Chapter 11

Techniques of Integration

11.1 Basic Substitution and Formulas

Definition

The *indefinite integral* of a function f, written

$$\int f(x)dx,$$

is the most general antiderivative of f; that is,

$$\int f(x)dx = F(x) + C \qquad (C \text{ is any constant})$$

if and only if $F'(x) = f(x)$.

Basic Formulas: (Memorize this list)

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1) \qquad\qquad \int \frac{du}{u} = \ln|u| + C, \quad \text{if} \quad u \neq 0$$

$$\int e^u du = e^u + C \qquad\qquad \int a^u du = \frac{a^u}{\ln a} + C$$

$$\int \sin u\, du = -\cos u + C \qquad\qquad \int \cos u\, du = \sin u + C$$

$$\int \sec^2 u\, du = \tan u + C \qquad\qquad \int \csc^2 u\, du = -\cot u + C$$

$$\int \sec u \tan u\, du = \sec u + C \qquad\qquad \int \csc u \cot u\, du = -\csc u + C$$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C \qquad\qquad \int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{du}{|u|\sqrt{u^2-1}} = \sec^{-1} u + C, \quad |u| > 1 \qquad\qquad \int \tan u\, du = -\ln|\cos u| + C \quad \text{or} \quad \ln|\sec u| + C$$

$$\int \cot u\, du = \ln|\sin u| + C \qquad\qquad \int \sec u\, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\, du = \ln|\csc u - \cot u| + C \qquad\qquad \int [f(u) + g(u)du] = \int f(u)du + \int g(u)du$$

$$\int k f(u)du = k \int f(u)du \qquad \text{if } k \text{ is a constant}$$

1. $\displaystyle\int (x^2 + a)^2 dx$ 2. $\displaystyle\int (x-3)^2 dx$

3. $\displaystyle\int x^{\frac{5}{2}} dx$ 4. $\displaystyle\int \frac{a\pi\, dx}{x}$

5. $\displaystyle\int [(x+1)^3 + e^x] dx$ 6. $\displaystyle\int \frac{x^2 e^x + x}{x^2} dx$

7. $\displaystyle\int 10 \sec^2 x\, dx$ 8. $\displaystyle\int 16(2^x + 1) dx$

9. $\displaystyle\int (\sin x + \cos x) dx$ 10. $\displaystyle\int (\sqrt{x} + \sec x \tan x) dx$

11. $\displaystyle\int (\sec^2 x + 1) dx$ 12. $\displaystyle\int \left(x^2 + \frac{3x^{\frac{2}{3}}}{2} + \frac{1}{x^2} \right) dx$

13. $\displaystyle\int \left(5x - \frac{4}{x^3} + \frac{3}{x^5} \right) dx$ 14. $\displaystyle\int \sqrt{x}\, dx$

15. $\displaystyle\int x\sqrt{x}\, dx$ 16. $\displaystyle\int \left(x\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx$

17. $\displaystyle\int \frac{x}{\sqrt{x}} dx$ 18. $\displaystyle\int \frac{x^5 + x^3 + 2}{1 + x^2} dx$ (*Hint:* divide)

19. $\displaystyle\int \frac{x^3 - a^3}{x - a} dx \qquad \left[\frac{x^3}{3} + \frac{a}{2}x^2 + a^2 + x + A \right]$ 20. $\displaystyle\int \frac{(x^3 + 27)dx}{x^2 - 3x + 9}$

21. $\displaystyle\int \frac{1 - x^2}{1 - x^4} dx \qquad [\tan^{-1} x + c]$ 22. $\displaystyle\int \left(x + \frac{1}{\sqrt{1-x^2}} \right) dx$

23. $\displaystyle\int \left(\frac{x\sqrt{1-x^2} + 4}{\sqrt{1-x^2}} \right) dx$ 24. $\displaystyle\int \left(\frac{x^3 - 8}{x^2 + 2x + 4} \right) dx$

25. $\displaystyle\int \frac{x^3 + x^2 - x - 1}{x - 1} dx$ 26. $\displaystyle\int \frac{d\theta}{2\theta}$

