $t = 0$, the body crosses perpendicularly the x-axis: the body to be at perihelion ($\omega = 0$) or to aphelion ($\omega = \pi$). Consequently, we have two different families of resonant symmetric elliptic periodic orbits. These families are presented in the space $x - h$ by the energy integral

$$-\frac{Gm_0}{2a_{p/q}} - n' \sqrt{Gm_0 a_{p/q}(1 - e^2)} = h,$$

where $x = a_{p/q}(1 - e)$ is the perihelion ($e > 0$) or the aphelion ($e < 0$) distance.

The circular family and the 2/1 and 1/2 resonant elliptic families of symmetric periodic orbits of the unperturbed problem are presented in Fig. 1a. In the normalization we are using, the semi major axis along the 2/1 resonant family is $a_{2/1} = 0.6297$, called *internal resonance* and the semi-major axis along the 1/2 resonance is $a_{1/2} = 1.5874$, called *external resonance*. These families bifurcate from the points $x = 0.6297$ and $x = 1.5874$ of the circular family, respectively. At the bifurcation points, the tangent to the above resonant elliptic families is parallel to the x-axis.

Each family of the internal and the external resonance, is divided into two parts by the corresponding bifurcation point. One part corresponds to position of the small body at perihelion and the other at aphelion. In particular, for the internal family, the part $x < 0.6297$ corresponds to perihelion (family $\overline{II}_{(i)}$) and the part $x > 0.6297$ corresponds to aphelion (family $\bar{I}_{(i)}$). For the external family, the part $x < 1.5874$ corresponds to perihelion (family $\bar{I}_{(e)}$) and the part $x > 1.5874$ corresponds to aphelion (family $\overline{II}_{(e)}$).

2.2 The perturbed case: Non zero mass of Jupiter

Let us now assume that the mass of Jupiter is non zero i.e. $m_1 = \mu \neq 0$ and $m_0 = 1 - \mu$. The resonant elliptic families at the 2/1 and 1/2 resonance are continued to $\mu > 0$, but a gap appears at the bifurcation point of the unperturbed families . The topology at the bifurcation point of the resonant families and the gap that appears, is shown in Fig. 1b. There are two families of elliptic orbits in each resonance, one corresponding to the case where the small body is at perihelion and the other to the case where the small body is at aphelion. All these orbits are *symmetric* periodic orbits, i.e. out of the infinite set of all symmetric and asymmetric orbits of the unperturbed problem, only two orbits survive for $\mu > 0$, both symmetric. One of them is stable and the other unstable, as a consequence of the Poincaré-Birkhoff fixed point theorem, but along the family the stability may change.

In figure 2a we present the families of periodic orbits for the 2/1 *internal* resonance, for $\mu = 0.001$. The gap at the bifurcation point that we showed in Fig. 1b is presented in this figure by a small gap at the point $e_1 \approx 0$ ($x \approx 0.6297$). The families are presented in a plane where the horizontal axis indicates the x-coordinate of the massless body in the rotating frame and the vertical axis corresponds to the osculating eccentricity of the nearly Keplerian orbit of the massless body. As in the unperturbed case, there are two families, family $I_{(i)}$, where the small body is at aphelion and family $II_{(i)}$,

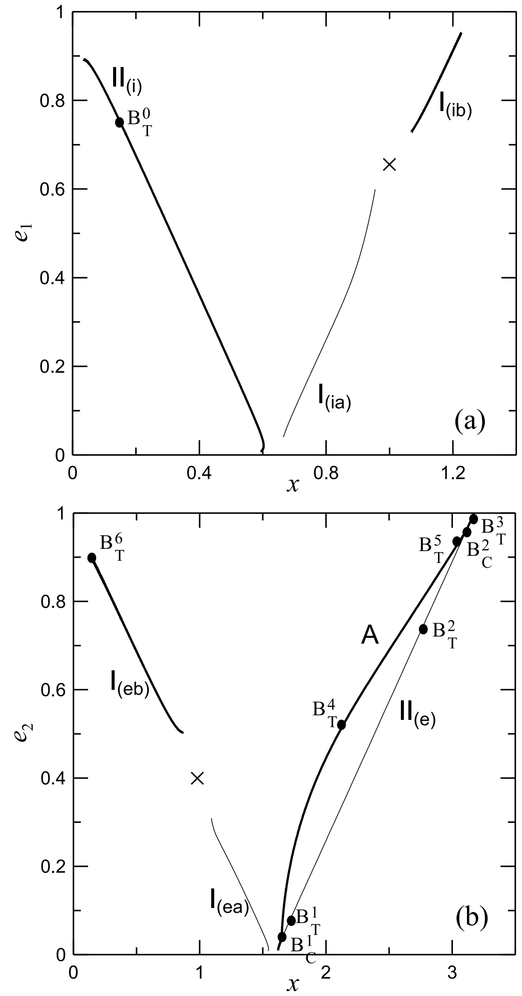

Figure 2. Resonant families of elliptic periodic orbits of the circular restricted problem. a) the case of internal 2/1 resonance. b) the case of external 1/2 resonance. Thin or bold curves indicate unstable or stable orbits, respectively. The symbol "×" indicates a region of close encounters between Jupiter and the small body. Note that we denote with e_1 the eccentricity of the small body in the inner orbits and with e_2 the eccentricity of the small body in the outer orbits.

where the small body is at perihelion. Family $I_{(i)}$ consists of two parts: $I_{(ia)}$ and $I_{(ib)}$ separated by a collision (see Fig. 1a). The first part consists of unstable orbits (thin curve) while the second part consists of stable orbits (thick curve). The orbits of the family $II_{(i)}$ are all stable.

