One qubit gates

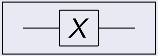
Basis of eigenvectors

$$|0\rangle \sim \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \quad |1\rangle \sim \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \sim \left[\begin{array}{c} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{array} \right]$$

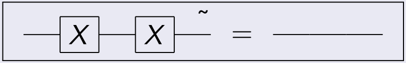
NOT gate

$$\mathrm{NOT} = \sigma_x \sim \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \sim$$

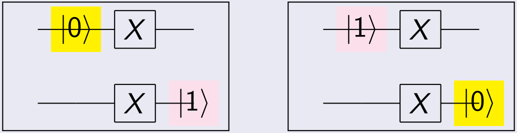


$$\sigma_x|0\rangle = |1\rangle\,, \quad \sigma_x|1\rangle = |0\rangle \rightsquigarrow \sigma_x|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle$$

$$\sigma_x \cdot \sigma_x = \mathbb{I} \sim \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \rightsquigarrow$$

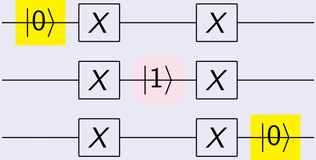


Definition NOT gate



Proposition

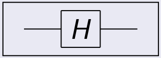
Proof



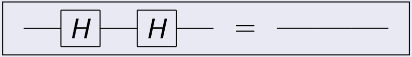
Hadamard gate

Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sim$$



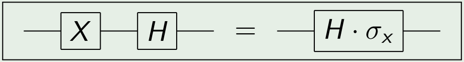
$$H \cdot H = H^2 = \mathbb{I} \rightsquigarrow$$

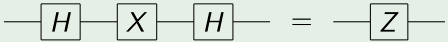


$$\left.\begin{aligned} H|0\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle \\ H|1\rangle &= \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle \end{aligned}\right\} \rightsquigarrow \left\{\begin{aligned} H|+\rangle &= |0\rangle \\ H|-\rangle &= |1\rangle \end{aligned}\right.$$

Examples of one qubit circuits

$$H \cdot \sigma_x = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \sim$$





$$\sim \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Tensor Product

U and V linear spaces

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in U, \quad v = \begin{bmatrix} v_1 \\ u_2 \\ \vdots \\ v_m \end{bmatrix} \in V \quad \rightsquigarrow \quad u \otimes v = \left[\begin{array}{c} u_1 v_1 \\ u_1 v_2 \\ \vdots \\ u_1 v_m \\ \hline u_2 v_1 \\ u_2 v_2 \\ \vdots \\ u_2 v_m \\ \hline \vdots\vdots \\ \hline u_n v_1 \\ u_n v_2 \\ \vdots \\ u_n v_m \end{array}\right] \in U \otimes V$$

Example: Basis for the two qubits states

$$\left|\widehat{0}\right\rangle = |00\rangle = |0\rangle \otimes |0\rangle \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|\widehat{1}\right\rangle = |01\rangle = |0\rangle \otimes |0\rangle \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|\widehat{2}\right\rangle = |10\rangle = |1\rangle \otimes |0\rangle \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left|\widehat{3}\right\rangle = |11\rangle = |1\rangle \otimes |1\rangle \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbb{U} \in \mathrm{End}(U), \quad \mathbb{V} \in \mathrm{End}(V)$$

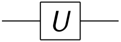
$$\mathbb{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1p} \\ u_{21} & u_{22} & \cdots & u_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ u_{q1} & u_{q2} & \cdots & u_{qp} \end{bmatrix}$$

$$\mathbb{V} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nm} \end{bmatrix}$$

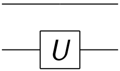
$$\mathbb{U} \otimes \mathbb{V} = \left[\begin{array}{c|c|c|c} u_{11}\mathbb{V} & u_{12}\mathbb{V} & \cdots & u_{1p}\mathbb{V} \\ \hline u_{21}\mathbb{V} & u_{22}\mathbb{V} & \cdots & u_{2p}\mathbb{V} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline u_{q1}\mathbb{V} & u_{q2}\mathbb{V} & \cdots & u_{qp}\mathbb{V} \end{array}\right]$$

$$\mathbb{U} \otimes \mathbb{V} \in \mathrm{End}\left(U \otimes V\right)$$

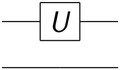
$$\left(\mathbb{U} \otimes \mathbb{V}\right)\left(u \otimes v\right) \equiv \mathbb{U}u \otimes \mathbb{V}v$$



$$\sim \mathbb{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$



$$\sim \mathbb{I} \otimes \mathbb{U} = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix}$$

