ORBITAL DYNAMICS - CHAOS IN[1]

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Article Outline

Glossary

Reference Frame

A *reference frame* can be determined by a set of solid bodies, through which we can define a three dimensional geometric figure, for example a triedron (three non planar axes starting from a point). The surface of the Earth can be used to define a reference system. A moving car can be also used to define a reference frame, different from the first one.

Inertial Frame

An *inertial frame* is a special class of reference frames, in which the basic laws of motion (*Newton's laws*) are valid. According to *Galileos' Principle of Relativity*, any frame of reference moving uniformly (with constant velocity without rotation) with respect to an inertial frame is also an inertial frame. A frame of reference which is rotating with respect to an inertial frame is not inertial. The criterion for a frame to be inertial is Newton's first law to be valid. This means that in an inertial frame a free body is either at rest or moves in a straight line with constant velocity. The best approximation in nature of an inertial frame is that frame which is defined by a triedron whose origin is at the center of mass of our Solar System and its three axes are in three fixed directions in space, defined by three distant stars.

Degrees of freedom

The number of independent variables that are needed to determine the position of a dynamical system is called the number of *degrees of freedom*. For example, a particle moving freely in space has three degrees of freedom, since its position is determined by its three Cartesian coordinates (x_1, x_2, x_3), which are independent.

Phase space

Consider a space whose coordinates determine exactly the state of the system. This space is

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called the *state space* or the *phase space* of the system. Each point of the phase space determines uniquely the initial conditions of the motion. The evolution of the system in the phase space is represented by a smooth curve, which is called the *phase curve*. The phase curves do not intersect, otherwise the point of intersection would correspond to two different solutions. The set of all phase curves is called the *phase diagram* and gives important information of the stable and unstable regions of the phase space. For gravitational systems, the phase space is the space of coordinates and velocities of all the bodies. Usually, instead of the velocities, the momenta are used in the definition of the phase space. In a gravitational system with n degrees of freedom, the phase space has $2n$ dimensions. For example, a body moving in the plane under the action of a force, has two degrees of freedom (coordinates x, y) and its phase space is the four-dimensional space x, y, $p_x = m\dot{x}$, $p_y = m\dot{y}$.

Orbit
An *orbit* of a body, or a set of bodies, considered as point masses, is the path that the bodies describe in a reference frame. The orbit of the *same* body or set of bodies is different in different frames of reference.

Periodic Orbit
A *periodic orbit* is the orbit of one or more bodies that repeats itself after a certain time, which is called the *period* of the periodic orbit. The periodicity property is closely related to the frame of reference to which the motion is referred to. For example, an orbit may be periodic in a rotating frame, but not in the inertial frame. In this latter case, for two or more bodies, it is the relative configuration that is repeated in the inertial frame.

Poincaré map
The Poincaré map is a method by which we transform the continuous flow of a dynamical system in its n-dimensional phase space, into a discrete map in a reduced phase space. The map is obtained by taking the intersections of the continuous flow in the original phase space with a *surface of section*, defined properly. This surface of section is $(n-1)$-dimensional, in general, or $(n-2)$-dimensional if an integral of motion exists, which is the energy integral in gravitational systems. These will be explained in detail in sections 8 and 9. A periodic orbit appears as a fixed point on the Poincaré surface of section. The Poincaré map is very useful in the study of ordered and chaotic motion in a dynamical system, especially in systems with few degrees of freedom.

Stability
The notion of *stability* refers to the behavior of the orbits in the vicinity of a periodic orbit. If a slight change in the initial conditions results to a new orbit, called the *perturbed orbit*, which deviates much from the periodic orbit, then the periodic orbit is called *unstable*. In the gravitational systems that we will study, this deviation is exponential. If on the other hand, the perturbed orbit stays close to the periodic orbit, the periodic orbit is called *stable*. But there are different aspects of stability. For example, if the perturbed orbit, considered as a geometrical figure, is close to the periodic orbit, then the periodic orbit is called *orbitally stable*. However, in this latter case it may happen that two bodies, one on the periodic orbit and one on the perturbed orbit, which start very close to each other, may deviate much as each one moves on its own orbit, although the geometric figures of the two orbits are close to each other. In this aspect, the orbit is considered as unstable. A Keplerian elliptic orbit, in the two-body problem, belongs to this latter category. A different type of stability is the *asymptotic stability*. In this case any perturbed orbit, not only stays in the vicinity of the periodic orbit, but tends asymptotically to the periodic orbit. In gravitational systems asymptotic stability does not appear, unless there exists a dissipation to the system.

Ordered and Chaotic Motion
The notion of *chaoticity* is related to the deviation of a perturbed orbit from a given orbit. It may happen that the perturbed orbit does not deviate much as time goes on. In this case we say that we are in an *ordered region*. The prediction of the evolution of the system in this case is possible. In some cases however, the perturbed orbit deviates exponentially from the original orbit. Prediction is not possible for a long time. In this latter case we are in a *chaotic region*. In general, both ordered and chaotic regions exist in the same dynamical system.

Keywords: Asteroid, Chaos, Extrasolar systems, Periodic Orbits, Resonance, Stability

1 Definition of the subject and its importance

By the term *orbital dynamics* we mean the study of the motion of one or more bodies. Motion is one of the first things that a human being noticed, since the very early stages of human life. Apart from the motion of himself, walking around, he also realized that everything around him is not still, but changes position, being it a wild animal, a dry leaf drifting in the wind, the motion of clouds in the sky, or the change of the position of the celestial bodies, most notably of the Sun and the Moon.

Evidently, motion is one of the most important aspects in everyday life. By the term *motion* we mean the change of the position of one or more bodies in space, with respect to the other bodies in that region. If only one body existed in the universe, motion could not be defined. This makes necessary the introduction of an important notion in physics, the *frame of reference*. The surface of the Earth, for example, defines a frame of reference, with respect to which we define the position of a body and its motion, as the body changes position. But a bus moving on the road is also a frame of reference, different from that defined by the surface of the Earth, i.e. the road. And it is a different thing if the bus moves on a straight line with constant velocity, or makes rapid turns following a difficult mountain road. Among all possible frames of reference, the *inertial frames of reference* have a special status in the study of motion. It is in these frames that the basic laws of motion (Newton's laws) are valid.

If the dimensions of the body can be considered as negligible, with respect to its surroundings, we can consider it as a point mass. However, in many cases, the finite dimensions of a body cannot be ignored. In this case its motion cannot be described by the motion that a point describes, but we have to consider also the rotation of the body. Whether we consider a body as a point mass or as a body with finite dimensions, depends on the particular study. For example the Earth is considered as a point mass in the study of its motion around the Sun, but as a finite body when we study the motion of an artificial satellite. In the present study we restrict ourselves in the motion of point masses. The path that such a body describes, is called the *orbit* of the body.

The motion of the bodies takes place under the action of forces which follow definite laws. In everyday life, the dominant force is the gravitational interaction between the bodies, according to *Newton's law of gravity*. Although it is, by far, the weakest force in nature, it is the main force that we feel in everyday life, in addition to the electromagnetic force, which also manifests itself in macroscopic phenomena. These forces affect the motion of the bodies through definite laws, expressed by differential equations, which are *deterministic* equations, i.e. to a certain set of initial conditions there exists one and only one final result, which in our case is a definite orbit. In classical physics, these laws are *Newton's laws of motion*. They are expressed by differential equations of the *second order*, which implies that the initial conditions that define the motion are the *initial position* and the *initial velocity*. This is the essential property of Newton's laws.

In classical physics we assume that it is possible to know exactly, *at the same time*, the position and the velocity of a body. In some other world, where the laws of motion were expressed by differential equations of the third order, the initial acceleration would be also necessary to define the motion. Alternatively, if the laws of motion were expressed as differential equations of the first order, only the initial position would be enough to determine the motion.

As we mentioned, the equations of motion are deterministic. This implies that motion would be exactly defined and that we could predict the motion of one or more bodies, for example the motion of an asteroid in our Solar System, if we knew its initial conditions. This idea prevailed classical physics until the sixties of the 20th century. But what will happen if we make a small error in the initial conditions? Does it have a great effect on the final position, after a certain time (for example a few million years for the asteroid), or the final error will be of the same order as the initial error? In the latter case the small error, due for example to a not very accurate measurement of the position and the velocity of the asteroid, is not important. In many cases however, including an asteroid in certain regions in the asteroid belt close to some mean motion resonances with Jupiter, the orbit is very sensitive to a change in the initial conditions. In this latter case, after a certain time, the orbit which corresponds to the slightly changed initial conditions, differs very much from the original orbit, because it deviates exponentially. These orbits, which are very sensitive to the initial conditions, are called *chaotic orbits*. In such a case prediction of the final position of the body, after a certain time, is not possible, because a very small error in the initial conditions, beyond the accuracy of the observations for the initial conditions, will give a completely different final position, due to the exponential deviation between the two orbits. As we will see, all the physical systems are non integrable and consequently they present chaotic behavior, at least for some initial conditions, and for this reason prediction of the evolution of such a system is not possible, after a certain time interval. This time interval is different in different systems and may be two weeks for meteorological systems or some million years for the motion of an asteroid.

Among all possible orbits in a dynamical system, the *periodic orbits* play a dominant role in the study of the evolution of the system, although it is known that they form a set of measure zero. This is so because, as it will become clear in the following, the periodic orbits are the "backbone" of the topology of the phase space, because their position and their stability character (stable or unstable) determine the structure of the phase space. It is close to the unstable periodic orbits that chaotic motion appears. A special class of periodic orbits in dynamical systems that describe the motion in the Solar System are the *resonant* periodic orbits, because around the stable resonant orbits islands of stable motion exist and the system can be trapped in these regions. In addition, since in a system there exist more that one resonances, the overlap of these resonances, as a perturbation increases, will generate chaotic motion.

In the following we restrict ourselves to the study of motion under gravitational forces, focusing on our Solar System and on extrasolar planetary systems, but the theory is applicable in all cases of motion, under any force field.

2 Introduction. The gravitational N-Body problem

The Newtonian gravitational force is the dominant force in the N-Body systems in the universe, as for example in a planetary system, a planet with its satellites, a multiple stellar system, or a galaxy.

In many cases, there is only one massive body, whose gravitational attraction provides the dominant force, as is the case with a planetary system, where the Sun is the main attracting body, or a planet surrounded by satellites. In this case the motion of the small bodies (planets or satellites) follow Keplerian orbits, perturbed by the gravitational interaction between the small bodies. This is a *nearly integrable* dynamical system. In these systems resonances exist between the small bodies in their motion around the massive body, as will be explained in the following. These correspond to periodic motion, and this makes clear the importance of the resonances in the dynamical properties of a nearly integrable system.

The simplest model of a gravitational system is a system of two bodies moving in Keplerian orbits around their common center of mass. This is an integrable system. In such systems all motion is ordered and chaos never appears. We consider now a hierarchy of models, starting from the above mentioned integrable system and adding more bodies to the system. We have different models, which are used to study particular systems. All these systems are not integrable, although they are nearly integrable. In these latter systems both ordered and chaotic regions appear, as we will see in the following. We consider two basic models:

The restricted three-body problem: Two bodies of finite masses, called *primaries*, revolve around their common center of mass in *circular* or *elliptic* orbits and a third body with *negligible* mass moves under their gravitational attraction, but does not affect the orbits of the two primaries. In most astronomical applications the second primary has a small mass compared to the first primary (the Sun), and consequently the motion of the third, massless, body is a perturbed Keplerian orbit. This is a model for the study of an asteroid (Jupiter being the second primary) or a trans-Neptunian object (Neptune being the second primary).

The general planetary three-body problem: Three bodies with finite masses moving under their gravitational attraction. This is a model for a triple stellar system. In many astronomical applications one of the three bodies has a large mass and the other two bodies have small, but not negligible masses. This is a model for an extrasolar planetary system, or a system of two satellites moving around a major planet. In the latter two cases the two small bodies move in perturbed Keplerian orbits.

The long term evolution of the system depends on the topology of its phase space and the existence of ordered or chaotic regions. The topology of the phase space is determined by the position and the stability character of the periodic orbits of the system (fixed points on a Poincaré map on a surface of section). Islands of stable motion exist around the stable periodic orbits. Chaotic motion appears at the unstable periodic orbits. This makes clear the importance of the periodic orbits in the study of the dynamics of such systems.

We will start with the basic elements of gravitational systems in general. Then we will focus our attention to the study of systems of two degrees of freedom, and then extend the results to three degrees of freedom. The study will be for a general dynamical system, with particular emphasis on Hamiltonian systems, because the gravitational systems are Hamiltonian.

Basic Equations and Integrals of motion

The gravitational force between two bodies, N_i, N_j, is given by Newton's law of gravitation

$$F_{ij} = -\frac{Gm_i m_j}{r_{ij}^2},$$

where G is the gravitational constant, m_i, m_j are the masses of the bodies N_i and N_j and r_{ij} is their distance. The minus sign indicates attraction. We have $3N$ degrees of freedom and the evolution in space is given by the system of differential equations

$$m_i \ddot{\vec{r}}_i = \vec{F}_i,$$

where

$$\vec{F}_i = -\sum_{j=1}^{N} \frac{Gm_i m_j(\vec{r}_i - \vec{r}_j)}{r_{ij}^3} = -\frac{\partial V}{\partial \bar{r}_i},$$

and

$$\vec{r}_i(x_i, y_i, z_i) \quad (i = 1, 2, ...N)$$

is the position vector of the body N_i. The system is conservative, and the potential function is

$$V(\vec{r}_m - \vec{r}_n) = -\sum_{ij} \frac{Gm_i m_j}{r_{ij}}. \quad 1 \le i < j \le N$$

The gravitational system of N bodies can be formulated in Hamiltonian dynamics, and the Hamiltonian function is

$$H = \sum \frac{\vec{p}_i^2}{2m} + V, \quad \vec{p}_i = m_i \dot{\vec{r}}_i.$$

We have the following integrals of motion:

$$\vec{r}_{cm} = \left(\sum m_i \vec{r}_i\right) \qquad \text{Center of mass}$$

$$\vec{p} = \sum m_i \vec{v}_i = constant \qquad \text{Linear momentum}$$

$$\vec{L} = \sum \vec{r}_i \times m_i \vec{v}_i = constant \qquad \text{Angular momentum}$$

$$E = T + V = constant \qquad \text{Energy integral}$$

The total momentum of the system is equal to $\vec{p} = m\vec{v}_{cm}$, $\vec{v}_{cm}$ being the velocity of the center of mass. We can assume that the total momentum is equal to zero, $\vec{p} = 0$, which implies that the center of mass is at rest, $\vec{v}_{cm} = 0$. Consequently, in the system where the center of mass is at rest, we have $3N - 3$ degrees of freedom. In this latter case we can take $\vec{r}_{cm} = 0$.