In the case of the *external* 1/2 resonance, the corresponding families are shown in Fig. 2b. These families arise from the 1/2 elliptic family of Fig. 1a when the last one is continued to $\mu > 0$. As in the internal resonance 2/1, a small gap appears at the point $e_2 = 0$, $x = 1.5874$. There are two families, family $I_{(e)}$, where the small body is at perihelion and family $II_{(e)}$, where the small body is at aphelion. As in the internal resonance 2/1, the family $I_{(e)}$ consists of an unstable part ($I_{(ea)}$) and a stable part ($I_{(eb)}$), which are separated by a collision (see Fig. 1a). The family $II_{(e)}$ starts and ends with stable orbits, but along the family the stability changes and an unstable part exists between the critical points B_C^1 and B_C^2. From each one of these critical points there bifurcates a family of *asymmetric* periodic orbits. It turns out that these two asymmetric families coincide to a single asymmetric family A, which starts from the point B_C^1 and ends to the point B_C^2. We have found that this family is stable for $\mu < 5.2\,10^{-3}$. We remark that the asymmetric family exists only for the external resonances of the form $1/q$, called asymmetric resonances (Beauge, 1994; Voyatzis et al., 2005).

3 THE ELLIPTIC RESTRICTED MODEL

Along the resonant elliptic families of the circular restricted problem the period varies. In the unperturbed case ($\mu = 0$) the period is exactly equal to 2π along the family, for the normalization we are using. When $\mu > 0$, the period along the elliptic resonant families varies, but is close to the value 2π. If it happens that for a particular orbit of the family the period is *exactly* equal to $T_0 = 2\pi$, this point is a bifurcation point for a family of single periodic orbits of the elliptic restricted TBP, along which the eccentricity of the second primary (Jupiter) varies (Hadjidemetriou, 1993; Broucke, 1969).

Concerning the stability, we remark that the monodromy matrix of the variational equations has two pairs of eigenvalues (λ_1,λ_2) and (λ_3, λ_4). In the circular model one pair, (λ_1, λ_2) is the unit pair, $\lambda_1 = \lambda_2 = 1$, because of the existence of the energy integral. The other pair, (λ_3, λ_4), may lie on the unit circle in the complex plane ($\lambda_{3,4} = e^{\pm i\phi}$), corresponding to stability, or may be on the real axis ($\lambda_3 = 1/\lambda_4 \in R$) corresponding to instability. In the elliptic model there is no energy integral, so we have four different cases: (i) *stable* orbits when all eigenvalues are on the unit circle (ii) *simply unstable* orbits when one pair of eigenvalues is on the unit circle and one on the real axis (iii) *doubly unstable* orbits when both pairs of eigenvalues are on the real axis and (iv) *complex instability*, where all eigenvalues are outside the unit circle, not on the real axis, arranged in reciprocal pairs and complex conjugate pairs $\lambda_{1,2} = Re^{\pm i\phi}$, $\lambda_{3,4} = R^{-1}e^{\pm i\phi}$ (Broucke 1969). We can also define the stability indices $b_1 = \lambda_1 + \lambda_2$ and $b_2 = \lambda_3 + \lambda_4$. A periodic orbit is stable if

$$|b_i| < 2, \ \ \forall i = 1, 2.$$

If one of the indices b_i does not satisfy the stability condition, the periodic orbit is simply unstable. If both indices do not satisfy the stability condition the periodic orbit is doubly unstable. Complex instability is not present in the particular model.

In the case of 2/1 and 1/2 resonant families, the critical points, i.e. periodic orbits with period exactly equal to 2π, are indicated on the families $II_{(i)}$ and $II_{(e)}$ in Fig. 2 by the points $B_T^i, i = 0, 1, 2, \ldots 6$. Note that for the internal resonance 2/1, there is only one bifurcation point, B_T^0, on the stable family $II_{(i)}$. For the external resonance 1/2 there are five bifurcation points. Three of them, B_T^1, B_T^2 and B_T^3 belong to the symmetric family $II_{(e)}$ and two more points, B_T^4, B_T^5 belong to the asymmetric family A. There are no bifurcation points on the families $I_{(i)}$ while on the family $I_{(e)}$ there is one bifurcation point, B_T^6 at high eccentricity value.

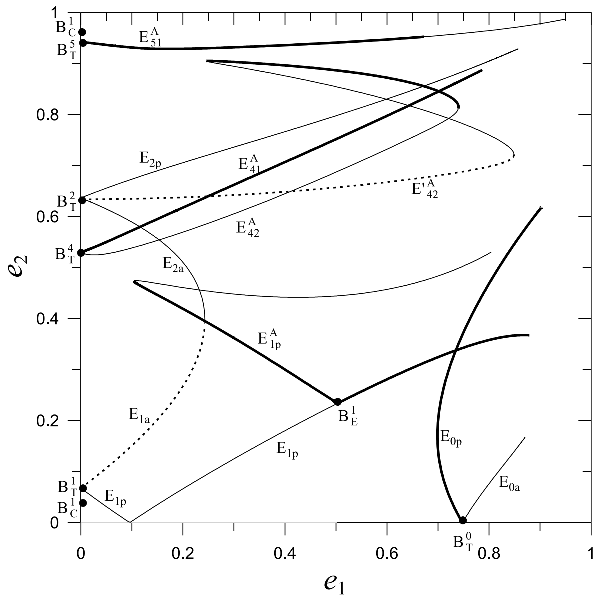

Figure 3. Families of 2/1 resonant periodic orbits in the elliptic restricted problem. The critical orbits B_T^i, $i = 0, 1, 2, 4, 5$ of the circular problem are indicated. Bold, thin and dotted face curves indicate stable, unstable and doubly unstable orbits respectively.

Let us consider a critical periodic orbit B_T^i of the circular problem. The continuation of a family of periodic orbits of the elliptic model, which bifurcates from a point B_T^i, is obtained by increasing the eccentricity of Jupiter, starting from the zero value and keeping its semimajor axis equal to unity, $a' = 1$, so that $n' = 1$. This is a family of resonant periodic orbits, along which *the eccentricity of Jupiter increases*. There are two possibilities (Hadjidemetriou, 1993): at $t = 0$ the elliptic orbit of Jupiter corresponds (i) to perihelion or (ii) to aphelion. So, there are two families of resonant periodic orbits of the elliptic model which bifurcate from each point B_T^i. In one family, denoted by E_p, Jupiter is at perihelion and in the other family, denoted by E_a, Jupiter is at aphelion. Additionally, as we will show in the following, families of asymmetric periodic orbits also exist for the elliptic problem and can be generated in three different ways. We will denote the asymmetric orbits by E^A.