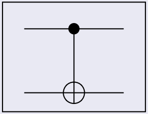


$$\sim \mathbb{U} \otimes \mathbb{I} = \begin{bmatrix} u_{11} & 0 & u_{12} & 0 \\ 0 & u_{11} & 0 & u_{12} \\ u_{21} & 0 & u_{22} & 0 \\ 0 & u_{21} & 0 & u_{22} \end{bmatrix}$$

CNOT gate

CNOT gate

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim$$

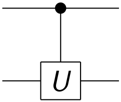


$$\mathrm{CNOT}\left(\alpha\left|00\right\rangle + \beta\left|01\right\rangle + \gamma\left|10\right\rangle + \delta\left|11\right\rangle\right) = \\ = \left(\alpha\left|00\right\rangle + \beta\left|01\right\rangle + \gamma\left|11\right\rangle + \delta\left|10\right\rangle\right)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$$

Controlled gates

$$\Lambda_1(U) =$$



$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{bmatrix}$$

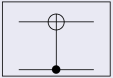
$$\Lambda_1(U)\left(\alpha\left|00\right\rangle + \beta\left|01\right\rangle + \gamma\left|10\right\rangle + \delta\left|11\right\rangle\right) =$$
$$\left(\alpha\left|00\right\rangle + \beta\left|01\right\rangle + (\gamma u_{11} + \delta u_{12})\left|10\right\rangle + (\gamma u_{21} + \delta u_{22})\left|11\right\rangle\right)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u12 \\ 0 & 0 & u_{21} & u22 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ (\gamma u_{11} + \delta u_{12}) \\ (\gamma u_{21} + \delta u_{22}) \end{bmatrix}$$

"Generalized" CNOT

"Generalized" CNOT gate

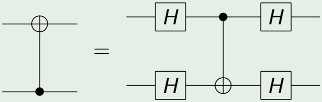
$$
''\mathrm{G}'' - \mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim
$$



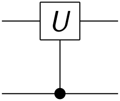
$$
\begin{aligned} \mathrm{CNOT}\,(\alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle) &= \\ &= (\alpha\,|00\rangle + \beta\,|11\rangle + \gamma\,|10\rangle + \delta\,|01\rangle) \end{aligned}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \delta \\ \gamma \\ \beta \end{bmatrix}
$$

Decomposition of the Generalized CNOT



Generalized Controlled gates

$G_1(U) =$ 

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{21} \\ 0 & 0 & 1 & 0 \\ 0 & u_{12} & 0 & u22 \end{bmatrix}$

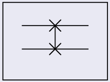
$$G_1(U)\left(\alpha\left|00\right\rangle + \beta\left|01\right\rangle + \gamma\left|10\right\rangle + \delta\left|11\right\rangle\right) =$$
$$\left(\alpha\left|00\right\rangle + (\beta u_{11} + \delta u_{12})\left|01\right\rangle + \gamma\left|10\right\rangle + (\beta u_{21} + \delta u_{22})\left|11\right\rangle\right)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{21} \\ 0 & 0 & 1 & 0 \\ 0 & u_{12} & 0 & u22 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ (\beta u_{11} + \beta ltau_{12}) \\ \gamma \\ (\beta u_{21} + \delta u_{22}) \end{bmatrix}$$

SWAP gate

SWAP gate

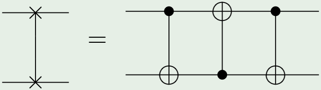
$$\mathrm{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim$$