27. $\displaystyle\int \sqrt{2px}\,dx$

28. $\displaystyle\int \frac{d\theta}{\sec\theta}$

29. $\displaystyle\int (1+x^2)^{\frac{1}{2}}x\,dx$

30. $\displaystyle\int x^2(2x^3+1)^2dx$

31. $\displaystyle\int \frac{x\,dx}{(x^2-1)^{\frac{3}{2}}}$

32. $\displaystyle\int \frac{x\,dx}{3x^2+1}$

33. $\displaystyle\int \frac{(x+1)dx}{x^2+2x+2}$

34. $\displaystyle\int \frac{2dy}{3y-4}$

35. $\displaystyle\int \frac{4dt}{5t+2}$

36. $\displaystyle\int \frac{x^2dx}{x^3+4}$

37. $\displaystyle\int \frac{(2x-5)dx}{x^2-5x+3}$

38. $\displaystyle\int \frac{v\,dv}{6v^2-1}$

39. $\displaystyle\int \frac{x^3}{x-1}\,dx$

40. $\displaystyle\int \frac{(y-3)dy}{y^2-6y+1}$

41. $\displaystyle\int \frac{x+6}{(x+2)^2}\,dx \qquad \left[Hint:\ \frac{x+6}{(x+2)^2}=\frac{(x+2)+4}{(x+2)^2}\right]$

42. $\displaystyle\int \frac{y\,dy}{(1+y^2)^4}$

43. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

44. $\displaystyle\int \frac{v+3}{v-1}\,dv$

45. $\displaystyle\int x\sqrt{x+1}dx$

46. $\displaystyle\int \frac{x^3dx}{(x^4+16)^2}$

47. $\displaystyle\int \frac{dt}{1+\sqrt{t}}$

48. $\displaystyle\int \frac{(x^2+1)^2}{x^3}\,dx$

49. $\displaystyle\int \frac{(1-2x)^2}{x}\,dx$

50. $\displaystyle\int \frac{\cos\beta\,d\beta}{2+3\sin\beta}$

51. $\displaystyle\int \frac{\csc^2 x\,dx}{1+\cot x}$

52. $\displaystyle\int \frac{\sin 2t\,dt}{4-3\cos 2t}$

53. $\displaystyle\int \frac{\cos x}{\sin x}\,dx$

54. $\displaystyle\int \sin^2\theta\cos\theta\,d\theta$

55. $\displaystyle\int \sin\theta\sqrt{\cos\theta}\,d\theta$

56. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

57. $\displaystyle\int x\sin x^2\,dx$

58. $\displaystyle\int \frac{\sec^2 2t\,dt}{1+\tan 2t}$

59. $\displaystyle\int \frac{\cos x\,dx}{\sin^4 x}$

60. $\displaystyle\int \sin 4x\cos 4x\,dx$

61. $\displaystyle\int \frac{\sin t\,dt}{(4+\cos t)^{\frac{3}{2}}}$

62. $\displaystyle\int \sec^5 x\tan x\,dx$

63. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

64. $\displaystyle\int x^3e^{2x^2}\,dx$

65. $\displaystyle\int \frac{e^y\,dy}{e^y-1}$

66. $\displaystyle\int \frac{e^{-x}dx}{1+e^{-x}}$

67. $\displaystyle\int \frac{dx}{e^x+1}$

68. $\displaystyle\int \frac{e^x+1}{e^x-1}\,dx$

69. $\displaystyle\int \frac{dx}{e^{-x}+e^x}$

70. $\displaystyle\int e^t\sec e^t\tan e^t\,dt$

71. $\displaystyle\int \frac{\sin(\ln x)dx}{x}$

72. $\displaystyle\int e^{\sin\theta}\cos\theta\,d\theta$

73. $\displaystyle\int \frac{e^{\ln x}}{x}\,dx$

74. $\displaystyle\int e^{\ln x}\,dx$

75. $\displaystyle\int \frac{\ln x\,dx}{x}$

76. $\displaystyle\int e^x(4-e^x)^{\frac{3}{2}}dx$

77. $\displaystyle\int \frac{dx}{x(1+2\ln x)}$

78. $\displaystyle\int \frac{x^2\ln^3(1+x^3)}{1+x^3}\,dx$

79. $\displaystyle\int \frac{(1+2\ln x)^4}{x}\,dx$

80. $\displaystyle\int \tan\theta\ln(\cos\theta)d\theta$

81. $\displaystyle\int \frac{(\ln y)^{\frac{3}{2}}}{y}\,dy$

82. $\displaystyle\int \frac{dx}{\sqrt{6x-4x^2}}$

Solution to #82: Let $I=\displaystyle\int \frac{dx}{\sqrt{6x-4x^2}}$

$6x-4x^2=-4\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)+\frac{9}{4}=\frac{9}{4}-4\left(x-\frac{3}{4}\right)^2$