The resonant families of the elliptic model will be presented in the space of the eccentricities of Jupiter and the small body. There are two bodies involved (Jupiter and the small body) and in all the following we will call e_1 the eccentricity of the inner of these bodies and e_2 the eccentricity of the outer body, irrespectively of whether the corresponding body is Jupiter or the massless body. We use this notation in order to have a direct comparison with the families of the general TBP presented in the following sections.

Let us start from the family $II_{(i)}$ of the circular model, and present the family of the elliptic model that bifurcates from the orbit B_T^0 (Fig. 2a), in the space e_1-e_2. Now e_1 denotes the eccentricity of the small body (inner body) and e_2 denotes the eccentricity of Jupiter (outer body). The Family $II_{(i)}$ of the 2/1 resonant family of the *circular* model is located in Fig. 3 on the axis $e_2 = 0$ (circular orbit of Jupiter). On this axis we present the point B_T^0 and show the two families of the *elliptic* model that bifurcate from this point. As we mentioned before, there are two families, E_{0a} and E_{0p}, corresponding to perihelion and aphelion of Jupiter, respectively. Both families start from the eccentricity $e_1 = 0.75$ of the small body and the eccentricity e_2 of Jupiter starts from zero and increases along the family.

We come now to the families of the elliptic model that bifurcate from the 1/2 external resonant family $II_{(e)}$ of the *circular* model (Fig. 2b). In this case, e_1 is the eccentricity of Jupiter (inner body) and e_2 is the eccentricity of the small body (outer body). The family $II_{(e)}$ is located in the Fig. 3 on the axis $e_1 = 0$. Let us consider first the bifurcation from the symmetric families of the circular problem. We study the two critical points B_T^1 and B_T^2 (the point B_T^3 corresponds to very high eccentricities and we do not study it here). From the point B_T^1 there bifurcate two resonant families of symmetric periodic orbits of the elliptic model, the family E_{1p}, which starts having stable orbits, and the family E_{1a} where the orbits are doubly unstable. From the critical point B_T^2 there also bifurcate two families of resonant symmetric periodic orbits of the elliptic model, the families E_{2a} and E_{2p}; both are unstable. It turns out that the family E_{1a} that bifurcates from the point B_T^1 and the family E_{2a} that bifurcates from the point B_T^2 coincide and form a single family that starts from B_T^1 and ends to B_T^2.

We come next to the asymmetric families of the elliptic model. There exist two critical points on the asymmetric family A in Fig.2, the points B_T^4 and B_T^5. As in the previous cases, from each one of these points we have a bifurcation of a resonant family of the elliptic model, which is asymmetric. From the point B_T^4 we have the bifurcation of the family E_{41}^A (stable) and the family E_{42}^A (unstable). This latter family has a complicated form, its stability changes three times and ends at the point B_T^2. So it seems that from this latter point there bifurcate, in addition to the two symmetric families, one more asymmetric family. From the asymmetric point B_T^5 there bifurcate two asymmetric families, E_{51}^A and E_{52}^A. The family E_{51}^A is stable but family E_{52}^A cannot be numerically continued due its strong instability and, thus, is not presented in Fig.2.

Along the resonant families of the elliptic model it may also happen to exist bifurcation points. Such a case is with the family E_{1p}, where the critical point B_E^1 appears. This happens because the stability type changes at this point. From this point we have a bifurcation of the family A_{1p}^A of asymmetric periodic orbits. Voyatzis and Kotoulas (2005) showed that many families of the external resonances have such critical points and conjectured the bifurcation of asymmetric orbits.

4 FROM THE RESTRICTED TO THE GENERAL PROBLEM

Let us consider three bodies with non zero masses, with one of them much more massive than the other two. The more massive body with mass m_0 is the Sun, S, and the two small bodies, P_1 with mass m_1 and P_2 with mass m_2, will be called *planets*. In the following, P_1 will be the inner planet and P_2 the outer planet. If $m_1 \neq 0$ and $m_2 = 0$ we have the restricted model where P_1 is the corresponding Jupiter and P_2 the small body, which moves initially in an outer orbit. If the Keplerian orbit of the bodies S and P_1 is circular, we have the circular restricted problem and if it is elliptic, we have the elliptic restricted model. If $m_1 = 0$ and $m_2 \neq 0$ then P_2 plays the role of Jupiter and the massless body P_1 evolves initially in an inner orbit.

In order to study the continuation of the periodic orbits from the restricted to the general problem we define a rotating frame of reference xOy, whose x-axis is the line $S - P_1$, with the center of mass of these two bodies at the origin O and the y-axis is in the orbital plane of the three bodies. In this rotating frame the body P_1 is always on the x-axis and P_2 moves in the xOy plane. We have four degrees of freedom and we use as coordinates the position x_1 of P_1, the coordinates x_2, y_2 of P_2 and the angle θ between the x axis and a fixed direction in the inertial frame.

The Lagrangian $\mathcal{L}$ in the above coordinates is given in Hadjidemetriou (1975). The variable θ is ignorable and, consequently, the angular momentum $L = \partial\mathcal{L}/\partial\dot{\theta}$ is constant. We can use as normalizing conditions to fix the units of mass, length and time the conditions

$$m = m_0 + m_1 + m_2 = 1, \quad G = 1, \quad L = \text{constant}.$$

Periodic orbits of period T exist in the rotating frame. By fixing the initial condition $\dot{x}_1(0) = 0$, a periodic orbit is represented by a point in the 5-dimensional space

$$\Pi = \{(x_1(0), x_2(0), y_2(0), \dot{x}_2(0), \dot{y}_2(0))\}. \quad (1)$$

When a periodic orbits is symmetric, i.e. it is invariant under the fundamental symmetry $\Sigma : (t, x, y) \rightarrow (-t, x, -y)$ (Voyatzis and Hadjidemetriou, 2005), we can always take as initial conditions $y_2(0) = 0$ and $\dot{x}_2(0) = 0$ and the dimension of the space of initial conditions Π is reduced to three. In the planetary problem, the orbits of P_1 and P_2 are almost Keplerian and we can present the families of periodic orbits in the projection plane of planetary eccentricities $e_1 - e_2$, which correspond to the initial conditions.