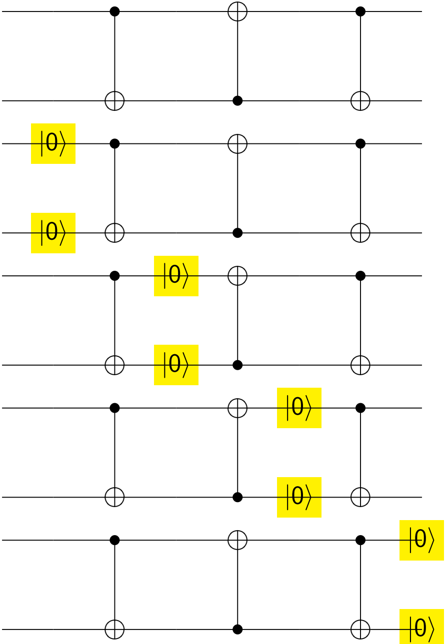


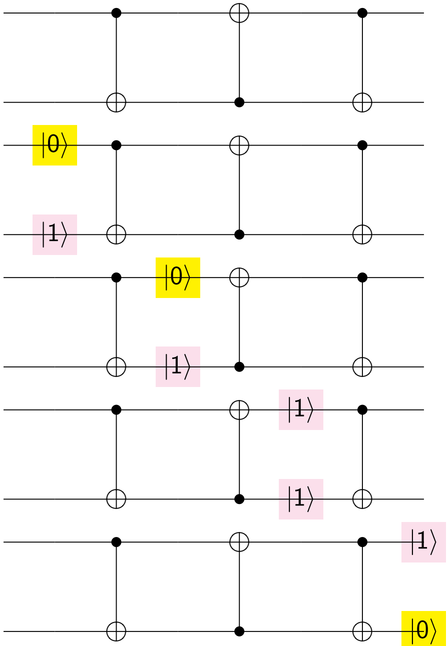
$$\mathrm{SWAP}\left(\alpha\left|00\right\rangle + \beta\left|01\right\rangle + \gamma\left|10\right\rangle + \delta\left|11\right\rangle\right) =$$
$$= \left(\alpha\left|00\right\rangle + \gamma\left|01\right\rangle + \beta\left|10\right\rangle + \delta\left|11\right\rangle\right)$$

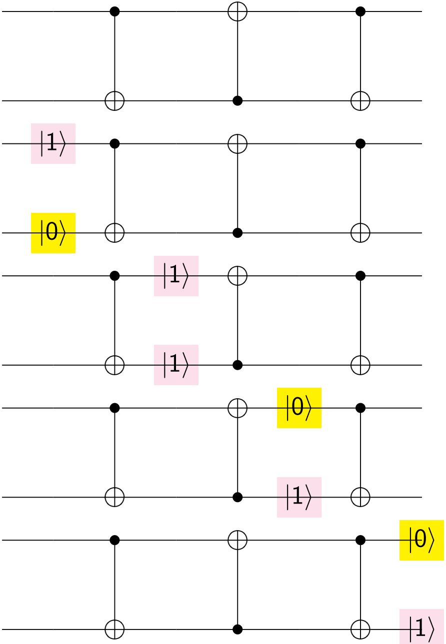
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma \\ \beta \\ \delta \end{bmatrix}$$

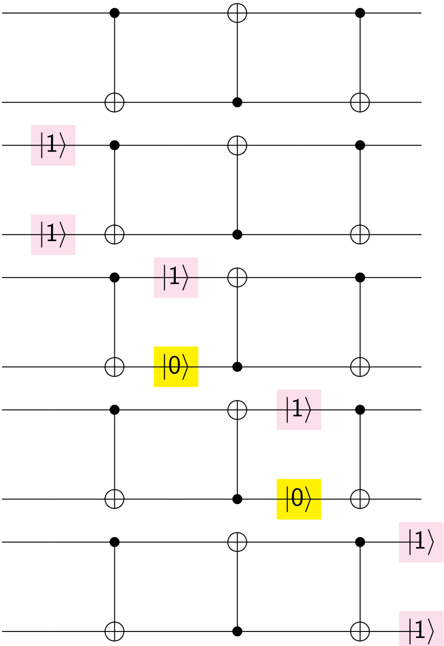
Decomposition of the SWAP




|0⟩
|0⟩
|0⟩
|0⟩
|0⟩
|0⟩
|0⟩
|0⟩


|0⟩
|1⟩
|0⟩
|1⟩
|1⟩
|1⟩
|1⟩
|0⟩


|1⟩
|0⟩
|1⟩
|1⟩
|0⟩
|1⟩
|0⟩
|1⟩



|1⟩
|1⟩
|1⟩
|0⟩
|1⟩
|0⟩
|1⟩
|1⟩