3 Periodic orbits in systems with two degrees of freedom

Periodic orbits

The periodic orbits play an important role in understanding the dynamics of a system, because they determine critically the topology of the phase space. This will become clear in the following. For this reason it is important to know the basic families of periodic orbits in a dynamical system, because they are the "backbone" of the phase space.

Let us consider a dynamical system with two degrees of freedom, defined by the set of two second order differential equations

$$\begin{aligned} \ddot{x}_1 &= F_1(x_1, x_2, \dot{x}_1, \dot{x}_2), \\ \ddot{x}_2 &= F_2(x_1, x_2, \dot{x}_1, \dot{x}_2). \end{aligned} \tag{1}$$

The initial conditions that determine a solution are (x_{10}, x_{20}, $\dot{x}_{10}$, $\dot{x}_{20}$) and the corresponding solution has the form

$$\begin{aligned} &x_1(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t), \\ &x_2(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t). \end{aligned}$$

The solution is periodic, with period T, if

$$x_i(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t + T) = x_i(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}; t),$$

for every t.

Existence of symmetric periodic orbits

We assume that the differential equations (1) are invariant under the transformation

$$x_1 \to x_1, \;\; x_2 \to -x_2, \;\; t \to -t.$$

This property appears in several models that are of astronomical interest. This means that if $x_1(t), \; x_2(t)$ is a solution, then $x_1(t), \; -x_2(-t)$ is also a solution. Note that this second solution is the symmetric of the first solution with respect to the x_1-axis. Consequently, if an orbit starts from the x_1-axis perpendicularly, $\dot{x}_{10} = 0$, and crosses again the x_1-axis perpendicularly, $\dot{x}_1 = 0$, the orbit is closed and is a *symmetric* periodic orbit with respect to the x_1-axis.

The initial conditions of a symmetric periodic orbit are ($x_{10}, \; x_{20} = 0, \; \dot{x}_{10} = 0, \; \dot{x}_{20}$), which means that a symmetric periodic orbit is determined only by the two nonzero initial conditions $x_{10}, \; \dot{x}_{20}$. From the above we see that the *periodicity conditions* are

$$\begin{aligned} x_2(x_{10}, 0, 0, \dot{x}_{20}; T/2) = 0, \\ \dot{x}_1(x_{10}, 0, 0, \dot{x}_{20}; T/2) = 0, \end{aligned}$$

which imply that the orbit starts perpendicularly from the x_1-axis ($x_{20} = 0, \;\; \dot{x}_{10} = 0$) and crosses again perpendicularly the x_1-axis after a time interval equal to half the period T. We remark that the second *perpendicular* crossing may take place after several (non perpendicular) crossings from the x_1-axis.

The periodic orbits are not isolated, in general. They belong to families, along which the period varies. A family of symmetric periodic orbits is represented by a continuous curve in the space of initial conditions $x_{10}, \dot{x}_{20}$. This curve is called a *characteristic curve.*

4 Variational equations

We study now the behavior of the system in the vicinity of a specific orbit, by considering perturbed initial conditions, i.e. initial conditions in the vicinity of the initial conditions of this orbit.

We express the system of differential equations (1) as a system of four differential equations of the first order,

$$\dot{x}_i = f_i(x_1, x_2, x_3, x_4), \quad (i = 1, ...4) \tag{2}$$

where $x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$. Let $x_i = x_i(x_{10}, x_{20}, x_{30}, x_{40}; t), \;\; (i = 1, ...4)$ be a solution of the system (2), nonperiodic in general, corresponding to the initial conditions $x_1(0), x_2(0), x_3(0), x_4(0)$. We consider new initial conditions, in the vicinity of these initial conditions, of the form $x_i(0)+\xi_i(0)$, where $\xi_i(0)$ are small. The new solution can be expressed in the form

$$x'_i(t) = x_i(t) + \xi_i(t), \quad (i = 1, ...4)$$

where $\xi(t)$ is the deviation vector between the initial solution $x_i(t)$ and the perturbed solution $x'_i(t)$, at the *same time* t, $\xi(t) = x'_i(t) - x_i(t)$. The behavior of the system in the vicinity of the solution $x_i(t)$ depends on the deviation vector $\xi(t)$.

We assume that the initial perturbation $\xi(0)$ is small, and consequently, for continuity reasons, the deviation $\xi(t)$ should be also small, at least for a finite time interval. For this reason we linearize the system of differential equations (2), to first order terms in the $\xi_i(t)$, by substituting the perturbed solution $x_i'(t)$ into the system (2) and keeping only the first order terms in ξ_i. We obtain the system of *variational equations*,

$$\dot{\xi}_i = \sum_{k=1}^{4} p_{ik}\xi_k, \quad p_{ik} = \left(\frac{\partial f_i}{\partial x_k}\right)_{x_i(t)}, \quad (i = 1, ...4) \tag{3}$$

which describes the evolution of the system (2) in the neighborhood of the orbit $x_i(t)$, to first order terms in the deviations. The partial derivatives are computed for the solution $x_i(t)$. This is a linear system with time dependent coefficients.

The general solution of the linear system (3) is expressed as a linear combination of four linearly independent solutions. In particular, let us consider a 4×4 matrix $\Delta(t)$ whose columns are four linearly independent solutions corresponding to the initial conditions $\Delta(0) = I_4$, where I_4 is the 4×4 unit matrix. This matrix is called *fundamental matrix of solutions* and the general solution of the variational equations is expressed in the form

$$\xi(t) = \Delta(t)\xi(0). \tag{4}$$

A basic property of the matrix $\Delta(t)$ is the Liouville-Jacobi formula (Yakubovich and Starzhinskii, 1975, Jordan and Smith, 1988)

$$\det \Delta(t) = \det \Delta(0) \exp \int_0^t \text{trace}(P) dt, \tag{5}$$

where P is the matrix of the coefficients of the variational equations (3), with elements p_{ij}. This relation gives the change in time of the determinant of the matrix $\Delta(t)$, which describes important properties of the evolution of the system in phase space, as we shall see in the following. Of special importance is the case where $\text{trace}(P) = 0$, because in this case the determinant of the matrix $\Delta(t)$ is constant.

Another important property is that the columns of the matrix $\Delta(t)$ are the partial derivatives of the solution $x_i(x_{10}, x_{20}, x_{30}, x_{40}, t)$ with respect to the initial conditions: The solution $x_i(x_0; t)$ satisfies the system (2),

$$\frac{\partial x_i(x_0, t)}{\partial t} = f_i(x_1(x_0, t), x_2(x_0, t), x_3(x_0, t), x_4(x_0, t)). \ (i = 1, ...4)$$

If we apply to the above equations the operator $\partial/\partial x_{j0}$, $j = 1, ...4$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial x_{jo}}\right) = \sum_k \left(\frac{\partial f_i}{\partial x_k}\right)\frac{\partial x_k}{\partial x_{j0}}, \ (i = 1, ...4) \tag{6}$$

for each x_{j0}. We note that the system (6) is the system of variational equations (3) satisfied by the vector $(\partial x_1/\partial x_{j0}, \partial x_2/\partial x_{j0}, \partial x_3/\partial x_{j0}, \partial x_4/\partial x_{j0})$. In addition, we note that $\partial x_i/\partial x_{j0} = \delta_{ij}$ for $t = 0$, which implies that these vectors, for $j = 1, ...4$, are the four columns of the fundamental matrix of solutions $\Delta(t)$. This means that the fundamental matrix of solutions $\Delta(t)$ is the Jacobian of the solution $x(t)$ with respect to the initial conditions,

$$\Delta(t) = \frac{\partial(x_1, x_2, x_3, x_4)}{\partial(x_{10}, x_{20}, x_{30}, x_{40})}. \tag{7}$$

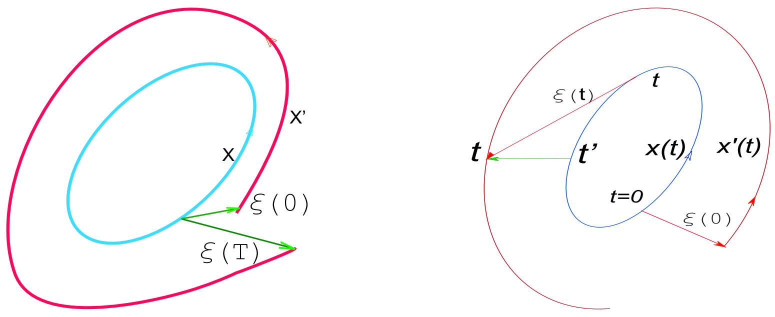

Figure 1: (a) Mapping at integral multiples of the period. (b) Orbital stability: The distance between two points *at the same time* t, $\xi(t) = x'(t) - x(t)$, is not small, but the distance between the points at two *different times*, t and t', $x'(t) - x(t')$, remains bounded.

5 Linear stability of a periodic orbit

The variational equations (3) are a system of four linear differential equations with time dependent coefficients. If the solution $x(t)$ is T-periodic, then the partial derivatives are also T-periodic. In this latter case the system of variational equations is a *linear system with periodic coefficients*. The theory related to the study of such systems is the *Floquet theory* (Jordan and Smith, 1988) and some elements of it will be presented in the following sections.

Existence of a periodic solution

We shall prove that the derivative $\dot{x}_i(t)$ of the periodic solution $x_i(t)$ is a solution of the variational equations. Indeed, the solution $x_i(t)$ satisfies the system (2)

$$\dot{x}_i(t) = f_i(x_1(t), x_2(t), x_3(t), x_4(t)), \quad (i = 1, ...4)$$

and if we apply the operator d/dt we obtain

$$\frac{d}{dt}(\dot{x}_i(t)) = \sum_{j=1}^{4} \left(\frac{\partial f_i}{\partial x_j}\right)_{x_i(t)} \dot{x}_j(t).$$

This is the system of variational equations (3), for the solution $\xi_i = \dot{x}_i(t)$. So we come to the conclusion that *the variational equations that correspond to a T-periodic orbit have always a T-periodic solution, which is the derivative $\dot{x}_i(t)$ of the periodic solution.*

Mapping at integral multiples of the period. The monodromy matrix

Let $x_i(t)$ be a periodic orbit and $x'(t)$ a perturbed orbit, which, to a linear approximation, can be expressed in the form

$$x'_i(t) = x_i(t) + \xi_i(t),$$

where $\xi_i(t)$ is the solution of the variational equations. This latter solution is expressed in the form

$$\xi(t) = \Delta(t)\xi(0), \tag{8}$$

and for $t = T$,

$$\xi(T) = \Delta(T)\xi(0). \tag{9}$$

From this expression we obtain, by induction,

$$\xi(nT) = [\Delta(T)]^n \xi(0). \tag{10}$$

Equations (9) and (10) give the deviation, to a linear approximation, of the perturbed orbit $x'(t)$ from the periodic orbit $x(t)$ after a time interval equal to n times the period T, due to an initial deviation $\xi(0) = x'(0) - x(0)$ at $t = 0$. In fact Equation (10) is a mapping of the initial deviation $\xi(0)$ at integral multiples of the period T (see Figure 1a). This is a linear mapping defined by the matrix $\Delta(T)$. It is clear that the stability of the periodic orbit $x(t)$ depends on the properties of the mapping (10), i.e. on the eigenvalues of the matrix $\Delta(T)$. The matrix $\Delta(T)$ is called the *monodromy matrix*.

Unit eigenvalue of the monodromy matrix

Existence of an Integral of Motion

We shall prove that if the system of differential equations (2) has an integral of motion,

$$G(x_1, x_2, x_3, x_4) = constant,$$

the system of variational equations (3) has a unit eigenvalue: Let $x_i(x_0, t)$ be a T-periodic solution. Since $G(x_1, x_2, x_3, x_4)$ is an integral, we have the relation

$$G(x_1(x_0,t), x_2(x_0,t), x_3(x_0,t), x_4(x_0,t)) = G(x_{10}, x_{20}, x_{30}, x_{40}).$$

We apply to the above relation the operator $\partial/\partial x_{j0}$, and we obtain

$$\sum_{k=1}^{4} \left(\frac{\partial G}{\partial x_k}\right)_t \left(\frac{\partial x_k}{\partial x_{jo}}\right)_t = \left(\frac{\partial G}{\partial x_{jo}}\right).$$

We set now $t = T$ and taking into account that

$$\left(\frac{\partial G}{\partial x_k}\right)_{t=T} = \left(\frac{\partial G}{\partial x_k}\right)_{t=0},$$

due to the fact that $x(t)$ is periodic, we obtain

$$(\Delta^\tau(T) - I)\,\nabla G = 0, \tag{11}$$

where Δ^τ is the transpose of Δ. From this relation we obtain that if $\nabla G \neq 0$ then $\Delta(T)^\tau$ has a unit eigenvalue. Thus finally, we come to the conclusion that *if the dynamical system has an integral of motion, which is not stationary along the periodic orbit, the monodromy matrix $\Delta(T)$ has a unit eigenvalue.*

Existence of a periodic orbit of the Variational Equations

We shall also prove that if the system of variational equations has a periodic solution $\xi(t)$, the monodromy matrix has a unit eigenvalue: We have $\xi(t+T) = \xi(t)$, for any t and consequently, for $t = 0$, $\xi(T) = \xi(0)$. Due to this latter relation, Equation (8) takes the form, for $t = T$, $\xi(0) = \Delta(T)\xi(0)$, and finally

$$(\Delta(T) - I)\,\xi(0) = 0. \tag{12}$$

Thus we come to the conclusion that *if the system of variational equations has a periodic solution, the monodromy matrix has a unit eigenvalue.*

We have proved above that the system of variational equations has the T-periodic solution $\xi(t) = \dot{x}_i(t)$, where $\dot{x}_i(t)$ is the periodic solution corresponding to the variational equations. Thus, the monodromay matrix $\Delta(T)$ has always a unit eigenvalue. The corresponding eigenvector is the vector $\xi(0) = \dot{x}_i(0)$, which is the tangent vector to the periodic orbit, in the phase space.