The monodromy matrix of the variational equations has now three pairs of eigenvalues (λ_1,λ_2), (λ_3, λ_4) and (λ_5, λ_6). Due to the existence of the energy integral it is $\lambda_5 = \lambda_6 = 1$ and the stability of the periodic orbits is defined by the first two pairs as in the elliptic restricted problem (see section 3).

4.1 The continuation from the restricted to the general problem

In general, the periodic orbits of the restricted problem are continued to the general problem. Particularly, it is proved by Hadjidemetriou (1975) that a periodic orbit of period T of the circular restricted problem is continued to the general problem with the same period, by increasing the mass (e.g. m_2) of the initially massless body. The continuation is not possible only in the case where the period is a multiple of 2π (the period of the primaries). Such continuation forms monoparametric families of periodic orbits with parameter the mass of the small body (m_2), provided that the masses of the other two bodies are fixed. If we keep all masses fixed (and non zero), we obtain a monoparametric family of periodic orbits of the general problem, of the planetary type, along which the elements of the two planetary orbits vary. This family is represented by a smooth curve in the space Π of initial conditions. The periodic orbits of the elliptic restricted problem are also continued to the general problem (Hadjidemetriou and Christides, 1975).

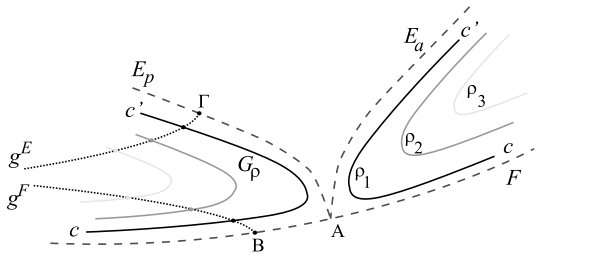

Figure 4. The schematic generation of families of periodic orbits in the general problem from the restricted problem. Dashed curves indicate the families of the restricted problem and the solid curves indicates the families continued in the general problem (see text for detailed description).

The evolution of the characteristic curves of the families, as m_2 increases, is studied by Bozis and Hadjidemetriou (1976). In the present paper we study the continuation using a slightly different approach. Provided that the planetary masses are small with respect to the mass of Sun, i.e. $m_1 \ll m_0$ and $m_2 \ll m_0$, it is found that the families of resonant periodic orbits in the space Π depends on the ratio $\rho = m_2/m_1$ of the planetary masses rather than by their actual values (Beauge et al., 2003). Thus, we can use ρ as a continuation parameter for a monoparametric family of periodic orbits. If we add one more dimension to the space of initial conditions Π in order to assign the value of ρ, we obtain an extended space Π'. In this extended space we can form characteristic surfaces of two-parametric families. In the following we will present families of periodic orbits considering sections of Π' defined by fixing ρ to a constant value.

In Fig. 4 it is shown schematically the continuation of the families F and E of the circular and the elliptic restricted problem, respectively, with e.g. $\rho = 0$. The point A of the family F of periodic orbits of the circular model corresponds to an orbit with period $T = 2k\pi$, where k is an integer. At this point the families E_p and E_a bifurcate to the elliptic restricted problem. By giving mass to the small body, any orbit of F with $T \neq 2k\pi$, e.g. the orbit B, is continued to the general problem with parameter the mass ratio ρ and the monoparametric family g^F is formed. A family g^F exist for any initial orbit of F, except for the orbit A. If we start now from a point on the family g^F and keep ρ fixed, we obtain a monoparametric family c. The same continuation holds for the periodic orbits of the families E_p and E_a of the elliptic restricted problem and for a fixed $\rho \neq 0$ we get the characteristic curve c'. The individual parts of c and c' join smoothly to each other forming the families G_ρ for the general problem and for any fixed mass ratio $\rho \ll 1$. The periodic orbit A is a *singularity* for the continuation resulting in the formation of a gap between the left and right characteristic curves cc'.

4.2 Families of 2/1 and 1/2 orbits continued from the restricted problem to the general problem

Concerning the continuation of the resonant families, which are presented in sections 2.2 and 3, to the general problem,

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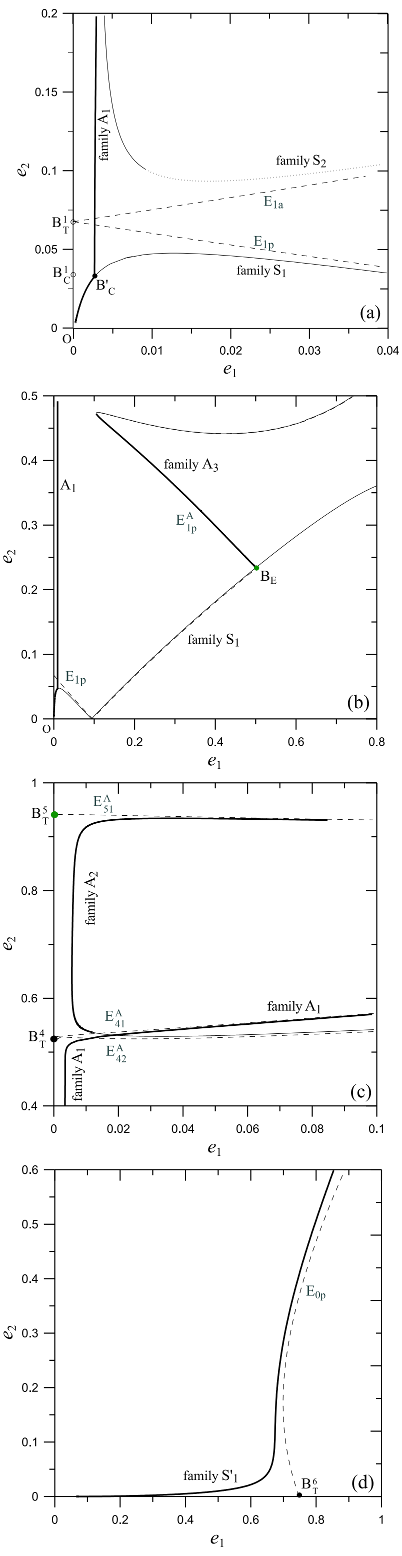