$\therefore I=\displaystyle\int \frac{dx}{\sqrt{\frac{9}{4}-4\left(x-\frac{3}{4}\right)^2}}$

Let $u=2\left(x-\frac{3}{4}\right)\ \ \therefore du=2dx$

Now $I=\frac{1}{2}\displaystyle\int \frac{du}{\sqrt{\frac{9}{4}-u^2}}=\frac{1}{2}\sin^{-1}\frac{u}{3/2}+C$

$\therefore I=\frac{1}{2}\sin^{-1}\frac{4x-3}{3}+C$

83. $\int \dfrac{x\,dx}{x^2 + x + 1}$

   Solution for #83:   Let $H = \int \dfrac{x\,dx}{x^2 + x + 1}$

   $x^2 + x + 1 = (x^2 + x + \frac{1}{4}) + \frac{3}{4} = (x + \frac{1}{2})^2 + \frac{3}{4}$;

   let $u = x + \frac{1}{2}$; $du = dx$;

   then $H = \int \dfrac{(u - \frac{1}{2})du}{u^2 + \frac{3}{4}} = \int \dfrac{u\,du}{u^2 + \frac{3}{4}} - \dfrac{1}{2}\int \dfrac{du}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

   In the first integral let $w = u^2 + \frac{3}{4}$ and we have $2u\,du = dw$

   $\therefore \int \dfrac{u\,du}{u^2 + \frac{3}{4}} = \dfrac{1}{2}\int \dfrac{dw}{w} = \dfrac{1}{2}\ln|w| + C_1$

   Also $\int \dfrac{du}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2u}{\sqrt{3}}\right) + C_2$

   $\therefore\ \ H = \frac{1}{2}\ln|x^2 + x + 1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

84. $\int \dfrac{dx}{x^2 + 3x + 5}$

85. $\int \dfrac{dx}{3x^2 - 12x + 16}$ $\left[\frac{\sqrt{3}}{6}\tan^{-1}\frac{\sqrt{3}}{2}(x-2) + C\right]$

86. $\int \dfrac{dx}{\sqrt{4x - x^2}}$

87. $\int \dfrac{dx}{\sqrt{9x - 4x^2}}$ $\left[\frac{1}{2}\sin^{-1}\frac{8x-9}{9} + C\right]$

88. $\int \dfrac{dx}{12x^2 + 56x + 72}$

89. $\int \dfrac{8dx}{9x^2 - 12x + 20}$ $\left[\frac{2}{3}\tan^{-1}\frac{3x-2}{4} + C\right]$

90. $\int \dfrac{dx}{\sqrt{8x - 25x^2}}$

91. $\int \dfrac{dx}{\sqrt{-x^2 - 2x + 8}}$ $\left[\sin^{-1}\frac{x+1}{3} + C\right]$

92. $\int \dfrac{dx}{2x^2 + 3x + 2}$

93. $\int \dfrac{x\,dx}{\sqrt{3 + 4x - 4x^2}}$

94. $\int \dfrac{x\,dx}{\sqrt{4x - x^2}}$ $\left[-\sqrt{4x - x^2} + \sin^{-1}\frac{x-2}{2} + C\right]$

95. $\int \dfrac{(3x - 1)dx}{9x^2 + 6x + 26}$

96. $\int \dfrac{x\,dx}{\sqrt{6 + x - 2x^2}}$

97. $\int \dfrac{dx}{(3x + 1)\sqrt{3x^2 + 2x - 5}}$ $\left[\frac{1}{4\sqrt{3}}\sec^{-1}\frac{4}{3x+1} + C\right]$

98. $\int \dfrac{dx}{(x + 1)\sqrt{x^2 + 2x - 8}}$

99. $\int \sin^{\frac{1}{2}}\theta \cos^3\theta\,d\theta$ $\left[\frac{2}{3}\sin^{\frac{3}{2}}\theta - \frac{2}{7}\sin^{\frac{7}{2}}\theta + C\right]$