Vertical stability of planar periodic orbits

In the previous sections we studied the stability of a planar periodic orbit with respect to perturbations of the initial conditions *in the plane*. We study now the stability with respect to perturbations of the initial conditions *perpendicular* to the plane of motion. This type of stability we call *vertical stability* and completes the study of the stability of a planar periodic orbit.

Consider a dynamical system of three degrees of freedom,

$$\begin{aligned}\ddot{x}_1 &= f_1(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3),\\ \ddot{x}_2 &= f_2(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3),\\ \ddot{x}_3 &= x_3 f_3(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3).\end{aligned} \tag{13}$$

This is the form of the differential equations of many gravitational systems, for example of the 3-dimensional restricted three body problem, described in section 11 (see Murray and Dermott 1999, p.67). In this model, a small body with negligible mass moves under the gravitational attraction of two main bodies, which describe Keplerian orbits around their center of mass, under their mutual gravitational attraction. The plane x_1x_2 is the plane of motion of the two main bodies (in the inertial frame) and the small body moves in the three dimensional space $x_1x_2x_3$. It is intuitively clear that if the small body starts from a position in the x_1x_2 plane and its velocity is *in* this plane, then its motion is restricted in the x_1x_2 plane, since the gravitational attraction from the two main bodies is in this plane. This physical property of the motion is described by the special mathematical form of the equations of motion (13) (third equation).

It is easy to verify that the equations (13) admit a planar solution, which we will assume to be periodic: $x_1(t),\ x_2(t),\ x_3(t) = 0$, corresponding to the initial conditions $x_{10},\ x_{20},\ x_{30} = 0,\ \dot{x}_{10},\ \dot{x}_{20},\ \dot{x}_{30} = 0$. We consider now a small perturbation ϵ_3 along the x_3 axis, $x_{10}+\epsilon_1,\ x_{20}+\epsilon_2,\ x_{30} = 0+\epsilon_3,\ \dot{x}_{10}+\epsilon_4,\ \dot{x}_{20}+\epsilon_5,\ \dot{x}_{30} = 0+\epsilon_6$, where ϵ_i are small, and we want to study the behavior of the perturbed solution. We define new variables $x_4 = \dot{x}_1$, $x_5 = \dot{x}_2$, $x_6 = \dot{x}_3$, and a simple calculation shows that the system of variational equations of the system (13), for the periodic solution $x_i(t)$, breaks into two uncoupled systems: a system in the planar displacements $\xi_1, \xi_2, \xi_4, \xi_5$, corresponding to the variational equations of the planar motion, and a system in the vertical displacements (along the x_3 axis) ξ_3, ξ_6. This latter system is

$$\begin{aligned}\dot{\xi}_3 &= \xi_6,\\ \dot{\xi}_6 &= f_{30}(t)\xi_3,\end{aligned} \tag{14}$$

where the function $f_{30}(t)$ is the T-periodic function $f_3(x_1(t), x_2(t), x_3 = 0, x_4(t), x_5(t), x_6 = 0)$, computed for the planar T-periodic solution $x_i(t)$. The system (14) is the system of variational equations for the displacements along the x_3 axis. The vertical stability depends on the eigenvalues $\lambda_5,\ \lambda_6$ of the monodromy matrix $\Delta_2(T)$ of this system.

6 Hamiltonian systems

The gravitational systems are Hamiltonian. For this reason, we study in this section the special properties that a Hamiltonian system has, in addition to the general properties obtained in the previous sections. We start with systems with two degrees of freedom.

A Hamiltonian system is defined by the Hamiltonian function

$$H(x_1, x_2, x_3, x_4),$$

where $x_1,\ x_2$ are the coordinates and $x_3,\ x_4$ the momenta.

The Hamiltonian equations are

$$\dot{x}_1 = \partial H/\partial x_3,\ \ \dot{x}_2 = \partial H/\partial x_4,\ \ \dot{x}_3 = -\partial H/\partial x_1,\ \ \dot{x}_4 = -\partial H/\partial x_2,$$

or

$$\dot{x} = -J\nabla H, \tag{15}$$

where ∇H is a column vector with elements $\partial H/\partial x_i$ and J the 4×4 symplectic matrix

$$J = \begin{pmatrix} 0 & -I_2 \\ +I_2 & 0 \end{pmatrix},$$

where I_2 is the 2×2 unit matrix. Note that $J^{-1} = -J$.

6.1 Variational equations of Hamiltonian systems

The variational equations of a Hamiltonian system (15) have the special form given by

$$\dot{\xi} = -JA\xi, \tag{16}$$

where the elements a_{ij} of the 4×4 matrix A are

$$a_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j}.\ \ (i,j = 1, ...4) \tag{17}$$

Note that the matrix A is symmetric. The system (16) is called a *linear Hamiltonian system.* A complete study of such systems can be found in Yakubovich and Starzhinskii (1975). It is easy to see that it can be expressed in the Hamiltonian form (15) with Hamiltonian

$$H = \frac{1}{2}\xi^\tau A\xi = \frac{1}{2}\sum_{i,j=1}^{4} a_{ij}\xi_i\xi_j.$$

From the relations (16), (17) we can verify that the trace of the matrix of the coefficients of the linear Hamiltonian system (16) is equal to zero. Consequently, due to the general property (5), the determinant of the fundamental matrix of solutions $\Delta(t)$ is equal to unity (see also Meyer and Hall, 1992), $\det\Delta(t) = \det\Delta(0) = 1$. For $t = T$ we obtain

$$\det\Delta(T) = 1,$$

from which we see that the determinant of the monodromy matrix is equal to one.

Using now the results of section 4, we find that

$$\det\Delta(t) = \det\left|\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(x_{10},x_{20},x_{30},x_{40})}\right| = 1. \tag{18}$$

This means that the determinant of the Jacobian of the flow in phase space is equal to one. Consequently, *the volume in phase space is conserved* (Liouville theorem).

The monodromy matrix of a Hamiltonian system is symplectic (see for example, Hadjidemetriou 2006b)

$$\Delta^\tau(T)\ J\ \Delta(T) = J, \tag{19}$$

where the superscript τ means transpose. This is an important property of the monodromy matrix of a Hamiltonian system, which is called the *symplectic* property. Thus we come to the conclusion that *the monodromy matrix of a Hamiltonian system is symplectic.*

The eigenvalues of a symplectic matrix have some special properties. We express the property (19) as

$$\Delta^\tau(T) = J\Delta^{-1}(T)J^{-1},$$

from which we see that the matrix $\Delta^\tau(T)$ is related to the matrix $\Delta^{-1}(T)$ by a similarity transformation. Consequently, they have the same set of eigenvalues. Thus finally, we come to the conclusion that the eigenvalues of $\Delta(T)$ are in reciprocal pairs. In addition, due to the fact that the matrix $\Delta(T)$ is real, they are also in complex conjugate pairs.

From the above we see that the four eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the monodromy matrix have the property

$$\lambda_1\lambda_2 = 1, \quad \lambda_3\lambda_4 = 1.$$

We note that the variational equations correspond to a periodic orbit $x(t)$. So, $\xi(t) = \dot{x}(t)$ is a periodic solution of the variational equations and according to section 4, one eigenvalue is equal to one, $\lambda_1 = 1$. Using now relation $\lambda_1\lambda_2 = 1$, we come to the conclusion that *the monodromy matrix of a Hamiltonian system corresponding to a periodic orbit has a double unit eigenvalue*,

$$\lambda_1 = 1, \quad \lambda_2 = 1.$$

6.2 Stability of Hamiltonian systems

The stability of the periodic orbit depends on the eigenvalues of the monodromy matrix, as we showed in section 5. Instability appears if at least one eigenvalue is outside the unit circle in the complex plane. Since two of the eigenvalues are always equal to unity, it is the other two eigenvalues, λ_3, λ_4, that determine the stability. As we proved, these eigenvalues are reciprocal and also complex conjugate, so they are either *on the unit circle*, or *on the real axis*, one inside the unit circle and the other outside. If they are real, the orbit is unstable, because one of them will be larger than $+1$ or smaller than -1. A special case is $\lambda_3 = \lambda_4 = +1$ or $\lambda_3 = \lambda_4 = -1$.

A remark is necessary at this point for the double unit eigenvalue. In Hamiltonian systems, in general, to the double unit eigenvalue there exists only one eigenvector. This introduces a secular term in the general solution of the variational equations. The two linearly independent solutions corresponding to the double unit eigenvalue are

$$\begin{aligned} \xi^1 &= f_1(t), \\ \xi^2 &= f_2(t) + t\, f_1(t), \end{aligned} \tag{20}$$

where $f_1(t), f_2(t)$ are T-periodic. This implies that the orbit is always unstable, due to the secular term $tf_1(t)$. We will show however that this secular term introduces a time shift only along the perturbed orbit, and thus we have *orbital stability*, provided that the other two eigenvalues are on the unit circle: Taking into account that $\xi^1(t) = \dot{x}(t)$, where $x(t)$ is the periodic solution corresponding to the unit eigenvalue, we note that the perturbed orbit has a term $\epsilon\xi^2$ and the corresponding part of the solution is expressed as $x'(t) = x(t) + \epsilon t\dot{x}_i(t) + \epsilon f_2(t)$ and, to a linear approximation in ϵ,

$$x'(t) = x(t+\epsilon) + \epsilon f_2(t + \epsilon t).$$

Thus, if we define a new time $t' = t + \epsilon t$, we obtain (see Figure 1b)

$$x'(t) - x(t') = \text{bounded}.$$

Thus we come to the conclusion that the secular term introduces a phase shift only along the orbit. This means that the two orbits, $x(t)$ and $x'(t)$, considered as geometrical curves, are close to each other. In this case we say that we have *orbital stability*, provided that the eigenvalues λ_3, λ_4 are on the unit circle and consequently the corresponding solution is bounded.

For the other two eigenvalues λ_3, λ_4 we have the solutions

- eigenvalues real and positive: $\xi^{3,4} = f_{3,4}(t)\, e^{\pm\alpha t}$,

- eigenvalues real and negative: $\xi^{3,4} = f_{3,4}(t)\, e^{\pm\alpha t}\, e^{\pm i\pi t/T}$,

- eigenvalues complex conjugate on the unit circle $\xi^{3,4} = f_{3,4}(t)\, e^{\pm i\beta t}$,

where α, β are real and the functions $f_3(t)$, $f_4(t)$ are T-periodic. The exponent α is called the *characteristic exponent*. The general solution in the vicinity of the periodic solution is a linear combination of the above four solutions ξ^1, ξ^2, ξ^3, ξ^4.

The stability criteria can be obtained from the elements of the monodromy matrix as follows: The eigenvalues are the roots of the characteristic equation of $\Delta(T)$ and consequently

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \text{trace}\Delta(T),$$

$$\lambda_1\lambda_2\lambda_3\lambda_4 = \det\Delta(T) = 1.$$

Taking into account that $\lambda_1 = \lambda_2 = 1$ we find that the two nonzero eigenvalues λ_3, λ_4 are the roots of the quadratic equation

$$\lambda^2 - K\lambda + 1 = 0,$$

where

$$K = \text{trace}\Delta(T) - 2.$$

The stability depends on the value of K, which is called the *stability index*. Note that the stability index depends only on the trace of the monodromy matrix.

Asymptotic stability never appears, because it is not possible for the eigenvalues λ_3, λ_4 to be *both* inside the unit circle. This is also a consequence of the fact that the volume in phase space is conserved.

Let us assume that a periodic orbit is stable, which implies that the eigenvalues λ_3, λ_4 are on the unit circle and we assume that they are not equal to $+1$ or -1. If a parameter varies, then the eigenvalues λ_3, λ_4 are restricted to move *on the unit circle*, because they must be both inverse, $\lambda_3 = 1/\lambda_4$ and complex conjugate. Consequently, the stability is conserved. However, if λ_3, λ_4 meet at the points $+1$ or -1, then it is possible for them to go outside the unit circle and thus generate instability. For this reason the orbits with $\lambda_3 = \lambda_4 = \pm 1$ are called *critical* as far as the stability is concerned. This is the mechanism by which instability is generated at the 3:1 resonance in the asteroid belt, where the eigenvalues λ_3, λ_4 meet at the point -1 (Hadjidemetriou 1982).

7 Extension to three or more degrees of freedom

All the above results concerning the eigenvalues and the stability of a periodic orbit, obtained for Hamiltonian systems with two degrees of freedom, can be easily extended to three or more degrees of freedom.

In a Hamiltonian system with n degrees of freedom the monodromy matrix is a $2n\times 2n$ symplectic matrix, and the eigenvalues are in reciprocal pairs (because of the symplectic property), and in complex conjugate pairs (because the elements of the matrix are real).

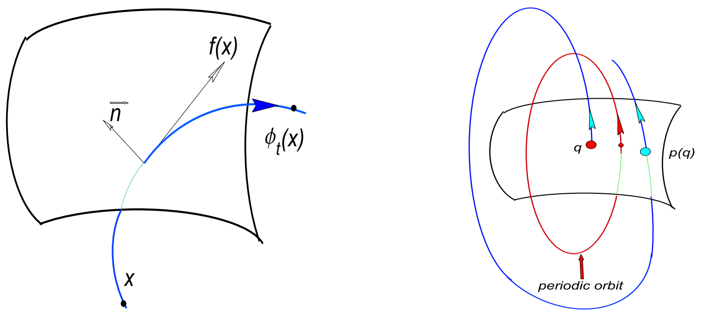

Figure 2: (a) The surface of section, (b) The Poincaré map on a surface of section

There is always a unit pair of eigenvalues, due to the existence of the energy integral $H = h =$ constant (see section 5). For the other eigenvalues we have the following possibilities :

- Complex conjugate, on the unit circle, $e^{\pm i\phi}$: STABILITY.