Figure 5. Families of periodic orbits of the general problem (solid curves) for $\rho = 0.01$ in cases (a)-(c) (external resonance) and $\rho = 100$ in case (d) (internal resonance). The families of the circular restricted problem lie along the axis $e_1 = 0$ and $e_2 = 0$ for the external and internal resonance, respectively. The families of the elliptic restricted problem are indicated by the dashed curves.

singularities are expected at the bifurcation points B_T (see Fig. 3) for both internal (2/1) and external (1/2) resonances, where is $T = 2k\pi$. In the following, we study four different cases of continuation presented in Fig. 5. We remind that the families of the circular restricted problem are continued as curves close to the vertical and the horizontal axes in the plane $e_1 - e_2$, for the external ($e_1 = 0$) and the internal resonances ($e_2 = 0$), respectively. In our computations we vary ρ by changing the value of one of the planetary masses and such that $\max(m_1, m_2) = 10^{-3}$.

In the panel (a) of Fig. 5 we show the continuation close to the bifurcation point B_T^1, which belongs to the symmetric family $II_{(e)}$ of the circular problem and is the starting point for the families E_{1p} and E_{1a} of the elliptic problem. By giving mass to the small body (i.e. $\rho \neq 0$) the part (OB_T^1) of the family $II_{(e)}$ and the family E_{1p} are continued and they join smoothly forming the symmetric family S_1. The change of stability that is shown in family $II_{(e)}$ at the point B_C^1 is also obtained in the family S_1. Namely, the bifurcation point B_C^1 (periodic orbit of critical stability) continues as a bifurcation point B_C' in the general problem too. The asymmetric family A of the circular restricted problem, which bifurcates from B_C^1, is continued smoothly to the general problem as an asymmetric family which bifurcates from the B_C'. Similarly to the formation of S_1, the unstable part of the family $II_{(e)}$ above the point B_T^1 and the family E_{1a}, which is doubly unstable, are continued and form the family S_2. Note that the different stability types of the two families results to a periodic orbit of critical stability in the family S_2.

In family E_{1p} there is a change in stability at a point B_E, where the asymmetric family E_{1p}^A bifurcates for the elliptic problem. As ρ takes a positive value, B_E continues as a critical point in the general problem and the asymmetric family A_3 bifurcates from it (Fig. 5b). Actually, A_3 can be considered as the continuation of the family E_{1p}^A. No singularities are obtained in this case. Such a type of continuation explains the existence of the asymmetric planetary corotations found by Michtchenko et al. (2006) at the 3/2 resonance, which in the elliptic restricted problem shows an asymmetric family similar to E_{1p}^A. We note that asymmetric periodic orbits were known only for resonances p/q with $q = 1$.

As it is mentioned above, the asymmetric family A continues smoothly to the general problem (as family A_1) starting from its bifurcation point B_C^1 (see Fig. 5a) . However, the family A contains the critical point B_T^4 (see Fig. 2), which admits a singularity for the continuation to the general problem. In the panel (c) of Fig. 5, the gap formed in the above singularity is shown. The asymmetric family E_{41}^A, which bifurcates from B_T^4, continues and completes the asymmetric family A_1 of the general problem. Additionally, the part of family A located between the bifurcation points B_T^4 and B_T^5, the family E_{42}^A and the family E_{51}^A continue in the general problem, join smoothly and form the asymmetric family A_2. Again we obtain a change of stability along A_2 due to the different types of stability on the families of the restricted problem (family A is stable while family E_{42}^A is unstable).

In the case of the internal resonance, the continuation to the general problem shows the same characteristics as in the external resonances. We remark that in this case only symmetric periodic orbits appear. In Fig. 5d we show the formation of the symmetric family S_1' of the general problem,

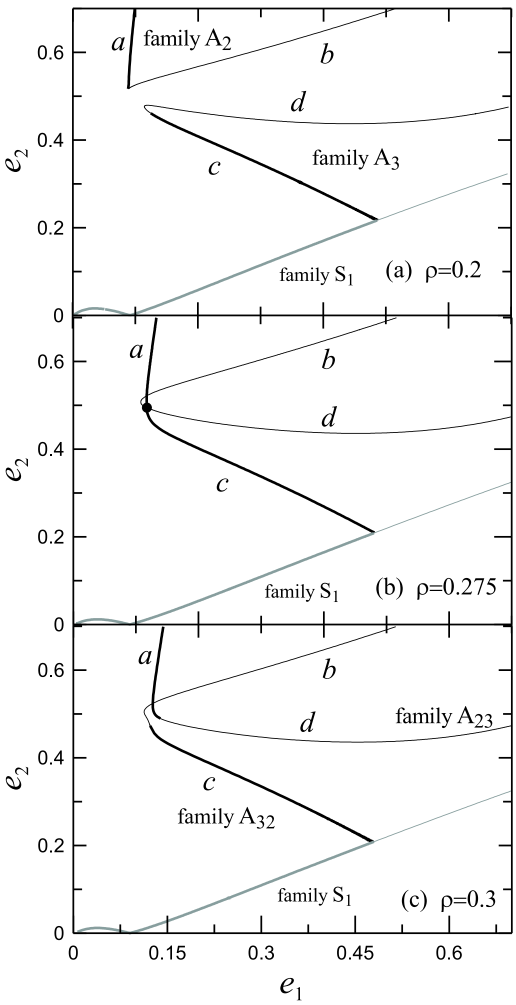

Figure 6. The "collision–bifurcation" of the families A_2 and A_3 at $\rho = \bar{\rho}_2$ and the formation of the new families A_{23} and A_{32}. a) $\rho = 0.2 < \bar{\rho}_1$ b) $\rho = 0.275 \approx \bar{\rho}_1$ and c) $\rho = 0.3 > \bar{\rho}_1$.

