Πρ. 8 $f(x)$ μονότονη & φραγμένη στο $[a,b]$

$\rightsquigarrow$ $f(x)$ Darboux - ολοκληρώσιμη.

$f(x)$ αύξουσα $\qquad x_{k-1} < x < x_k$

$$\Downarrow$$

$$\underset{\text{minimum}}{m_k = f(x_{k-1})} \leq f(x) \leq \underset{\text{maximum}}{f(x_k) = M_k}$$

$$U(P,f) - L(P,f) = \sum_{k=1}^{n} \underbrace{\left(f(x_k) - f(x_{k-1})\right)}_{\geq 0} \Delta x_k$$

$$\Delta x_k \leq |P|$$

$$U(P,f) - L(P,f) \leq |P| \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right)$$

$$\leq |P| \left(f(b) - f(a)\right)$$

$$\forall \epsilon > 0 \quad \exists \ \delta = \frac{\epsilon}{f(b) - f(a)} > 0 \ :$$

$$|P| < \delta \rightsquigarrow U(P,f) - L(P,f) < \epsilon$$

$$\lim_{|P| \to 0} \left(U(P,f) - L(P,f)\right) = 0$$

$$\Downarrow$$

$$\exists \ I_D(f) = U(f) = L(f)$$

Πρ. 9 $\quad f(x)$ (ομοιόμορφα) συνεχής στο $[a,b]$

$\Downarrow$

$f(x)$ Darboux ολοκληρώσιμη

$f(x)$ ομοιόμορφα συνεχής

$$\forall \epsilon>0 \quad \exists\, \delta>0 : \quad |x-y|<\delta \Rightarrow |f(x)-f(y)| < \frac{\epsilon}{b-a}$$

$$m_k = \inf_{x_{k-1}\le x \le x} f(x) = f(\xi_k)$$

$$|\xi_k-\eta_k| \leq |P|$$

$$M_k = \sup_{x_{k-1}\le x \le x_k} f(x) = f(\eta_k)$$

$$\forall \epsilon>0 \quad \exists\, \delta>0 : \quad |P|<\delta \rightsquigarrow$$

$$U(P,f) - L(P,f) \leqslant |P| \frac{\epsilon}{b-a} < \epsilon$$

$$\| \qquad \sum_k (M_k - m_k)\,\Delta x_k$$

$\Downarrow$

$$\lim_{|P|\to 0} \big(U(P,f) - L(P,f) \big) = 0$$

$\Downarrow$

$f(x)$ Darboux ολοκληρώσιμη

Πρ. 10 $\quad \exists$ ολοκλήρωμα Darboux $U(f)=L(f)=I_D(f)$ $\Rightarrow$ $\exists$ ολοκλήρωμα Rieman $I_R(f)$

$$L(P,f) \leqslant S(P,T,f) \leqslant U(P,f)$$

$$\sum_{\kappa} m_\kappa \Delta x_\kappa \leqslant \sum_{\kappa} f(\xi_\kappa)\Delta x_\kappa \leqslant \sum_{\kappa} M_\kappa \Delta x_\kappa$$

$$\forall \epsilon>0 \quad \exists \delta>0 \; : \; |P|<\delta \qquad U(P,f)-L(P,f)<\epsilon$$

$$\left.\begin{array}{l} U(P,f) < L(P,f)+\epsilon \leqslant L(f)+\epsilon \\ U(f)-\epsilon \leqslant U(P,f)-\epsilon < L(P,f) \end{array}\right\} \Rightarrow$$

$$U(f)-\epsilon < S(P,T,f) < L(f)+\epsilon$$

οπότε $\quad |S(P,T,f) - I_Q(f)| < \epsilon$

$$I_R(f) = \lim_{|P|\to 0} S(P,T,f) = I_Q(f)$$

Πρ. 10 $\quad \exists$ Ολοκλήρωμα Riemann $I_R(f)$ $\Rightarrow$ $\exists$ Ολοκλήρωμα Darboux $I_D(f)$