- Real, on the real axis, in reciprocal pairs (positive or negative), λ, $1/\lambda$: INSTABILITY.

- Complex, inside and outside the unit circle, in reciprocal and in complex conjugate pairs, $Re^{i\phi}$, $Re^{-i\phi}$, $R^{-1}e^{i\phi}$, $R^{-1}e^{-i\phi}$: COMPLEX INSTABILITY.

Note that in three, or more, degrees of freedom we have a new type of instability, the *complex instability*, which cannot appear in systems with two degrees of freedom.

8 The Poincaré map

This a very useful method in the study of the evolution of a dynamical system. By the Poincaré map we transform the continuous flow in the n-dimensional phase space of a dynamical system to an equivalent discrete flow (map) in a phase space of $(n-1)$-dimensions (or $(n-2)$-dimensions for Hamiltonian flows).

Consider the dynamical system in $\mathcal{R}^n$: $\dot{x} = f(x)$, where x, $f(x)$: vectors in $\mathcal{R}^n$ and $\phi_t(x)$ is the flow. We consider the surface of section

$$\Sigma \subset \mathcal{R}^n : \quad (n-1) - \mathrm{dim}$$

and we assume that the flow is transverse: The velocity vector of the flow is not tangent to this surface (Figure 2a): $f(x) \cdot n(x) \neq 0$, where $n(x)$ is the normal unit vector to the surface.

The Poincaré map is defined as:

$$q \rightarrow p(q),$$

$$p(q) = \phi_\tau(q),$$

where q is the position on the surface of section at a $t = 0$ and $p(q)$ is the position on this surface at the next intersection at $t = \tau$, (Figure 2b).

The following properties apply:

- The vector $p(q)$ defines accurately the state.

- The vector $p(q)$ is a continuous function of q.

- If $\bar{x}(t)$ is a T-periodic orbit, the corresponding Poincaré map is a *fixed point*, maybe multiple (it repeats itself after several intersections) as seen in Figure 2b (for the simple case).

9 Poincaré map in Hamiltonian systems

In this case the differential equations of motion are the canonical equations

$$\dot{q} = \partial H/\partial p, \quad \dot{p} = -\partial H/\partial q. \quad q, p \subset \mathcal{R}^n$$

Let us consider the $(2n-2)$-dimensional surface of section Σ, defined as

$$H = h, \quad f(q,p) = 0 \quad \text{(for example } q_2 = 0\text{)}.$$

The continuous Hamiltonian flow in the $2n$-dimensional phase space is transformed to an equivalent discrete flow (map), on a $(2n-2)$-dimensional surface of section. In addition to the general properties of the Poincaré map mentioned above, we also have the properties:

- The Poinaré map of a Hamiltonian flow is symplectic.

- The stability of the fixed points of the Poincaré map is the same as the stability of the corresponding periodic orbit. We have the same set of eigenvalues, except the double unit eigenvalue which corresponds to the periodic orbit (the phase space now has two dimensions less). Note that this double unit eigenvalue is responsible for the phase shift along the perturbed orbit, which implies that this shift is eliminated by the Poincaré map. Thus, in the Poincaré map, the stability of the fixed point (periodic orbit) means *orbital stability.*

Hamiltonian systems with two degrees of freedom

The Poincaré map is particularly useful in systems with two degrees of freedom, where the phase space is four dimensional and the Poincaré map is in a two dimensional phase space. This makes the study very easy because we present the evolution of the system in a two dimensional space, where we can have a direct view.

We define the variables x_i as $x_1 = q_1$, $x_2 = q_2$, $x_3 = p_1$, $x_4 = p_2$. The energy integral is $H(x_1, x_2, x_3, x_4) =$ constant. We consider the surface of section

$$H(x_1, x_2, x_3, x_4) = h, \quad x_2 = 0, \text{ with } \; x_4 > 0.$$

The map is in the space $x_1\, x_3$. The consecutive points of the map may lie on a smooth curve, called *invariant curve* (ordered motion), or be scattered (chaotic motion).

Let us assume that another first integral of motion exists, in addition to the energy integral $H = h$ =constant:

$$G(x_1, x_2, x_3, x_4) = c.$$

Then all the consecutive points of the map lie on smooth invariant curves: Let (x_1, x_3) be a point of the map on the two-dimensional surface of section. We have $x_2 = 0$ and x_4 is expressed in terms of x_1, x_3, through the energy integral $H = h$, as $x_4 = x_4(x_1, x_2 = 0, x_3)$. The points x_1, x_3 satisfy also the integral $G(x_1, x_2 = 0, x_3, x_4(x_1, x_2 = 0, x_3)) = c$, or

$$F(x_1, x_3) = 0,$$

which implies that the consecutive points (x_1, x_3) of the map lie on a smooth curve.

10 The gravitational two-body problem

The differential equations of the *relative motion* of two point masses m_1, m_2 are given by

$$\ddot{\vec{r}} = -\frac{GM}{r^2}\,\vec{e}_r,$$

where $M = m_1 + m_2$. The orbit is a conic section and in particular, for bounded motion, it is a Keplerian, elliptic orbit. The two bodies describe in the inertial frame two similar orbits around their common center of mass, whose dimensions are inversely proportional to their masses. This is one of the few integrable problems in nature. Its importance is that many real systems, as for example the asteroid problem, or the planetary systems, can be considered as perturbed two-body problems. For this reason it is important to know the basic properties of this simple two body problem and then study the evolution as a perturbation is applied.

The two-body problem in a rotating frame

Consider a body, S, with mass m_1 and a second body, J, with mass m_2, which describe circular orbits around their common center of mass. We define a rotating frame of reference xOy, whose $x-$axis is the line SJ, the origin is at their center of mass and the xy plane is the orbital plane of the circular motion of the these two bodies (Figure 3b).

Our aim is to study the motion of a massless body A in the rotating frame xOy, under the gravitational attraction of S and J. We start with a zero mass of the body J, $m_2 = 0$. In this approximation, the second body J is used only to define the rotating frame xOy, which rotates with constant angular velocity n'. Evidently, the motion of the body A is a Keplerian orbit, presented in the rotating frame. We shall give later mass to the body J, thus perturbing the Keplerian orbit of A.

The Hamiltonian function H that describes the unperturbed motion of A, in polar coordinates, r, ϕ (in the rotating frame), is

$$H_0 = \frac{p_r^2}{2} + \frac{p_\phi^2}{2r^2} - n'p_\phi - \frac{GM}{r}. \tag{21}$$

The momenta are $p_r = \dot{r}$ and $p_\phi = r^2(\dot{\phi} + n')$. Note that the angle ϕ is an ignorable coordinate and consequently, in addition to the energy integral $H_0 = h$ =constant, we also have the angular momentum integral p_ϕ =constant.

The orbit of the body A (in the inertial frame) is a Keplerian orbit, which we assume to be elliptic. In terms of the elements of the orbit, the Hamiltonian (in the rotating frame) and the angular momentum are expressed as

$$H_0 = -\frac{GM}{2a} - n'p_\phi, \quad p_\phi = \sqrt{GMa(1-e^2)}.$$

<u>Circular orbits</u>

In the rotating frame there exist circular orbits of the body A with an arbitrary radius r_0, which correspond to the periodic solution

$$r = r_0,\ p_r = 0,\ \dot{\phi} = n - n',\ p_\phi = nr_0^2,$$

where

$$n = p_\phi / r_0^2$$

is the angular velocity of the circular orbit (in the inertial frame). The following relations also hold:

$$\frac{p_{\phi 0}^2}{r_0^3} = \frac{GM}{r_0^2} \quad \rightarrow \quad \frac{GM}{r_0^3} = n^2.$$

The period of the circular orbit in the rotating frame is

$$T = \frac{2\pi}{(n-n')}.$$

A circular orbit in xOy is a Keplerian orbit in the inertial frame, with semi major axis $a = r_0$ for any r_0. Consequently, a *family of circular periodic orbits* exists, which evidently is symmetric with respect to the x-axis. The parameter along the family is the semi major axis a (the radius), or the angular velocity (in the inertial frame) n. This family is represented by a smooth curve, in the space $h - r_0$, given by

$$-\frac{GM}{2a} - n'\sqrt{GMa} = h,$$

obtained from the energy integral for $e = 0$ (Figure 3a). Note that from the energy integral $H_0 = h$ we can obtain the value of $\dot{y}_0$, which together with x_0 define exactly the initial state, because $y_0 = 0$ and $\dot{x}_0 = 0$, due to the symmetry of the orbit with respect to the $x-$axis.

Elliptic orbits

An elliptic orbit in the inertial frame is periodic in the rotating frame only if it is resonant:

$$\frac{n}{n'} = \frac{p}{q} = \text{rational},$$

which means that the semi major axis must be given by

$$\frac{(GM)^{1/2}a^{-3/2}}{n'} = \frac{p}{q}.$$

Let us consider now a particular resonance p/q, which means that we keep fixed the semi major axis $a_{p/q}$. The orbit is resonant periodic for *any* eccentricity e, so a *family of elliptic periodic orbits* exists, with the eccentricity as a parameter along the family. There is however another parameter, defining the *orientation* of the elliptic orbit, which is the angle ω of the line of apsides with a fixed direction. In general, an elliptic orbit is not symmetric with respect to the rotating x-axis, contrary to the circular orbits, which are symmetric.

In the space $h - r_0$, where $r_0 = a_{p/q}(1 - e)$ is the pericenter distance ($r_0 = x_0$), an elliptic family is represented by a smooth curve (Figure 3a), given by the energy integral

$$-\frac{GM}{2a} - n'\sqrt{GMa(1 - e^2)} = h.$$

The value of a is fixed, equal to the corresponding resonance and the eccentricity is a parameter along the family. Note that this presentation is not unique: a point on the elliptic family represents *all* the elliptic resonant orbits with the same eccentricity, but arbitrary orientation ω. An elliptic periodic orbit in the rotating frame is also periodic in the inertial frame. The resonant families of periodic orbits bifurcate from the family of circular orbits, at those points corresponding to the resonant values of the radius $a = a_{p/q}$.

Note that along the circular family the value of the semimajor axis varies, and consequently the ratio n/n' varies and passes through resonant (rational) values. It is at these points that we have a bifurcation to an elliptic family. Evidently, all the circular and the elliptic orbits are stable, as they are Keplerian orbits.

In the following we study how the above mentioned nice picture of the families of periodic orbits change, when a perturbation is applied, and how instabilities and chaotic regions appear. We consider two cases: the restricted 3-body problem, both circular and elliptic, and the planetary problem, including our Solar System and the extrasolar planetary systems. In all these cases the Hamiltonian is expressed in the form

$$H = H_0 + \epsilon H_1, \tag{22}$$

where H_0 is the integrable Hamiltonian of the two body problem.

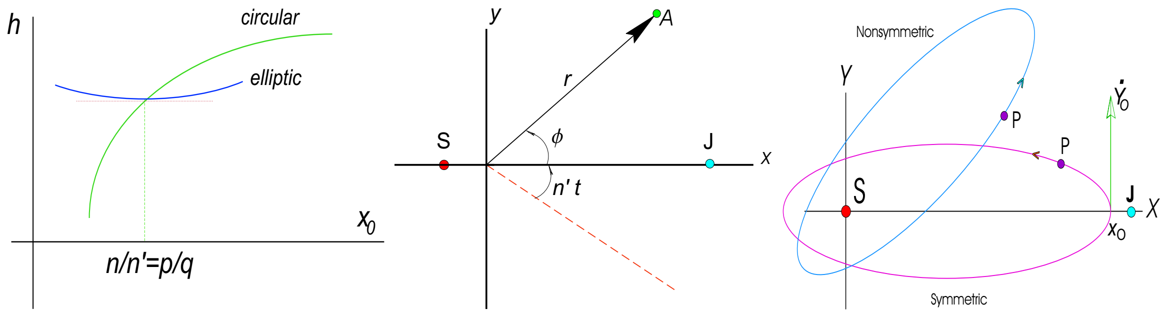

Figure 3: (a) The families of circular and of resonant elliptic periodic orbits in the unperturbed problem. The tangent to the elliptic family at the bifurcation point is parallel to the x-axis. (b) The rotating frame of the restricted problem. The mass, μ, of the second body, J, is equal to zero in the unperturbed problem, and is used only to define the rotating frame. In this case, the first body, S, is at the origin. (c) Two elliptic orbits of the small body, in the inertial frame, for $\mu = 0$. One is symmetric, corresponding to $\omega = 0^0$, and the other is asymmetric, corresponding to an arbitrary value of ω.

11 Application to the Solar System

11.1 A global view of the families of periodic orbits

We consider the Sun and Jupiter revolving around their common center of mass in *circular orbits* or in *elliptic orbits* and a third body, with negligible mass, moving under the gravitational attraction of these two bodies. We make the approximation that the small body does not affect the motion of the two main bodies, Sun and Jupiter, which we will call *primaries*. This model is the *restricted three-body problem* and an extended study is in the books of Szebehely (1967) and Roy (1982). This is a non integrable system, which is a good model to study the motion of a small body in our Solar System, for example an asteroid, a comet, or a small body in the Kuiper belt, at the edge of our Solar System (Jupiter is replaced by Neptune in this latter case).

Let us start with the study of the motion of an asteroid in the asteroid belt, a zone of small bodies between the orbits of Mars and Jupiter. For this reason we define a *rotating* frame xOy, with O the center of mass of the Sun and Jupiter and the x-axis along the line Sun-Jupiter (Figure 3b). We start our study with the simplest case, considering that the orbits of the Sun and of Jupiter are circular (circular restricted three body problem). In this case the system xOy rotates with *constant* angular velocity n'. We start with planar motion of the asteroid and then extend the study to motion in space. Based on this model, we extend our study by assuming that the orbits of the Sun and Jupiter are elliptic.

In our study we normalize the units of length, mass and time by the relations

$$G = 1, \ (m_{sun} = 1 - \mu, \ m_J = \mu), \ r_0 = 1, \quad \text{which implies} \quad n' = 1,$$

where G is the gravitational constant, r_0 is the radius of the circular orbit of Jupiter around the Sun, and μ, the mass of Jupiter, is considered a small parameter, of the order of 10^{-3} in our case. The Hamiltonian for the motion of the small body is of the form (22), with $\epsilon = \mu$, where H_0 is the Hamiltonian (21) of the two body problem in the rotating frame.