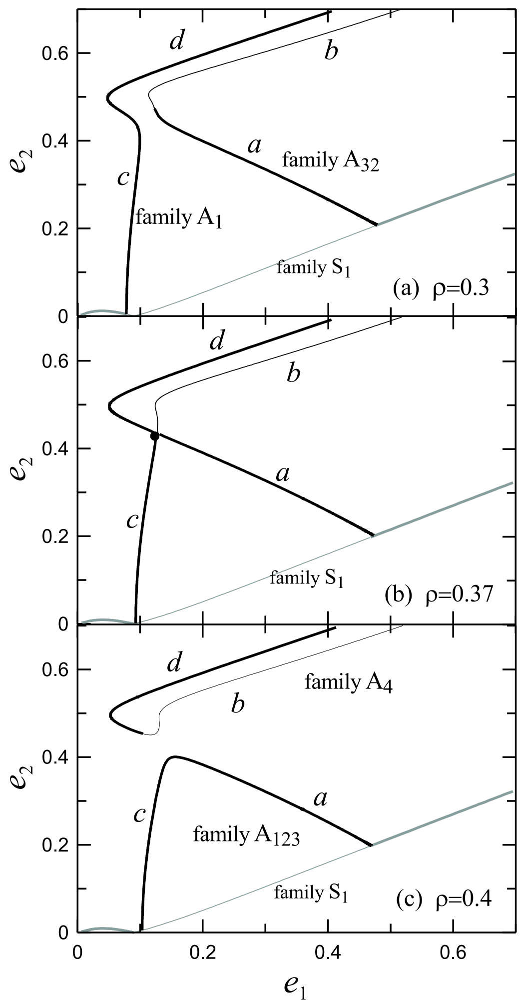

Figure 7. The "collision–bifurcation" of the families A_1 and A_{32} at $\rho = \bar{\rho}_2$ and the formation of the new families A_4 and A_{123}. a) $\rho = 0.3 < \bar{\rho}_2$ b) $\rho = 0.37 \approx \bar{\rho}_2$ and c) $\rho = 0.4 > \bar{\rho}_2$.

after the continuation of the families $II_{(i)}$ and E_{0p} of the restricted problem. We note that since $\rho = \infty$ in this case, the continuation maybe assumed as varying $1/\rho$.

5 CONTINUATION WITHIN THE FRAMEWORK OF THE GENERAL PROBLEM

In the previous section we considered the continuation of families of periodic orbits when we pass from the restricted to the general problem. The situation, which was described, is referred to small values of the mass of one of the planets, namely $\rho \ll 1$ or $\rho \gg 1$. Now we consider the case of the external resonance and its families generated for $\rho > 0$. As the parameter ρ increases, the families evolve and new bifurcations and structure changes are possible. We follow the evolution of the 1/2 resonant families, which are presented in the previous section, and we show in the following how the new structures are formed by increasing ρ. We remind that as $\rho \to \infty$ we approach the internal resonance of the restricted problem.

Figure 6a shows the asymmetric families A_2 and A_3 for $\rho = 0.2$. Up to this value not any structural change of the families occur. In both families the stability changes and, consequently, we distinguish in each family two parts, parts a and b for A_2 and c and d for A_3. For the critical value $\rho = \bar{\rho}_1 \approx 0.275$ the two families collide at a point in the space of initial conditions Π (Fig. 6b). This point corresponds to the orbit of critical stability and is the intersection point of the parts a–d. For $\rho \gg \bar{\rho}_1$ (see Fig. 6c) the part a of family A_2 and the part d of family A_3 join together and form the family A_{23}. Similarly, the parts c and b of the families A_3 and A_2, respectively, form the family A_{32}. Note that the new families are separate and their intersection in the plane $e_1 - e_2$ is due to the projection.

The evolution of the family A_1, as ρ increases, take place smoothly without structural changes up to $\rho = \bar{\rho}_2 \approx 0.37$. In figure 7a, which corresponds to $\rho = 0.3$, it is shown that the family A_1 has come close to the family A_{32}, which is generated after the bifurcation at $\rho = \bar{\rho}_1$. At $\rho = \bar{\rho}_2$ the two families collide and, similarly to the previous case, for $\rho > \bar{\rho}_2$ we obtain two new families, the family A_4 and the family A_{123} (Fig. 7c). In this case only the family A_{32} has an orbit of critical stability, which separates the family in

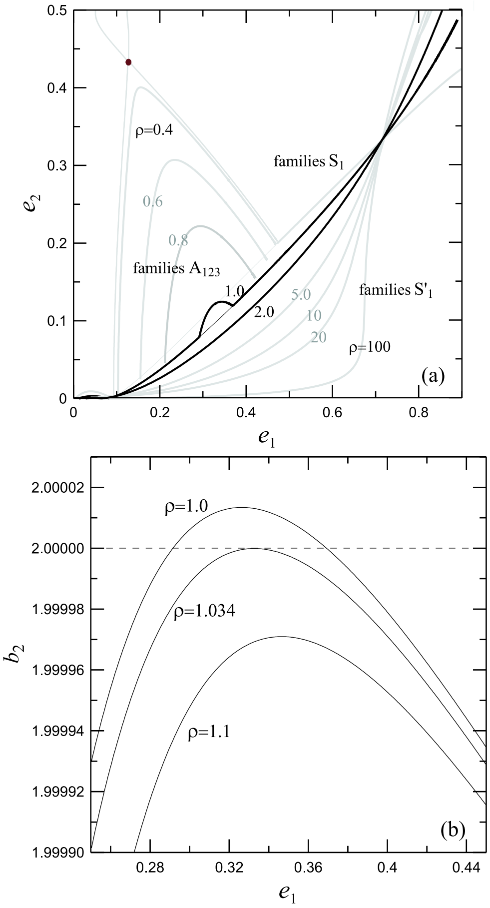

Figure 8. a) The evolution of the families A_{123} and S_1 as ρ passes the critical value $\bar{\rho}_3 = 1.034$ and takes large values. b) The stability index b_2 along the family S_1, which determines the interval of instability ($b_2 < -2$) and the bifurcation points (at $b_2 = -2$) of the family A_{123}.

a stable (a) and in an unstable part (b). The family A_1 is whole stable and there is no clear border between its parts c and d. After the bifurcation, an orbit of critical stability is shown only in the new family A_4, while the family A_{123} is whole stable and starts and ends at bifurcation points of the symmetric family S_1. Note that these bifurcation points originate to the bifurcation points B_C^1 and B_E of the circular and the elliptic, respectively, restricted problem.

