Σύγκλιση ακολουθιών

Ορισμός: ΣΥΓΚΛΙΝΟΥΣΑ ΑΚΟΛΟΥΘΙΑ

Η ακολουθία $\{x_n\}_{n\in\mathbb{N}}$ συγκλίνει στο x $\lim\limits_{n\to\infty} x_n = x$ ή $x_n \underset{n\to\infty}{\longrightarrow} x$ αν για κάθε περιοχή του x "τελικά" όλοι οι όροι της ακολουθίας περιέχονται σε αυτή

$$\boxed{\lim_{n\to\infty} x_n = x} \equiv \boxed{\forall\, \epsilon > 0 \quad \exists\, N(\epsilon) > 0 \,:\, \forall\, n > N(\epsilon) \ \rightsquigarrow \ x_n \in B(x,\epsilon)}$$

Εψιλοντικός ορισμός

$$\boxed{\lim_{n\to\infty} x_n = x} \equiv \boxed{\forall\, \epsilon > 0 \quad \exists\, N(\epsilon) > 0 \,:\, \forall\, n > N(\epsilon) \ \rightsquigarrow \ |x_n - x| < \epsilon}$$

Μιά συγλίνουσα ακολουθία έχει **μόνο** ένα οριακό σημείο.

$$\{\lim_{n\to\infty} x_n = x\} \equiv \{\forall\, \epsilon > 0 \quad \exists\, N(\epsilon) > 0 \,:\, \forall\, n > N(\epsilon) \;\rightsquigarrow\; |x_n - x| < \epsilon\}$$

Παράδειγμα:

$$x_n = \frac{1}{n} \;\rightsquigarrow\; \lim_{n\to n} x_n = 0$$

ΠΡΟΧΕΙΡΟ

$$n > N \;\rightsquigarrow\; |x_n - x| = \left|\frac{1}{n} - 0\right| = \boxed{\underbrace{\frac{1}{n}}_{\text{φθίνουσα;}}} < \boxed{\frac{1}{N} = \epsilon}$$

$$\frac{1}{N} = \epsilon, \quad N(\epsilon) = \frac{1}{\epsilon}$$

$$\forall\, \epsilon > 0 \;\exists\, N(\epsilon) = \frac{1}{\epsilon} \,:\, \forall\, n > N(\epsilon) \;\rightsquigarrow\; |x_n| < \epsilon$$

$$\{\lim_{n\to\infty} x_n = x\} \equiv \{\forall\,\epsilon > 0 \quad \exists\,N(\epsilon) > 0\ :\ \forall\,n > N(\epsilon)\ \rightsquigarrow\ |x_n - x| < \epsilon\}$$

Παράδειγμα:

$$x_n = \frac{1}{n^2+\mathrm{e}^n} \ \rightsquigarrow\ \lim_{n\to n} x_n = 0$$

ΠΡΟΧΕΙΡΟ

$$n > N \ \rightsquigarrow\ |x_n - x| = \frac{1}{n^2+\mathrm{e}^n} \le \boxed{\underbrace{\frac{1}{\mathrm{e}^n}}_{\text{φθίνουσα;}}} < \boxed{\frac{1}{\mathrm{e}^N} = \epsilon}$$

$$\frac{1}{\mathrm{e}^N} = \epsilon \ \rightsquigarrow\ N(\epsilon) = \ln\frac{1}{\epsilon}$$

$$\forall\,\epsilon > 0\ \exists\,N(\epsilon) = \ln\frac{1}{\epsilon}\ :\ n > N(\epsilon)\ \rightsquigarrow\ |x_n| < \epsilon$$

$$a > 1, \rightsquigarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$$

$$1 + x > 0 \rightsquigarrow (1+x)^n \geq 1 + nx \text{ (Bernoulli)} \rightsquigarrow$$

$$a > 1 \rightsquigarrow a = \left(\underbrace{\sqrt[n]{a} - 1}_{x} + 1\right)^n \geq 1 + n\left(\underbrace{\sqrt[n]{a} - 1}_{x}\right) \rightsquigarrow$$

$$\boxed{\sqrt[n]{a} - 1 \leq \frac{a-1}{n}}$$

Πρόχειρο

$$n > N \rightsquigarrow \sqrt[n]{a} - 1 \leq \frac{a-1}{n} < \frac{a-1}{N} = \epsilon \rightsquigarrow N(\epsilon) = \frac{a-1}{\epsilon}$$