Planar orbits

Let us start with the unperturbed problem, $\mu = 0$, for planar motion, which is the two-body problem in the rotating frame. As we mentioned in the previous section, there exists a family of

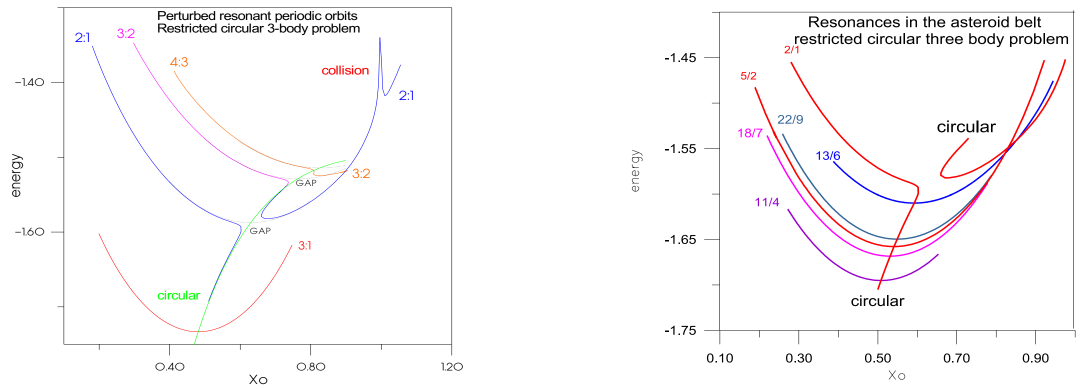

Figure 4: (a) The circular family and the bifurcation to resonant elliptic families at the resonances 2/1, 3/2, 4/3,.. for $\mu = 0.000954786$. The orbits on the elliptic families are symmetric, corresponding to $\omega = 0^0$ or $\omega = 180^0$. These are the only orbits that were continued to $\mu \neq 0$. All the other unperturbed orbits, corresponding to any other value of ω (see Figure 3c), did not survive the perturbation, as a consequence of the Poincaré-Birkhoff fixed point theorem. (b) A closer look at some resonant families, for different higher order resonances.

circular orbits, along which the resonance n/n' varies (n is the mean motion (angular velocity) of the orbit of the asteroid) and families of resonant *elliptic* periodic orbits, which bifurcate from the circular family at all the resonant circular orbits $n/n' = p/q$, as shown schematically in Figure 3a. Evidently, all the orbits of these families are stable, as they are Keplerian, elliptic, orbits. We study now how all these families evolve and where instabilities appear, when $\mu > 0$, i.e when the gravitational effect of Jupiter in taken into account. A complete analysis is given in Hadjidemetriou (2006b).

As we mentioned before, there is an infinite set of resonant periodic orbits along the circular unperturbed family. The continuation of the above mentioned circular family from $\mu = 0$ to $\mu > 0$ and the generation of instabilities depends on the resonances that appear on this family. These resonances belong to three topologically different cases, as far as the continuation to $\mu > 0$ is concerned. These are the cases (i) $n/n' = (\nu + 1)/\nu$, (ii) $n/n' = (2\nu + 1)/(2\nu - 1)$, $(\nu = 1, 2, 3, ...)$ and (iii) all other resonances.

-(i) All the circular orbits that are not at the resonance $n/n' = 2/1$, $3/2$, ..., are continued as nearly circular orbits in the rotating frame. The resonant circular orbits $n/n' = 2/1$, $3/2$, ... are not continued as periodic orbits in the rotating frame. At these resonances, a gap appears and the single unperturbed family of circular orbits breaks into a set of disconnected families of periodic orbits. From these gaps we have a bifurcation of two families of resonant elliptic periodic orbits (Figure 4a). The stability of the circular orbits at $\mu = 0$ is preserved, except at the resonances 3/1, 5/3, .., as we explain below. The resonant elliptic families at the 2/1, 3/2, ... resonances may be stable or unstable, depending on the phase (perihelion or aphelion at $t = 0$) and other factors (for example, close approaches).

-(ii) At the circular orbits at the resonances $n/n' = 3/1$, $5/3$, ... the continuation to nearly circular orbits is possible, but the stability is destroyed. A small unstable region appears at these resonances, on the family of circular orbits. At the critical points, at the two ends of this unstable region, we have a bifurcation of two families of symmetric resonant elliptic periodic orbits, which differ in phase. One is stable and the other is unstable (Figure 13a).

-(iii) In all other resonances on the circular family, for example 5/2, 4/1, 7/3, ... the circular orbits are continued as nearly circular orbits and in addition the stability is preserved. At these

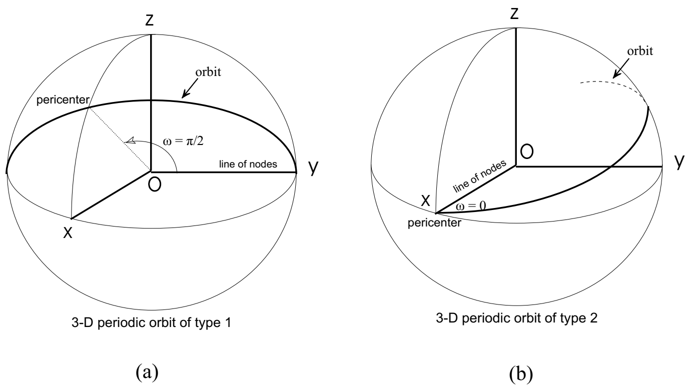

Figure 5: The two different symmetries in three-dimensional orbits: (a) Type 1. (b) Type 2.

points we have a bifurcation of two families of symmetric resonant elliptic periodic orbits which differ in phase (see Figure 4b). In this case also, one family is stable and the other unstable (but the stability may change along a family).

A remark is necessary at this point for the families of elliptic periodic orbits for $\mu > 0$. The elliptic unperturbed resonant families, shown in Figure 3a, are two parametric, with the eccentricity e and the angle of apsides ω as the two parameters. The eccentricity increases along the elliptic family, starting from zero values, but to a fixed eccentricity there corresponds an infinity of values of ω (Figure 3c). What happens to this two-parametric family of unperturbed periodic orbits as $\mu > 0$? It is proved (Hadjidemetriou 2006b) that, for a fixed eccentricity, out of the infinite set of periodic orbits, for different omegas, only a finite, even, number survive (usually just two), half stable and half unstable. This is a consequence of the Poincaré-Birkhoff fixed point theorem (see Arnold and Avez 1968). This theorem refers to perturbed twist mappings: In the unperturbed case there exist resonant invariant curves where *all* points are fixed points, so that on this unperturbed invariant curve there exists an infinite number of fixed points. As soon a a perturbation is applied, only a finite number of fixed points survive, half of them stable and half unstable.

Thus, all the elliptic resonant families are monoparametric families along which the eccentricity increases, starting from zero values. In most cases the orbits are symmetric with respect to the rotating x-axis ($\omega = 0$ or $\omega = \pi$) and the eccentricity can be considered as a parameter. Some of these families are stable and others are unstable. The stability depends on the phase, i.e. on whether the asteroid is at perihelion or aphelion when it crosses the x-axis, but also on other factors as, for example, to a close encounter with Jupiter. Along a family of resonant elliptic periodic orbits the resonance is almost constant. A global picture of the circular and the elliptic families is shown in Figures 4a,b.

Three-dimensional orbits in the circular model

We study now three-dimensional periodic orbits in the model of the circular restricted problem. These families bifurcate from the planar families at those points which are critical with respect to the vertical stability. It is only at these points that the vertical deviations $\xi_3(t)$ of a perturbed orbit, given by the variational equations (14), have a period equal to the period of the planar periodic orbit. We remark at this point that along a resonant family of elliptic periodic orbits, the vertical stability index is very close to critical. (It is exactly critical on the unperturbed elliptic family). Depending on the particular resonance, such a critical point may or may not exist.

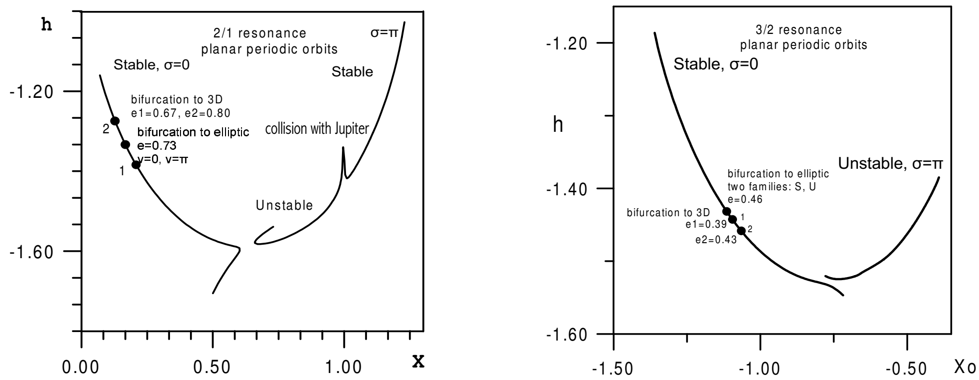

Figure 6: (a) The two elliptic families of resonant periodic orbits at the 2/1 resonance and the bifurcation points to three-dimensional orbits, at $e = 0.67$ and $e = 0.80$ and also to the elliptic model, at $e = 0.72$ on the stable family. (b) The resonant elliptic periodic family at the 3/2 resonance and the bifurcation points to three-dimensional orbits, at $e = 0.39$ and $e = 0.43$ and also to the elliptic model, at $e = 0.46$.

The three-dimensional periodic orbits are, in general, symmetric and there exist two types of symmetry. Their initial conditions are given below and are shown in Figure 5:

Type 1: $x_{20} = \dot{x}_{10} = \dot{x}_{30} = 0, \quad x_{10}, x_{30}, \dot{x}_{20} \neq 0.$

Type 2: $x_{20} = x_{30} = \dot{x}_{10} = 0, \quad x_{10}, \dot{x}_{20}, \dot{x}_{30} \neq 0.$

In Figure 6a we present the two families of 2/1 resonant planar periodic orbits, corresponding to $\mu = 0.000954786$. The stable family, for $\sigma = 0$ ($\omega = 0$), corresponds to the case where the asteroid is at perihelion at $t = 0$. The other family, for $\sigma = \pi$ ($\omega = \pi$), corresponds to position of asteroid at aphelion at $t = 0$ and starts as unstable up to the point where we have a collision with Jupiter. After that point the family continues, for larger eccentricities, as stable. On the stable family, corresponding to perihelion at $t = 0$, there exist two critical points, at high eccentricities, $e = 0.67$ and $e = 0.80$, as far as the vertical stability is concerned, as shown in Figure 6a. From each one of these two points we have a bifurcation of a family of three-dimensional periodic orbits. One family, starting from $e = 0.67$, belongs to type 1 and is stable (Figure 7a), while the other family, starting from $e = 0.80$, belongs to type 2 and is unstable (Figure 8a). Typical three-dimensional periodic orbits on these two families are shown in Figures 7b and 8b. Also, on the stable family in Figure 6a, there exists a bifurcation point, at $e = 0.73$, to two families of periodic orbits of the elliptic problem (see Figure 9), as explained in the next paragraph. Note that the bifurcation points to three dimensional periodic orbits and to the elliptic problem exist only on the stable family. On the unstable families such bifurcation points do not exist. A remark is necessary at this point: In the unperturbed case ($\mu = 0$) all points on the families of elliptic periodic orbits are critical as far as the vertical stability is concerned and also critical as far as the bifurcation to the elliptic problem is concerned (period equal to 2π). The existence or not of such critical points when $\mu \neq 0$ depends on the particular resonance. In the case we studied here, only the above critical points appeared. In other resonances the situation may be quite different.

A similar situation exists for the 3/2 resonance, as shown in Figure 6b.

Families in the elliptic restricted problem

Families of resonant periodic orbits in the case where the orbits of the Sun and Jupiter are *elliptic*, with eccentricity e_J (elliptic restricted three-body problem) exist, which bifurcate from the

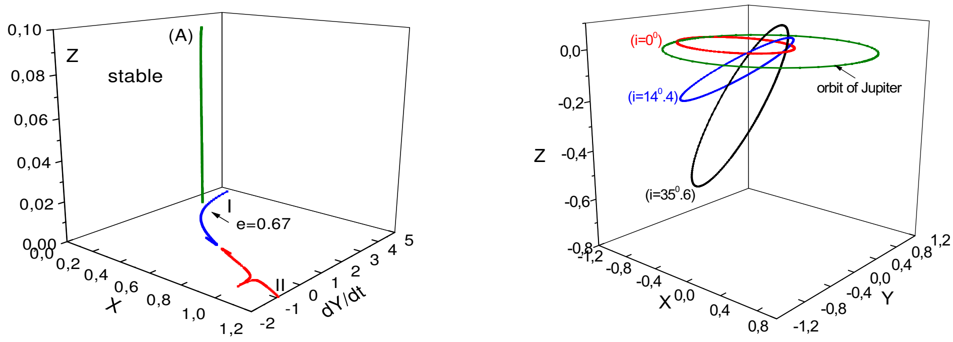

Figure 7: (a) The stable family of three-dimensional periodic orbits bifurcating at $e = 0.67$ (b) Three-dimensional periodic orbits on the stable family (type 1).

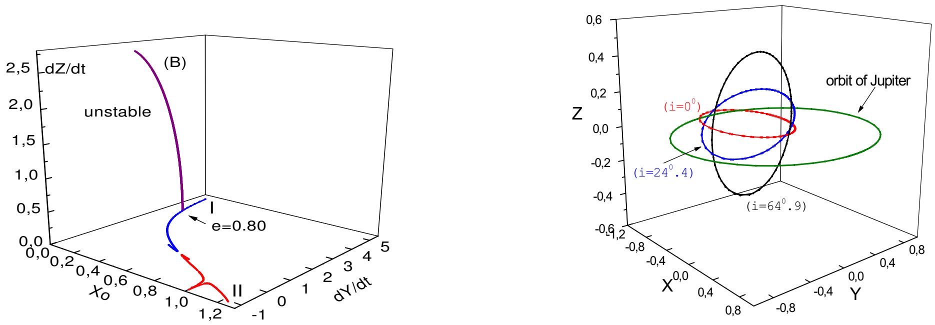

Figure 8: (a) The unstable fa s bifurcating at $e = 0.80$.
(b) Three-dimensional period .