Now we restrict our study to the evolution of the family A_{123} for $\rho > \bar{\rho}_2$. As it is shown in Fig. 8a, as ρ increases, the ending points of the family move on along the family S_1 in opposite direction and the family shrinks and, finally disappears at $\rho = \bar{\rho}_3 \approx 1.034$. In Fig. 8b we present the above transition by considering the stability indices b_1, b_2 for the orbits along the family S_1. The horizontal axis indicates the eccentricity of the periodic orbits along the family S_1 and the value of the corresponding stability index b_2 is presented on the vertical axis. For all orbits it holds $|b_1| < 2$ and, thus, the condition $|b_2| < 2$ is sufficient and necessary for linear stability. In Fig. 8b we obtain that unstable orbits exist for $\rho < \bar{\rho}_3$ in a part of the family S_1 defined by the e_1 interval where $b_2 < -2$. As ρ increases, the curve of b_2 values is raised continually and for $\rho > \bar{\rho}_3$ is located above the value

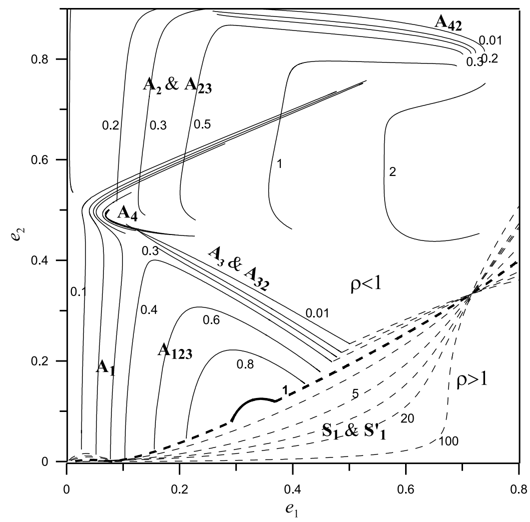

Figure 9. Stable segments of families of periodic orbits for various values of the planetary mass ratio $\rho = m_2/m_1$. Dashed and solid curves indicate symmetric and asymmetric families, respectively. The family S_1 and A_{123} for $\rho = 1$ is indicated by a bold style as a border between the internal and external resonances.

$b_2 = -2$. Therefore, the unstable part of S_1 disappears and, consequently, the family A_{123} disappears too.

Actually for $\rho > \bar{\rho}_3$, the planet P_1 becomes the small body and we pass to the case of the internal resonance. The remaining family S_1 evolves smoothly, as ρ increases, and should be assumed as a family S_1' of the internal resonance, which approaches the families $II_{(i)}$ and E_{0p} of the restricted problem as $\rho \to \infty$ (see sections 2.2,3 and Fig. 5d).

The above described bifurcation scheme explains completely the origin and the structure of corotations found by Beauge et al. (2006). Following the other asymmetric families, which are coming from the restricted problem, we obtain new "collision–bifurcations" as ρ increases and new structures of characteristic curves are formed. We have found e.g. that the families A_4 and A_{23} collide for $\rho \approx 0.45$ and generate the miscellaneous family found by Voyatzis and Hadjidemetriou (2005) and called "A_2" there in.

6 STABLE PERIODIC CONFIGURATIONS AND RESONANT PLANETARY SYSTEMS

From a dynamical point of view a stable periodic orbit indicates an *exact resonance* in phase space and in its neighborhood the motion is quasiperiodic and takes place on invariant tori. In this domain, the resonant angles of the associated mean motion resonance $n_1/n_2 = (p+q)/q$, defined as $\theta_1 = (p+q)\lambda_2 - p\lambda_1 - \varpi_1$ and $\theta_2 = (p+q)\lambda_2 - p\lambda_1 - \varpi_2$, generally librate. Consequently, the apsidal difference $\Delta\varpi = \varpi_2 - \varpi_1$ also librates. Far from the periodic orbit a passage from libration to circulation is possible, for at least one the resonant angles, but the system can still remain stable. A study of the stability in the 2:1 resonance is presented in Michtchenko et al. (2008) and Marzari et al (2006). The dynamics of the 3:1 resonant domain is studied in Voyatzis (2008).

Considering the continuation scheme described in the

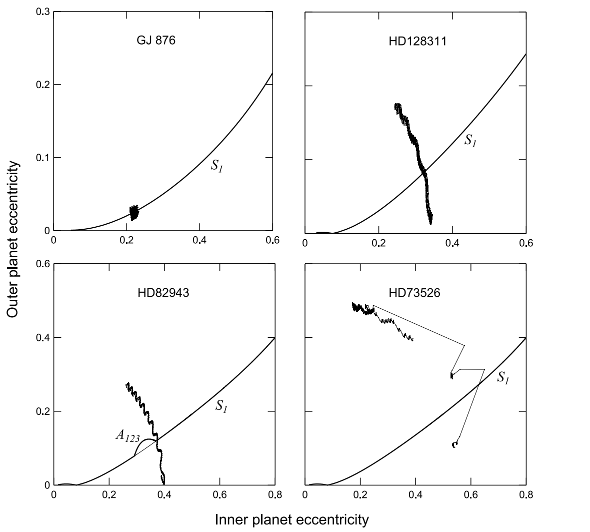

Figure 10. Trajectory projection on the eccentricity plane of the four potential 2/1 resonant planetary systems. The families of resonant periodic orbits, which correspond to the particular planetary mass ratio value ρ, are also shown.

previous section we have obtained a set of families that include parts of stable periodic orbits. In summary, the characteristic curves of the stable parts of the families are presented in Fig.9. We obtain that the symmetric families S_1 (or (S_1') are located in a domain where $e_1 > e_2$ and correspond to (0,0) apsidal corotations (i.e. the resonant angles θ_1, θ_2 and the apsidal difference $\Delta\varpi$ librate around 0°). It is clear that for the external resonance ($\rho < 1$) the characteristic curves show a larger complexity in comparison with the internal resonance. Also the asymmetric families dominate in the external resonance. This is due to the fact that the circular restricted problem shows more rich dynamics with respect to the existence of periodic orbits and bifurcation points for the elliptic problem. Additionally, within the framework of the general problem, we have obtained collision-bifurcations only for $\rho < 1$.