$$A = I_R(f)$$

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad |P| < \delta \Rightarrow A - \frac{\epsilon}{4} < S(P,T,f) < A + \frac{\epsilon}{4}$$

$$m_k = \inf_{x_{k-1} \leq x \leq x_k} f(x) \rightsquigarrow \exists\ \xi_k \in [x_{k-1}, x_k] : \quad f(\xi_k) < m_k + \frac{\epsilon}{4(b-a)}$$

$$M_k = \sup_{x_{k-1} \leq x \leq x_k} f(x) \rightsquigarrow \exists\ \eta_k \in [x_{k-1}, x_k] : \quad M_k - \frac{\epsilon}{4(b-a)} < f(\eta_k)$$

$$\Xi = \{\xi_1, \xi_2, \cdots, \xi_n\}$$

$$H = \{\eta_1, \eta_2, \cdots, \eta_n\}$$

$$A - \frac{\epsilon}{4} < S(P, \Xi, f) < L(P,f) + \frac{\epsilon}{4}$$

$$U(P,f) - \frac{\epsilon}{4} < S(P, H, f) < A + \frac{\epsilon}{4}$$

$$\Downarrow$$

$$U(P,f) - L(P,f) - \frac{\epsilon}{2} < \frac{\epsilon}{2}$$

$$\Downarrow$$

$\exists$ ολοκλήρωμα Darboux

Πρ. 11. (Θεώρημα Darboux)

f ολοκληρώσιμη στο $[a,b] \rightsquigarrow$

αν $\xi_{k,n} \in \left[a + \frac{k-1}{n}(b-a),\ a + \frac{k}{n}(b-a) \right]$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(\xi_{k,n})$$

Αποδ. $P_n = \{ x_0, x_1, \dots, x_n \}$, $x_k = a + \frac{k}{n}(b-a)$

$\Delta x_k = \frac{b-a}{n}$

$T_n = \{ \xi_{0n}, \xi_{01}, \dots, \xi_{nn} \}$.

$\forall \epsilon > 0 \quad \exists \delta > 0 : \ |P| < \delta$

$|S(P,T,f) - I_R(f)| < \epsilon$

Υπαρξη Ολοκληρώματος Riemann

$\forall \ \delta > 0 \quad \exists \ n_0 \in \mathbb{N} : \ \underbrace{|P_n|}_{= \frac{b-a}{n}} < \delta \quad \forall \ n > n_0$

$\Rightarrow \quad |S(P_n, T_n, f) - I_R(f)| < \epsilon.$

$\downarrow$

$\lim_{n\to\infty} S(P_n, T_n, f)$

Πρ. 12

Γραμμικότητα $\displaystyle\int_a^b \left(c_1 f(x) + c_2 g(x)\right) dx =$

$$= c_1 \int_a^b f(x)\,dx + c_2 \int_a^b g(x)\,dx$$

Για μιά διαμέριση P και μιά επιλογή T

$$S(P, T, c_1 f + c_2 g) = \sum_{k=1}^{n} \left(c_1 f(\xi_k) + c_2 g(\xi_k)\right) \Delta x_k$$

$$= c_1 \underbrace{\sum_{k=1}^{n} f(\xi_k) \Delta x_k}_{S(P, T, f)} + c_2 \underbrace{\sum_{k=1}^{n} g(\xi_k) \Delta x_k}_{S(P, T, g)}$$

"Θετικότητα" $f_1(x) \leq f_2(x) \rightsquigarrow I_R(f_1) \leq I_R(f_2)$

$$\sum f_1(\xi_k) \Delta x_k \leq \sum_k f_2(\xi_k) \Delta x_k$$

"Τριγωνική ιδιότητα"

$$\boxed{\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx}$$

$$\left|\sum_k f(\xi_k) \Delta x_k\right| \leq \sum_k |f(\xi_k)| \Delta x_k$$