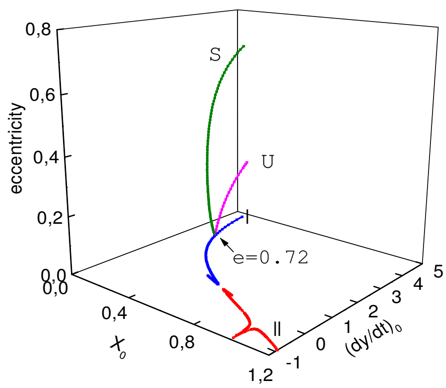

Figure 9: The two families of periodic orbit of the elliptic restricted three-body problem, bifurcating from the 2/1 resonant family at $e = 0.72$. One is stable and the other unstable.

families of the circular model, either the circular family or the elliptic families. The bifurcation can take place only at those points where the period of the periodic orbit on the families of the circular model is equal to the period of Jupiter (or a multiple of it). For a fixed value $e_J > 0$ the periodic orbits are isolated. We obtain a family by varying e_J.

Continuation from the family of circular orbits: Let us start from the unperturbed circular family. The period of a circular orbit ($\mu = 0$) is $T = 2\pi/(n-n')$ in the rotating frame and the period of Jupiter is $T_J = 2\pi/n'$, where n' is its mean angular velocity. We have $T = T_J/(n/n' - 1)$. At the resonance $n/n' = p/q$ we have $T = T_J q/(p-q)$ and if this orbit is described $p-q$ times, the period T^* of this orbit is an integral multiple of T_J, $T^* = q\,T_J$. If $n/n' \neq 2/1, 3/2, ..$ the region around this resonant orbit is continued to $\mu > 0$, as mentioned above. On the continued circular family there exists an orbit which, if described $p-q$ times, has a period exactly equal to $q\,T_J$. This means that a bifurcation from the circular family $\mu > 0$ to a family of the elliptic problem can take place close to a resonance. This is the case with the 3/1 resonance (Figure 13a).

Continuation from the family of nearly elliptic orbits: Consider a family of $n/n' = p/q$ resonant elliptic periodic orbits of the circular planar problem, for $\mu = 0$. The period all along the family is constant, and according to the above, if the orbits of the family are described $(p-q)$ times, the period is $T^* = q\,T_j$. This family is continued, when $\mu > 0$, to two families of elliptic periodic orbits, differing in phase. Along each family the eccentricity increases, starting from zero values. For continuity reasons, the (multiple) period along the continued family is close to $q\,T_J$. If at a certain point, corresponding to a value e of the eccentricity, it happens to be *exactly* equal to $q\,T_J$, then a bifurcation to the elliptic problem can take place. Two families of periodic orbits exist, along which the eccentricity of Jupiter increases. For a fixed eccentricity of Jupiter, for example $e_J = 0.048$, only two *isolated* periodic orbits exist. The above mentioned two families differ in the initial phase of Jupiter on its elliptic orbit at $t = 0$, i.e. if it is at perihelion or aphelion. In general, one family is stable and the other is unstable.

The numerical computations have shown that in certain resonances, for example 2/1, 3/2, 3/1, such bifurcation points do exist, at quite large values of the eccentricity (see Figures 6a,b for the 2/1 and 3/2 resonances and 13a for the 3/1 resonance). But in other resonances, for example 7/3, such bifurcation points do not exist. This plays an important role on the topology of the phase space close to a particular resonance, because the existence of a resonant periodic orbit/fixed point of the Poincaré map, determines the topology of the phase space. The non existence of periodic orbits in a region implies that the phase space is smooth and ordered regions exist. A systematic study along these lines has been made by Tsiganis et al. (2000, 2000a, 2000b). We present in Figure 9, as an example, two resonant families of the elliptic problem, at the 2/1 resonance, one stable and one unstable. These two orbits bifurcate from the point on the stable branch of the 2/1 resonant family of the circular problem, at $e = 0.72$, as shown in Figure 6a.

11.2 Generation of chaos at the unstable periodic orbits

Let us consider the simplest model, the circular restricted three-body problem, and study the topology of the phase space at the 2/1 and the 3/2 resonances, using as a guide the families of periodic orbits as presented in Figures 4a and 6a,b. We compute the Poincaré map on the surface of section $y = 0$, $H = h$, for different values of the energy h. These energy levels can be visualized by considering lines parallel to the x_0 axis in Figures 4a (or 6a,b) at different values of h. Note that these lines intersect the circular family and the resonant families, and these intersections correspond to the fixed points of the Poincaré map. For a better understanding of the physics, we mark on each map the value of the eccentricity of the stable resonant fixed point, instead of the energy h, since along the family the eccentricity increases. In Figure 10 we present

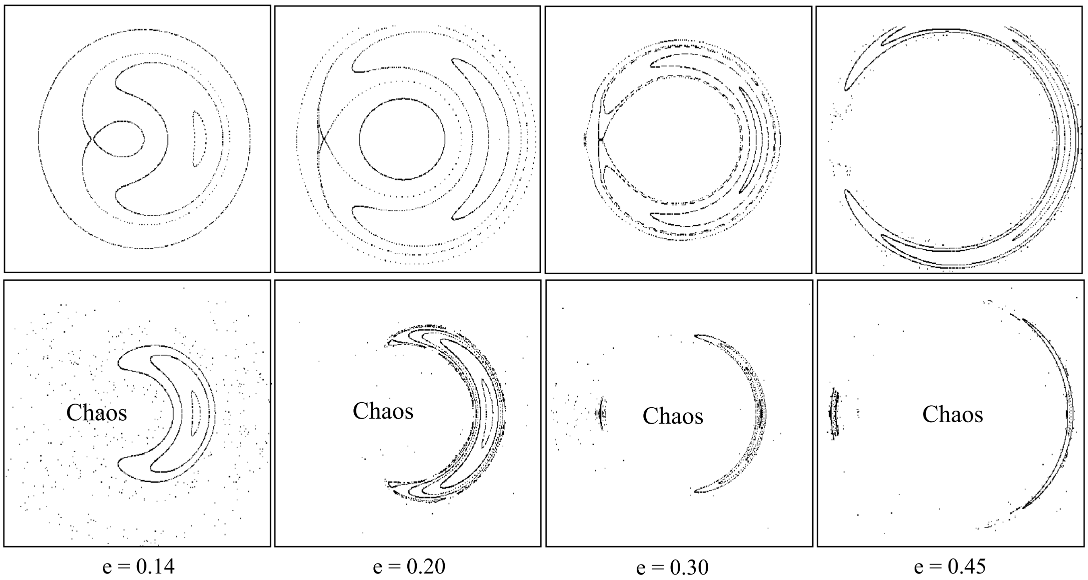

Figure 10: The Poincaré map at the 2/1 resonance (upper row) and the 3/2 resonance (lower row) for different energy levels, presented here by the corresponding eccentricity e of the resonant stable fixed point on the elliptic family. The stable and the unstable fixed points are clearly seen. The generation of chaotic motion at the unstable fixed points is evident. Note that the chaotic motion starts from the unstable fixed points.

several surfaces of section, at different energy levels, corresponding to different eccentricities, at the resonances 2/1 and 3/2. The fixed points corresponding to the circular periodic orbit (in the middle of the diagram) and to the stable and unstable resonant periodic orbits are clearly seen. The stable fixed points are surrounded by islands (closed invariant curves) of ordered motion, while the mapping close to the unstable fixed points is hyperbolic. Chaotic motion starts at these unstable points as the eccentricity increases as we move along the family. The chaotic orbits appear as scattered points, in contrast to the regular orbits, which are represented by smooth invariant curves. This phenomenon is stronger at the 3/2 resonance.

Note that the topology of the phase space, on the Poincaré map, is critically determined by the position of the fixed points and their stability character. This is the reason that the knowledge of the basic families of periodic orbits is so important for the study of the dynamics of the system.

11.3 Asteroid motion close to a resonance

It is known that in the region between the orbits of Mars and Jupiter there exists a zone of small bodies, revolving around the Sun, called *asteroid belt*. It has been observed that the distribution of these bodies is not smooth, but gaps exist at several resonances between the mean motion of the asteroid and Jupiter, the famous Kirkwood gaps (see Figure 12). The explanation of these gaps was an open question for many decades, and their existence was explained by realizing that the motion at the 3/1 resonance (and in many other resonances) is chaotic and consequently an asteroid could not stay in this region for a long time. The first study was made by Wisdom (1982, 1893, 1985). The study was based on the construction of a symplectic mapping model, by making use of the averaged Hamiltonian of the elliptic restricted three-body problem at the 3/1 resonance. It was shown that, due to the existence of chaos at this region, the eccentricity of an asteroid that starts its motion in a nearly circular orbit undergoes sudden jumps, after a

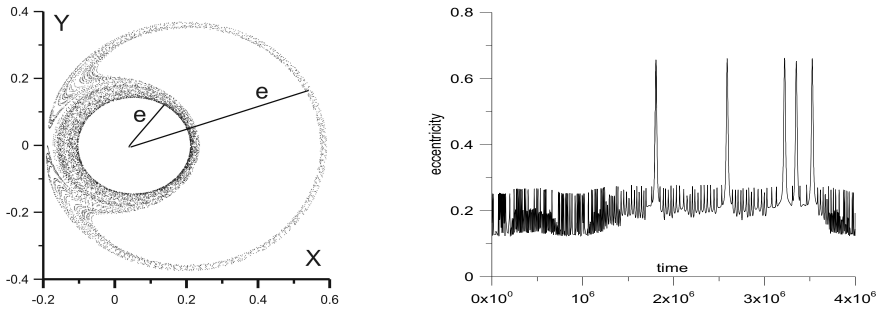

Figure 11: (a) The mapping, in the variables $X = e\cos(\sigma) - Y = e\sin(\sigma)$ for the motion of an asteroid at the 3/1 resonance. (b) The evolution of the eccentricity. Chaotic jumps of the eccentricity appear, at unpredictable times.

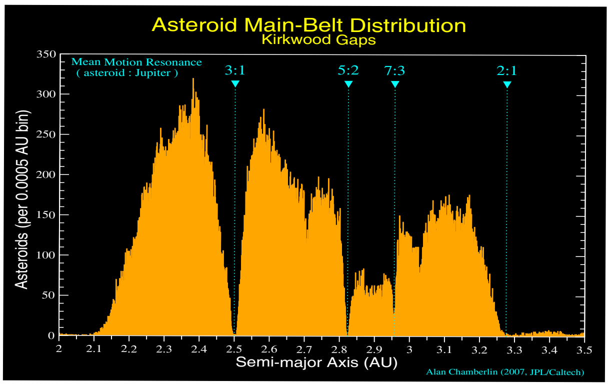

Figure 12: The distribution of the asteroids, obtained from 156929 asteroids, as given by JPL/Caltech in 2007. The Kirkwood gaps at the 3/1, 5/2, 7/3 and 2/1 mean motion resonances are clearly seen.

period which may be several million years (the semi major axis remaining almost constant), and thus the orbit of the asteroid may by Mars or even Earth crossing and thus undergo additional perturbations that will eventually drive it outside the 3/1 resonance region. Several papers followed this study, for many resonances in the asteroid belt, which used mapping models based on an averaged Hamiltonian at the corresponding resonance (Froeschlé 1991, Hadjidemetriou 1993, 1999, Hadjidemetriou and Voyatzis 2000). For the different methods used to transform the continuous flow to a mapping model see Hadjidemetriou (1998). Much work on the asteroid belt has been also made by making use of the averaging method or a combination of this method and numerical integrations (Henrard et al. 1995, Michtchenko and Ferraz-Mello 1996, Morbidelli and Giorgilli 1990a,b, Morbidelli 1996).

The variables used in the averaged models are the Delaunay variables, transformed to resonant action angle-variables (see Murray and Dermott 1999). For example, for the 3/1 resonance, for planar motion these variables are

$$S = \sqrt{\mu_1 a}\left(1 - \sqrt{1-e^2}\right), \quad \sigma = \frac{1}{3}(3\lambda' - \lambda) - \omega,$$

$$N = \sqrt{\mu_1 a}\left(3 - \sqrt{1-e^2}\right), \quad \nu = -\frac{1}{3}(3\lambda' - \lambda) + \omega',$$

where $\mu_1 = 1 - \mu$, $e' = 0.048$, and λ, ω, a are the mean longitude, the longitude of perihelion and the semimajor axis of the asteroid and the corresponding primed quantities refer to Jupiter.

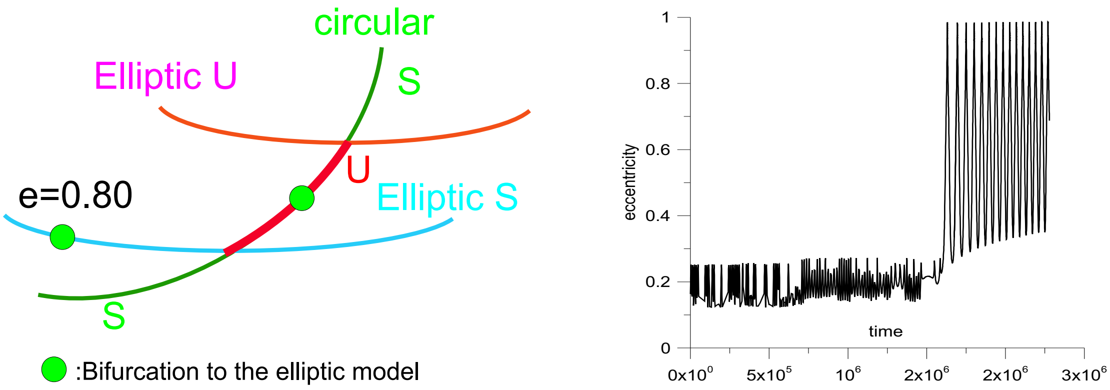

Figure 13: (a) The families of circular and elliptic periodic orbits of the circular model at the 3/1 resonance, and the bifurcation points to the elliptic model (schematically). S stands for stable and U for unstable. (b) The evolution of the eccentricity when the high eccentricity resonances are included in the model. The chaotic jumps are now up to eccentricities equal to 1.