100. $\int \cos^{-\frac{3}{2}}\theta \sin\theta\,d\theta$

101. $\int \cos^3 x \sin^3 x\,dx$

102. $\int \sin\theta \cos\theta\,d\theta$ $\left[\frac{1}{2}\sin^2\theta + C_1 = -\frac{1}{2}\cos^2\theta + C_2 = -\frac{1}{4}\cos 2\theta + C\right]$

103. $\int \cos^3 3\theta\,d\theta$

104. $\int \dfrac{\sin^3 x}{\cos^2 x}\,dx$

105. $\int \cos^3 x \csc^3 x\,dx$

106. $\int \sin^3 \dfrac{x}{2}\,dx$

107. $\int \tan^2 x \sec^4 x\,dx$

108. $\int \tan^3 \theta \sec^4 \theta\,d\theta$ $\left[\frac{1}{6}\sec^6\theta - \frac{1}{4}\sec^4\theta + C\right]$

109. $\int \tan^3 \theta \sec^{\frac{3}{2}}\theta\,d\theta$

110. $\int (\tan^2 \theta + \tan^4 \theta)d\theta$

111. $\int \cot^2 \dfrac{x}{3}\,dx$

112. $\int \csc^4 \alpha x\,dx$ $\left[-\frac{1}{\alpha}\cot\alpha x - \frac{1}{3\alpha}\cot\alpha x + C\right]$

113. $\int \cot x \csc^2 x \, dx$

114. $\int \sin^6 x \, dx$

115. $\int \sin^4 x \cos^2 x \, dx$

116. $\int \tan^2 x \cos^4 x \, dx$

117. $\int \sin^4 x \cos^4 x \, dx$

118. $\int \cos 2x \sin 3x \, dx$ $\qquad\qquad\qquad\qquad \left[\frac{1}{2}\sin 5x + \frac{1}{2}\sin x + C\right]$

119. $\int \cos 5x \cos 7x \, dx - \int \sin 5x \sin 7x \, dx$

120. $\int \sin^{-\frac{5}{2}} u \cos^3 u \, du$

11.2   Integration by Parts

121. $\int x \cos x \, dx$

122. $\int x \sin 2x \, dx$

123. $\int x^2 \sin 2x \, dx$

124. $\int \dfrac{x^2 \, dx}{\sec x}$

125. $\int \tan^{-1} x \, dx$

126. $\int x \tan^{-1} x \, dx$ $\qquad\qquad\qquad \left[\frac{1}{2}x^2 \tan^{-1} x + \frac{1}{2}\tan^{-1} x - \frac{1}{2}x + C\right]$

127. $\int \sin^{-1}\left(\dfrac{x}{a}\right) dx, \ a > 0$

128. $\int \sqrt{1 - x^2} \, dx$

129. $\int \ln x \, dx$

130. $\int x \ln x \, dx$

131. $\int x^2 \ln x \, dx$

132. $\int x^{-1} \ln x \, dx$

133. $\int \sec^3 \theta \, d\theta$

134. $\int \cos \sqrt{y} \, dy$

135. $\int \sqrt{y} \cos \sqrt{y} \, dy$ $\qquad\qquad \left[2y \sin\sqrt{y} + 4\sqrt{y}\cos\sqrt{y} - 4\sin\sqrt{y} + C\right]$

136. $\int \ln^2 x \, dx$

137. $\int e^x \sin x \, dx$

138. $\int e^{-2x} \cos x \, dx$

139. $\int 2^x \sin x \, dx$

140. $\int \dfrac{\sin^{-1}\sqrt{x} \, dx}{\sqrt{x}}$

141. $\int \tan^{-1}\sqrt{x} \, dx$ $\qquad\qquad\qquad \left[(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C\right]$

142. $\int x^3 e^{x^2} \, dx$

143. $\int x^3 e^{2x} \, dx$

144. $\int \theta \sec^2 a\theta \, d\theta$

145. $\int x^2 \sin^{-1} x \, dx$ $\qquad\qquad \left[\frac{1}{3}x^3 \sin^{-1} x + \frac{1}{9}(2+x^2)\sqrt{1-x^2} + C\right]$

146. $\int \sin^3 x \, dx$

147. $\int \cos 2x \sin x \, dx$

148. $\int x^2 e^{-x} \, dx$