From the chart of Fig. 9 the (π, π)- corotations, which first were indicated by Beauge et al (2006), are missing. We can show that such stable corotations correspond to families of symmetric periodic orbits, which are continued from the family $I_{(eb)}$ of the circular problem and the family of the elliptic problem that bifurcates from the point B_T^6 (see Fig. 5).

According to the present observations, planetary pairs whose orbital periods indicate 2/1 mean motion resonance exist round the stars GJ876, HD128311, HD82943 and HD73526. For these systems we examined their evolution in order to study if their dynamics is related with a particular resonant periodic orbit. For all systems we used the initial conditions estimated by Butler et al (2006). We consider again the plane $e_1 - e_2$ and project the system evolution in this plane. The results for the four planetary systems are shown in Fig. 10.

For the first three systems the evolution seems regular (quasiperiodic) and the trajectory is centered at point in the $e_1 - e_2$ plane that belongs on the family of periodic orbits. Thus the systems evolve around a particular periodic orbit and this fact indicates strongly the resonant configuration of the planetary pairs. Particularly, the system GJ876 is centered at the stable periodic orbit with $e_1 = 0.22$ and $e_2 = 0.2$ and the resonant angles θ_1, θ_2 and $\Delta\varpi$, as well, librate with amplitudes 12°, 38° and 35°, respectively. Similarly, the system HD128311 is centered at the stable periodic orbit with $e_1 = 0.32$ and $e_2 = 0.08$ and the resonant angles θ_1, θ_2 and $\Delta\varpi$ librate with amplitudes 43°, 83° and 103°, respectively. The system HD82943 seems to be centered close to the critical periodic orbit ($e_1 = 0.37$ and $e_2 = 0.12$) of the family S_1, which separates the stable and the unstable parts of S_1 and bifurcates the asymmetric family A_{123}. Now the angle θ_1 librates with amplitude 75° but the angle θ_2 (and, consequently, $\Delta\varpi$) circulates. The initial conditions given for the system HD73526 lead to close encounters and strongly chaotic motion, which disrupts the system in few hundred years.

7 DISCUSSION AND CONCLUSIONS

The method of continuation of periodic orbits has been applied in the present work in order to study the generation and the structure of families of periodic orbits in the general TBP, starting from the restricted problem. We considered the planar TBP of planetary type, referred to a rotating frame and studied resonant planetary motion. In particular, we studied the 2/1 resonant periodic orbits, but the method we used can be applied to all other resonances. We considered planar motion only.

We started our study from the unperturbed Keplerian problem. Then we passed to the circular restricted TBP and computed the basic families of 2/1 resonant periodic orbits, both for the inner orbits (inside Jupiter) and the outer orbits (outside Jupiter). The basic families are symmetric with respect to the rotating x-axis, but asymmetric families also exist in cases of resonances of the form $1/q$. On these families, we found the periodic orbits with period equal to 2π which, in the normalization we are using, are the bifurcation points to resonant 2/1 (or 1/2) families of periodic orbits of the elliptic restricted model. Two such families bifurcate from each of these critical points. In particular, we have the following cases: a) a symmetric (or asymmetric) periodic orbit of period 2π of the circular restricted problem is continued to the elliptic restricted problem as symmetric (or asymmetric) b) a symmetric periodic orbit of period 2π of the circular restricted problem can be continued to the elliptic problem as an asymmetric one - this exceptional case is verified only up to the accuracy of the numerical computations and c) a symmetric periodic orbit of the elliptic restricted problem, which is of critical stability, can be continued to the elliptic problem as an asymmetric one. The last case indicates the existence of asymmetric periodic orbits in the elliptic problem and in particular to resonances which are not necessarily of the form $1/q$.

The continuation of the families of periodic orbits of the restricted problem to the general problem follows the scenario described in the paper of Bozis and Hadjidemetriou (1976). Particularly, the families of the general problem originate from two families of the restricted problem, one of the circular problem and one of the elliptic problem. There is no essential difference in the continuation between symmetric and asymmetric periodic orbits. The asymmetric orbits, which bifurcate from symmetric orbits of the elliptic problem, are continued smoothly in the general problem. The stability of periodic orbits is preserved after the continuation from the restricted to the general problem.

After the continuation of the periodic orbits of the restricted problem to the general problem, by giving a very small mass to the massless body, we studied how these families evolve when we increase the mass of the small body. Particularly we studied such an evolution by considering as a parameter the planetary mass ratio $\rho = m_2/m_1$, keeping $m_i \ll 1, i = 1, 2$. Starting from $\rho = 0$ (external resonances of the restricted problem) and increasing its value, we found that the characteristic curves of two different families can collide in the space of initial conditions. At these points a bifurcation takes place (*collision–bifurcation*) causing a topological change in the structure of the colliding families and the formation of new families. As $\rho \to \infty$ we approximate the families of the internal resonances of the restricted problem without further bifurcations.

Following the method of continuation we obtained a large set of stable families of periodic orbits that consist exact resonances and centers of apsidal corotation resonances. The neighborhood of these orbits in phase space is occupied by invariant tori which guarantee the long term stability of the system providing also librations for the resonant angles. The planetary pairs in the systems GJ876, HD128311 and HD82943 seem to be associated with such 2/1 resonant domains in phase space, according to the current observations and fittings given by Butler et al (2006). The evolution of the system HD73526 cannot be associated with a stable periodic orbit and it appears strongly chaotic.

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