The variables used to present the mapping are the Poincaré variables $X = \sqrt{2S}\cos(\sigma)$, $\quad Y = \sqrt{2S}\sin(\sigma)$, which are also canonical variables.

In Figure 11a we present the mapping for an asteroid at the 3/1 resonance, from a mapping model used by Hadjidemetriou (1993), equivalent to the map used by Wisdom 1982. The variables are similar to the Poincaré variables, but instead of $\sqrt{2S}$ we used the eccentricity e (note that $\sqrt{2S}$ is proportional to e, for small values of e). At the beginning the asteroid moves along the inner "diffused" circle, with small radius, corresponding to low values of the eccentricity, but eventually comes close to the chaotic region which connects the inner circle with an outer circle, with larger radius, corresponding to larger values of the eccentricity. So, through this chaotic window we have a connection between the low eccentricity regions and the high eccentricity regions. This results to a chaotic jump between small and large eccentricities, in a chaotic, unpredictable, way. This behavior is called *intermittency*. This is clearly shown in Figure 11b.

At this point we draw the attention to an important point when we use the averaged Hamiltonian in the dynamical study. Since the averaging method is based on series expansions in a small parameter (in our case it is the eccentricity), it is not valid for high values of the parameter. In the present case, in the study of the asteroid at the 3/1 resonance, the averaged Hamiltonian used to construct the mapping which gives the evolution of the asteroid eccentricity does not contain the high eccentricity resonances. This is the case with the evolution of Figure 11. For this reason it is important, in constructing the averaged model, to know the topology of the *whole* phase space, and this can be done only if we know *all* the resonant families of periodic orbits at the 3/1 resonance (and of course in all other similar studies in other resonances). *A necessary criterion for the validity of the averaged Hamiltonian is its fixed points to coincide with the periodic orbits (fixed points of the Poincaré map) of the original model* (the elliptic restricted three-body problem in this case). This shows the importance of the periodic orbits in orbital dynamics. In Figure 13a we show, schematically, for the model of the circular restricted problem, the family of circular orbits and the unstable region which appears at the 3/1 resonance on the circular family, and also the two families of elliptic periodic orbits that bifurcate from the critical points at the two ends of this unstable region. One family is stable and the other is unstable. It is found that on the unstable part of the circular family there exists a bifurcation point to two families of 3/1 resonant periodic orbits of the elliptic model, which start with zero eccentricities. Both of them are unstable. These are the *low eccentricity resonances* that are included in the model of Figure 11. However, there exists one more bifurcation point, at the eccentricity $e = 0.80$ on the stable family of elliptic periodic orbits (of the circular model), from which two 3/1 resonant periodic orbits of the elliptic model appear, one stable and the other unstable, starting with high eccentricities, equal to $e = 0.80$. For a full description of the resonant structure of the restricted three-body problem at the 3/1 resonance see Hadjidemetriou (1992,1993). It is these *high eccentricity resonances* that are missing from the model of Figure 11. If these high eccentricity resonances are also included in the model, the jumps in the eccentricity are higher, up to $e = 1$ and thus the asteroid not only approaches the inner planets, but may also fall on the Sun. This evolution is shown in Figure 13b.

The study of the ordered and chaotic regions in the asteroid belt is not the only such study in our Solar System. A zone of small bodies, similar to the asteroid belt, exists at the edge of our Solar System, after the orbit of Neptune. This is the Kuiper belt, whose existence was conjectured to explain the source of low period comets. Since the last decade of the 20th century many small bodies were observed in the Kuiper belt and it was realized that ordered and chaotic regions exist in this region also, similar to those in the asteroid belt, at several resonances with Neptune. *Pluto* is one such body in the Kuiper belt, trapped at the 3/2 resonance with Neptune, together with many other smaller bodies at the same resonance, called *plutinos*. A good view of the dynamical structure in the Kuiper belt is given in Celletti et al. (2007).

All major planets, Jupiter, Saturn, Uranus, Neptune, have planetary rings, the most well known being the rings of Saturn. Although many of the properties of the ring systems can be understood by a fluid dynamics approach, several of their features are explained by resonant dynamics, as in the case of the asteroid belt or the Kuiper belt. The fine structure of the rings can be explained by resonances between the ring particles and small satellites of the planet. A description of the dynamics of the planetary rings can be found in Murray and Dermott, 1999, ch. 10.

The chaotic behavior of the Solar System, as a whole, is yet another interesting subject, and there are several numerical works on this problem, notably by Laskar 1988, 1989,1994 and by Wisdom 1987. There are not large scale chaotic orbits of the planets, although the system is non integrable and some chaos is expected. Especially the large planets do not show any significant change and their orbits stay, for some billion years, close to their present orbits. The inner planets, especially Mercury, have shown large deviations, but due to the chaotic nature of the Solar System and the fact that the numerical integrations are not with infinite accuracy, these results may not represent the actual evolution of the Solar System. It seems that the Solar System is stable and any chaotic motion is in a small scale and is bounded. Studies on the stability of extrasolar planetary systems have started recently, with many interesting results, as we explain in the next section.

Another interesting case of chaotic motion in the Solar System refers to the rotational motion of celestial bodies. Although the rotation of the planets is regular, there are small bodies, with irregular shape, that show chaotic rotation. Such a case is the satellite of Saturn, Hyperion, with approximate dimensions 180km $\times$ 140km $\times$ 112.5km. Although its orbit is stable, due to the fact that it is at the 4:3 resonance with the more massive satellite of Saturn, Titan, its rotation is chaotic (Wisdom et al. 1984).

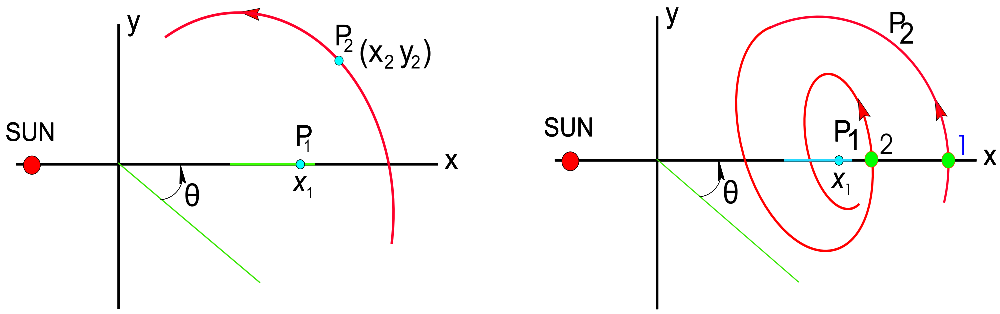

Figure 14: (a) The rotating frame. The planet P_1 moves on the x-axis and the planet P_2 in the xOy plane. The angle θ is an ignorable coordinate. (b) The Poincaré map at $y_2 = 0$.

12 Extrasolar planetary systems

12.1 Some general remarks

In the last decade of the 20th century it was discovered that our Solar System is not the only planetary system in the universe. Up to the present (May 2008) there are 281 observed extrasolar planetary systems, with 25 of them having two or more planets. In many planetary systems with two planets close to each other, the two planets are in mean motion resonance. Examples are: HD 82943 (Israelian et al. 2001, Mayor et al. 2004), GLIESE 876 (Marcy et al. 2001, Rivera and Lissauer, 2001), at the 2:1 resonance and 55Cnc at the 3:1 resonance (Marcy et al., 2002). Some of these systems have large eccentricities and are evidently stable.

There are different approaches to the study of the dynamical evolution of a planetary system and on the mechanisms that stabilize the system, or generate chaotic motion and instability: Beaugé and Michtchenko 2003, Beaugé et al. 2002, 2005, 2006, Ferraz-Mello et al. 2002, Gozdjiewski et al. 2003, Malhotra 2002, Lee and Peale 2002, Lee 2004. In these papers different methods have been applied, as the averaging method, direct numerical integrations of orbits, or various numerical methods which provide indicators for the exponential growth of nearby orbits. In this way the regions where stable motion exists have been detected, in the orbital elements space.

We present briefly a global view of the structure of the phase space of a planetary system with two planets, moving in the plane, as obtained from the set of the families of periodic orbits. As we have already mentioned before, the periodic orbits play a dominant role in understanding the dynamics of a system, because they determine critically the structure of the phase space. In this way, we can detect the regions where stable librations could exist. These will be the regions where a real planetary system could exist in nature. As we will see, stable regions corresponding to elliptic orbits of the two planets with relatively large eccentricities are associated with mean motion resonances. An early work on periodic orbits of the planetary type is by Hadjidemetriou (1976), well before the first extrasolar planetary systems were observed. Many papers followed on these lines, after the first extrasolar planetary systems were observed (Hadjidemetriou 2002, Psychoyos and Hadjidemetriou 2005, Voyatzis and Hadjidemetriou 2005,2006). We remark that stable motion could also exist far from resonances, if the eccentricities are small. This latter motion is close to a stable periodic orbit of the *circular family* of periodic orbits. We also remark that it is possible to have stable motion far from a periodic orbit, but in this latter case the two planets are not close to each other, so that their gravitational interaction is not very significant.

It can be proved (Hadjidemetriou 1975) that families of periodic orbits in the planar general three body problem exist, in a *rotating* frame xOy, whose x-axis is the line $S - P_1$, with origin at

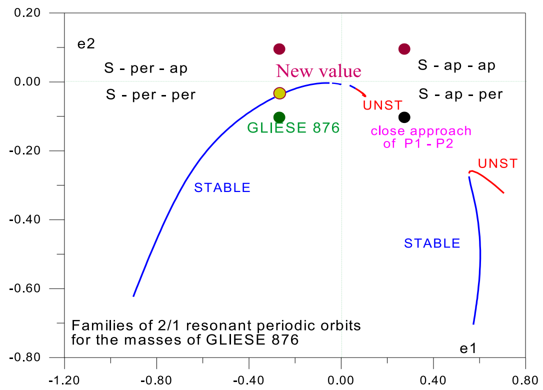

Figure 15: The families of periodic orbits at the 2/1 resonance in the space of the eccentricities. $e_i > 0$ means position at aphelion and $e_i < 0$ position at perihelion.

the center of mass of these two bodies, where S is the Sun and P_1 the inner planet. We assume that the center of mass of the whole system is at rest with respect to an inertial frame. We have four degrees of freedom, for planar motion, with generalized variables x_1, x_2, y_2, θ (Figure 14a). This is a non uniformly rotating frame, and the second planet P_2 moves in the plane xy. It turns out (Hadjidemetriou 1975) that the angle θ is ignorable, so we have three degrees of freedom in the rotating frame, with variables x_1, x_2, y_2. In the planetary three body problem (one big body, the star and two small bodies, the planets) the periodic orbits are similar to the families of the restricted problem as shown in Figure 4. There are two types of periodic orbits:

- Non resonant periodic orbits with nearly circular orbits of the two planets.

- Resonant periodic orbits with nearly elliptic orbits of the two planets.

The circular orbits are all symmetric but the elliptic orbits may be symmetric or asymmetric. There exist families of elliptic periodic orbits for every mean motion resonance. Close to a stable periodic orbit there exists a region of stable librations, and it is at these regions that a planetary system could be trapped.

Concerning the continuation of the unperturbed family of periodic orbits ($m_1 = m_2 = 0$), the situation is similar to that explained in the restricted three body problem. There are three topologically different resonant cases:

- The resonances of the form $(n + 1)/n$, (2/1, 3/2, ...) (Gaps on the circular family).

- The resonances $(2n + 1)/(2n - 1)$, (3/1, 5/3, ...) (Instability on the circular family).

- All other resonances, (5/2, 7/3, 8/3, ...) (Preservation of the stability on the circular family).

A global view of the resonant families of elliptic periodic orbits, for each one of the above resonance types can be found in Hadjidemetriou (2006a). There exist both *symmetric* and *asymmetric* families. The ratio of the planetary masses plays an important role on the stability and the existence of asymmetric families of periodic orbits. The sum of the masses of the planets also plays an important role on the stability and the existence of families of resonant periodic orbits. The stability of a symmetric periodic orbit depends, all other things being the same (semimajor axes, eccentricities), on the phase of the two planets, that is on whether the line of apsides are aligned or antialigned and on the position of the two planets at perihelion of aphelion at some epoch. The proper phase generates a *phase protection mechanism* so that stable planetary systems exist even for large eccentricities.

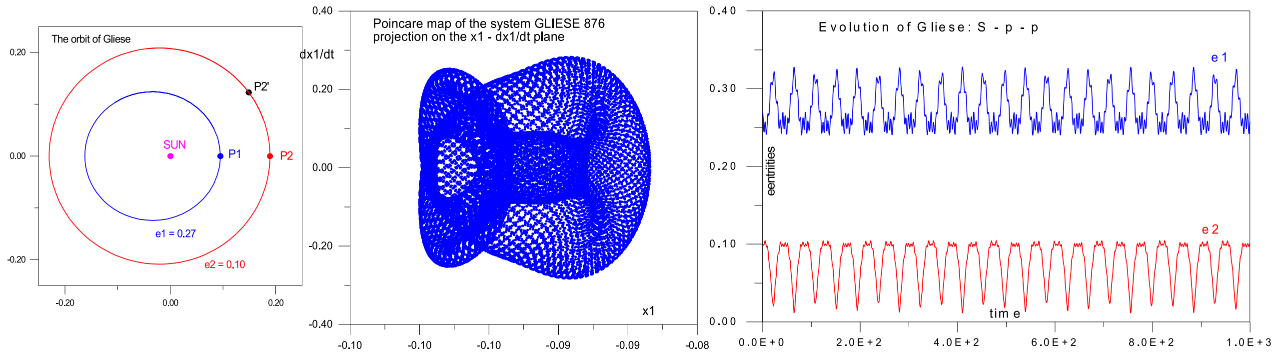

Figure 16: (a) The orbit, corresponding to $e_1 < 0$, $e_2 < 0$. (b) The Poincaré map: projection on the plane $x_1\,\dot{x}_1$. The motion is ordered.(c) The evolution of the eccentricities.