$$\forall \epsilon > 0 \; \exists N(\epsilon) = \frac{a-1}{\epsilon} :$$

$$n > N(\epsilon) \rightsquigarrow |\sqrt[n]{a} - 1| \leq \frac{|a-1|}{n} < \frac{|a-1|}{N} = \epsilon$$

$$\boxed{\lim_{n\to\infty} \frac{n!}{n^n} = 0}$$

$$\frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot\dots\cdot(n-2)\cdot(n-1)\cdot n}{n^n}$$

$$= \tfrac{1}{n}\cdot\tfrac{2}{n}\cdot(1-\tfrac{2}{n})\cdot(1-\tfrac{1}{n})$$

$$= \tfrac{1}{n}\cdot\prod_{k=1}^{n-2}\underbrace{\left(1-\frac{k}{n}\right)}_{<1} < \tfrac{1}{n}$$

Πρόχειρο

$$\frac{n!}{n^n} < \frac{1}{n} < \frac{1}{N} = \epsilon \;\rightsquigarrow\; N(\epsilon) = \frac{1}{\epsilon}$$

$$\forall\,\epsilon > 0\;\exists\, N(\epsilon) = \tfrac{1}{\epsilon}\;:$$

$$n > N(\epsilon)\;\rightsquigarrow\;\left|\frac{n!}{n^n}\right| \le \frac{1}{n} < \frac{1}{N(\epsilon)} = \epsilon$$

$$\boxed{\lim_{n\to\infty} \frac{n^3}{2^n} = 0}$$

$$2^n = \quad (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = \cdots + \binom{n}{4} + \cdots =$$

$$= \cdots + \tfrac{n(n-1)(n-2)(n-3)}{4!} + \cdots$$

$$2^n > \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$n > 6 \ \rightsquigarrow \ n - k > \frac{n}{2} \ \text{για } k = 1, 2, 3$$

$$2^n > \frac{n^4}{2^3\, 4!} \ \rightsquigarrow \ \frac{2^3 \cdot 4!}{n} > \frac{n^3}{2^n}$$

Πρόχειρο

$$\frac{n^3}{2^n} < \frac{2^3 \cdot 4!}{n} < \frac{2^3 \cdot 4!}{N} = \epsilon \ \rightsquigarrow \ N(\epsilon) = \frac{2^3 \cdot 4!}{\epsilon}$$

$$\forall \epsilon > 0 \quad \exists\, N(\epsilon) = \tfrac{2^3 \cdot 4!}{\epsilon} :$$

$$\forall\, n > N(\epsilon) \ \rightsquigarrow \ \left|\tfrac{n^3}{2^n}\right| < \tfrac{2^3 \cdot 4!}{n} < \tfrac{2^3 \cdot 4!}{N(\epsilon)} = \epsilon$$

ΜΗΔΕΝΙΚΕΣ ΑΚΟΛΟΥΘΙΕΣ

Ορισμός

$\{x_n\}_{n\in\mathbb{N}}$ είναι **μηδενική ακολουθία** $\Leftrightarrow \lim_{n\to\infty} x_n = 0$ ή $x_n \xrightarrow[n\to\infty]{} 0$

$\Leftrightarrow \forall\, \epsilon > 0,\ \exists\, N(\epsilon) > 0 \,:\, \forall\, n > N(\epsilon) \ \rightsquigarrow\ |x_n| < \epsilon$

Μηδ. 1: $x_n \xrightarrow[n\to\infty]{} 0 \Leftrightarrow |x_n| \xrightarrow[n\to\infty]{} 0$

Μηδ. 2: $x_n \xrightarrow[n\to\infty]{} 0 \Leftrightarrow \lambda\, x_n \xrightarrow[n\to\infty]{} 0$

Μηδ. 3: $\left\{ x_n \xrightarrow[n\to\infty]{} 0 \text{ και } y_n \xrightarrow[n\to\infty]{} 0 \right\} \Rightarrow \left\{ x_n + y_n \xrightarrow[n\to\infty]{} 0 \right\}$