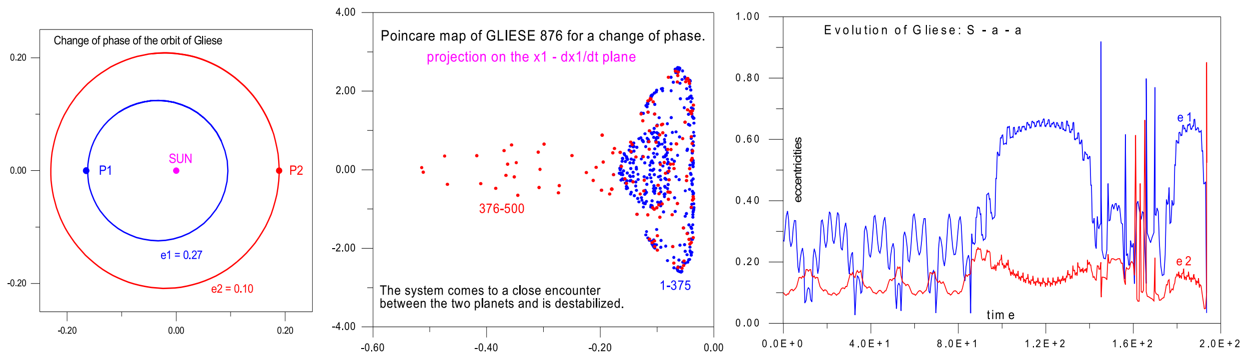

Figure 17: (a) The orbit, corresponding to $e_1 > 0$, $e_2 > 0$. (b) The Poincaré map: projection on the plane $x_1\,\dot{x}_1$. The motion is chaotic. (c) The evolution of the eccentricities.

The properties of motion close to a periodic orbit are studied by considering a Poincaré map on the surface of section $y_2 = 0$, $(\dot{y}_2 > 0)$, $H = h$=constant, (Figure 14b)). The phase space of the Poincaré map is the four dimensional space x_1, $\dot{x}_1$, x_2, $\dot{x}_2$ ($y_2 = 0$ and $\dot{y}_2$ is obtained from $H = h$, $\dot{y}_2 > 0$). Close to a stable periodic orbit we have stable librations and the motion in phase space takes place on a torus. On the contrary, close to an unstable periodic orbit we have irregular, chaotic, motion and in many cases the system disrupts into a binary system (the star and one planet) and an escaping planet.

The position of some real extrasolar planetary systems is compared with the above mentioned regions of stable librations. A detailed analysis of the dynamics of extrasolar planetary systems based on the families of periodic orbits is presented in Hadjidemetriou 2006b.

In the following, we present, as an example, the dynamics of a real extrasolar planetary system, Gliese 876, on the whole phase space, and study the stable configurations and the regions where chaotic motion appears.

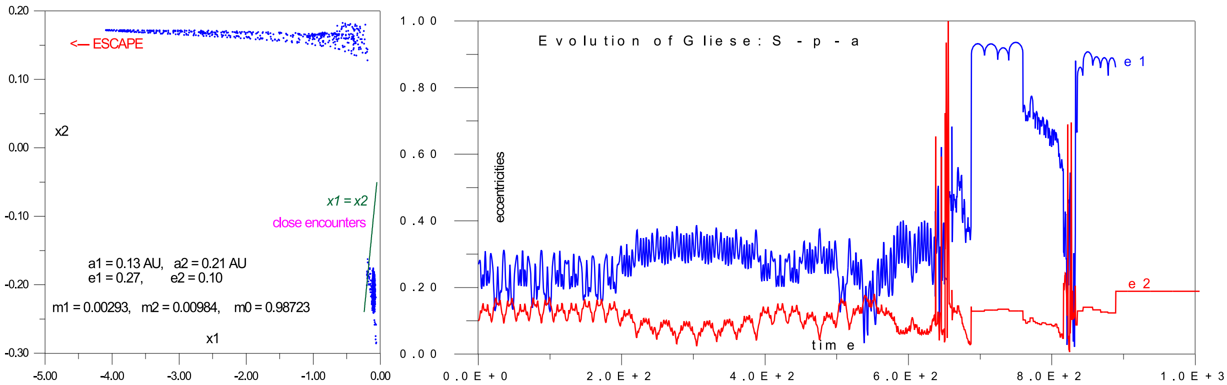

Figure 18: The orbit, corresponding to $e_1 < 0$, $e_2 > 0$. (a) The Poincaré map: projection on the plane $x_1\,x_2$. There exist points close to the $x_2 = x_1$ line, corresponding to close encounters between the two planets. The motion is chaotic. (c) The evolution of the eccentricities.

12.2 A real extrasolar planetary system: Gliese 876

Studies on the dynamical evolution of a planetary system, both theoretical and for real extrasolar planetary systems, have been made by different methods. One way to study the problem is to compute many orbits, for a set of initial conditions and study their behavior for a long time. A different method is to use the averaging method in order to obtain an averaged Hamiltonian, thus reducing the number of degrees of freedom. Analytic and numerical studies can then be made to find the stable regions in phase space (Beaugé and Michtchenko 2003, Beaugé et al. 2003, 2005, 2006, Ferraz-Mello et al. 2003, Gozdjiewski et al. 2003, Malhotra 2002, Lee and Peale 2002, Lee 2004, Sandor et al. 2007). A systematic study of the orbital dynamics in planetary systems can be made by finding all the basic families of periodic orbits. As we mentioned before, the position and the stability character of the periodic orbits define the topology of the phase space, and in this way we find all the stable regions, close to the stable periodic orbits, where a planetary system can be trapped, and we also find the chaotic regions, close to the unstable periodic orbits, where planetary system could not exist (Hadjidemetriou 2006a).

The ordered and chaotic regions in an extrasolar planetary system, the factors that affect the stability and the mechanism of generation of chaos, will be presented here by an example from a real extrasolar planetary system, Gliese 876 (Marcy et al. 2001). This is a planetary system 15.4 light years far from our solar system. The mass of the sun in this system is equal to $m_0 = 0.32$ solar masses and the masses of the planets P_1, P_2 are $m_1 \sin i = 1.89$ MJ and $m_2 \sin i = 0.56$ MJ, where MJ stands for the mass of Jupiter (i is the inclination of the orbital plane of this system with respect to the line of sight from us, and it is not known). The semimajor axes, the eccentricities and the periods of the planetary orbits are: $a_1 = 0.13$ AU, $a_2 = 0.21$ AU, $e_1 = 0.27$, $e_2 = 0.10$, $T_1 = 30.1$ days and $T_2 = 61.02$ days. The perihelia of the two planetary orbits are in the same direction. This is a system very close to the 2/1 resonance, $T_2/T_1 = 2.03$, and for this reason we study all the families of resonant periodic orbits at the 2/1 resonance, for the masses of this system (assuming $\sin i = 1$).

In Figure 15 we present the families of resonant 2/1 periodic orbits for the masses of Gliese 876, in the space of the planetary eccentricities $e_1\,e_2$. We used the convention that $e_i > 0$ means position of the planet at aphelion and $e_i < 0$ position at perihelion. In this way the space of the eccentricities is divided into four sections, according to the sign of the eccentricities, as shown in Figure 15. For $e_1 < 0$, $e_2 < 0$ and $e_1 > 0$, $e_2 > 0$ the perihelia of both planets are in the same direction, while for $e_1 > 0$, $e_2 < 0$ and $e_1 < 0$, $e_2 > 0$ the perihelia are in opposite directions.

We may also note that due to the 2/1 resonance, the phases where, for the same position of P_1, the position of P_2 is at perihelion or in aphelion are equivalent, corresponding to $t = 0$ and to $t = T/2$, respectively, where T is the period.

There are two families that start from the region $e_1 \approx 0$, $e_2 \approx 0$. At $e_1 = e_2 = 0$ there is a gap, similar to the gap on the family of circular orbits of the restricted three body problem, as shown in Figure 6. The first family corresponds to $e_1 < 0$, $e_2 < 0$, along which the eccentricities of the two planets increase. In all orbits of this family the perihelia are in the same direction and at $t = 0$ both planets are at perihelia. This family is stable, even for large values of the orbital eccentricities. Another family exists, for $e_1 > 0$ and $e_2 < 0$. In this family the perihelia of the two planets are in opposite directions and at $t = 0$ the planet P_1 is at aphelion and the planet P_2 is at perihelion. This family presents a gap at the region $e_1 = -0.2$, $e_2 = 0.4$, because the two planets are close to each other and the gravitational attraction between them is so strong (for the given masses) that a resonant 2/1 orbit cannot survive. This part of the family, from zero eccentricities up to the gap, which corresponds to small eccentricities, is unstable. But after this close approach region, the family continues with large eccentricities, and this part is now stable.

In the space of the eccentricities of Figure 15 we placed a planetary system with the same semimajor axes and eccentricities as Gliese 876, but with different phases. One of these positions, for $e_1 < 0$, $e_2 < 0$, is very close to the stable family. In Figures 16, 17 and 18 we present the evolution of each of these systems (with the same elements a_i, e_i as Gliese 876), by making use of the Poinaré map on the surface of section defined in Figure 14b. We note that the real system (green circle in Figure 15) is in an ordered region (Figure 16), with the eccentricities undergoing quasi periodic variations, and the projection of the Poincaré map on the $x_1\ \dot{x}_1$ plane is a nice surface (the same holds for the projection in all other planes of the four dimensional phase space of the Poincaré map). All other configurations however are unstable and present chaotic behavior, although the orbital elements are the same and only the phase differs. In Figure 17, corresponding to $e_1 > 0$, $e_2 > 0$, chaotic behavior develops after a rather ordered motion, and the system disrupts. In Figure 18, corresponding to $e_1 < 0$, $e_2 > 0$, also chaotic behavior develops after a long time of rather ordered motion. From the Poincaré map, which is given in its projection in the x_1x_2 space, we see that the mechanism of generation of chaos is the close encounters between the two planets, shown by the several points of intersection close to the line $x_2 = x_1$ (this is a real encounter and not just due to the projection from the four dimensional space $x_1\dot{x}_1x_2\dot{x}_2$ to the two dimensional plane x_1x_2, because P_1 is always on the x-axis and P_2 is also on the x-axis, which implies $y_2 = 0$, due to the definition of the map (see Figure 14b).

From the above we see that the phase of the two planets (perihelia in the same or in opposite directions, position of the planet at perihelion or aphelion at $t = 0$) plays a crucial role on the stability of the system. As we have seen, the stable regions are close to the stable periodic orbits, and this makes clear the importance of knowing all the families of periodic orbits. In this way we are in a position to know in what regions of the orbital elements a planetary system could exist in nature and what are the regions where a planetary system cannot exist. We note also that the orbital elements for Gliese 876 that we used in the above study were revised, as more accurate observations were taken into account. The new values correspond to a position almost *on* the stable family, as we show in Figure 15 (yellow circle).

13 Future Directions

The model of the restricted three-body problem has been studied for almost a century and most of its dynamical aspects are now known. This is not so for the general, planetary, three-body problem, where several aspects of the dynamics are not yet well studied. One reason is that the phase space has more dimensions than the restricted problem. Though the motion in the plane is quite well understood, because all the basic resonant and non resonant periodic orbits (symmetric and asymmetric) are well known, the three dimensional motion is not completely studied. The main reason for this is that the observational data for the extrasolar systems are not yet accurate enough to give information on three dimensional planetary motion. The knowledge of the basic three dimensional families will give a clear picture of the topology of the phase space and of the regions where a three dimensional planetary system could be trapped.

Another problem in the study of the extrasolar planetary systems is the explanation of large planetary eccentricities. Evidently, such systems are stable, since they are observed in nature, and we know from the studies up to now that such high eccentricity planetary systems can be stable, provided we have the right phase. But how did these systems reach their present configuration? It has been proposed that they were generated as low eccentricity systems and reached the present configuration following a *migration* process. A dissipation is needed for such an evolution and several mechanisms have been proposed. It is possible that a planetary system can trapped in a stable configuration, possibly with high eccentricities, due to the migration process. It is the stable periodic orbits that correspond to these stable configurations. Some work has been done on this problem (Beaugé et al. 2006), but more work is needed.

An open problem in orbital dynamics is the study of the early history of our Solar System. This study involves calculations of the N-body problem. It is believed that the orbits of the giant planets of our Solar System, from Jupiter and beyond, migrated due to the planetesimals which were left after the dispersal of the gas disk, in which the Solar System was formed. The idea is that the giant planets ejected the planetesimals and this resulted to a change of their orbits. Recent studies by Tsiganis et al. (2005), Gomes et al. (2005) and Morbidelli et al. (2005) suggest that all outer planets started in a different configuration than the present one, with Jupiter slightly further from the Sun than its present distance, while the rest giant planets were in a distance less than 15AU from the Sun. This is the so called *Nice model*, from the observatory of Nice where this group works. It is assumed that the planets were surrounded by a disk of planetesimals, which were ejected by the planets, and this resulted to a migration of their orbits. The assumption was made that Saturn was initially inside the 2/1 resonance with Jupiter, and as Saturn crossed this resonance, the eccentricity of the planets increased very much and the planets entered the outer planetesimal disk. This resulted to a heavy scattering of the planetesimals, which reached the inner Solar System and are responsible for the *Late Heavy Bombardment* on the surface of the Moon, which created its craters. More work is still to be done on this problem, including the effect of the giant planets on the orbits of the inner Solar System.

In the study of the long term stability and the chaotic behavior of our Solar System it is now realized that in addition to the Newtonian gravitational forces, the effects of the general theory of relativity must be taken into account. In addition, the effects of the loss of mass from the sun, although very small, plays a role in the long term evolution of the Solar System and must also be taken into account. This is a topic for future work.

It has been realized recently that very small nonconservative forces, as the effect from the mass loss of the sun, or the effects from the theory of general relativity, must be included in the study of the past history or the long term evolution of the solar system, for billions of years. This is important for the study of the evolution of the inner planets and most notably of Mercury. Work has now started on this matter, and it is expected to give interesting results.

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