Μηδ. 4: $\left\{ x_n \xrightarrow[n\to\infty]{} 0 \text{ και } y_n \xrightarrow[n\to\infty]{} 0 \right\} \Rightarrow \left\{ x_n \cdot y_n \xrightarrow[n\to\infty]{} 0 \right\}$

Μηδ. 5: $\left\{ |x_n| \leq |y_n| \text{ και } y_n \xrightarrow[n\to\infty]{} 0 \right\} \Rightarrow x_n \xrightarrow[n\to\infty]{} 0$

Μηδ. 6:

$x_n \xrightarrow[n\to\infty]{} 0 \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ φραγμένη $\Leftrightarrow \exists\, C > 0 \,:\, \forall\, n \rightsquigarrow |x_n| < C$

ΣΥΓΚΛΙΝΟΥΣΕΣ ΑΚΟΛΟΥΘΙΕΣ

Ορισμός συγκλίνουσας ακολουθίας

$$\{\ \{x_n\}_{n\in\mathbb{N}} \text{ συγκλίνει στο } x\ \} \Leftrightarrow \{\ \lim_{n\to\infty} x_n = x \text{ ή } x_n \underset{n\to\infty}{\longrightarrow} x\}$$
$$\Leftrightarrow \{\forall\,\epsilon > 0,\ \exists\, N(\epsilon) > 0\ :\ n > N(\epsilon)\ \rightsquigarrow\ |x_n - x| < \epsilon\}$$
$$\Leftrightarrow \{\ \{(x_n - x)\}_{n\in\mathbb{N}} \text{ μηδενική ακολουθία}\}$$

Πρ. 1: $x_n \underset{n\to\infty}{\longrightarrow} x \Rightarrow |x_n| \underset{n\to\infty}{\longrightarrow} |x|$

Πρ. 2: $x_n \underset{n\to\infty}{\longrightarrow} x \Rightarrow \lambda\, x_n \underset{n\to\infty}{\longrightarrow} \lambda x$

Πρ. 3: $x_n \underset{n\to\infty}{\longrightarrow} x \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ φραγμένη $\Leftrightarrow$
$\exists\, C,\ D > 0$ και $N > 0 : \forall\, n > N\ \rightsquigarrow\ D < |x_n| < C$

Πρ. 4: $\left\{x_n \underset{n\to\infty}{\longrightarrow} x \text{ και } y_n \underset{n\to\infty}{\longrightarrow} y\right\} \Rightarrow \left\{x_n + y_n \underset{n\to\infty}{\longrightarrow} x + y\right\}$

Πρ. 5: $\left\{x_n \underset{n\to\infty}{\longrightarrow} x \text{ και } y_n \underset{n\to\infty}{\longrightarrow} y\right\} \Rightarrow \left\{x_n \cdot y_n \underset{n\to\infty}{\longrightarrow} x \cdot y\right\}$

Πρ. 6α: $\left\{0 \le x_n \text{ και } x_n \underset{n\to\infty}{\longrightarrow} x\right\} \Rightarrow \{0 \le x\}$

Πρ. 6β: $\left\{x_n \le y_n \text{ και } x_n \underset{n\to\infty}{\longrightarrow} x, \quad y_n \underset{n\to\infty}{\longrightarrow} y\right\} \Rightarrow \{x \le y\}$

Πρ. 7:
$$\left\{\begin{array}{c} z_n \le x_n \le y_n \\ \text{και} \\ z_n \underset{n\to\infty}{\longrightarrow} a, \quad y_n \underset{n\to\infty}{\longrightarrow} a \end{array}\right\} \Rightarrow \left\{x_n \underset{n\to\infty}{\longrightarrow} a\right\}$$

Πρ. 8α: $\left\{x_n \underset{n\to\infty}{\longrightarrow} x \text{ και } x \neq 0\right\} \Rightarrow \left\{\frac{1}{x_n} \underset{n\to\infty}{\longrightarrow} \frac{1}{x}\right\}$

Πρ. 8β: $\left\{x_n \underset{n\to\infty}{\longrightarrow} x, \; y_n \underset{n\to\infty}{\longrightarrow} y \text{ και } y \neq 0, \right\} \Rightarrow \left\{\frac{x_n}{y_n} \underset{n\to\infty}{\longrightarrow} \frac{x}{y}\right\}$

Πρ. 9α: $\{|a| < 1\} \Rightarrow \left\{a^n \underset{n\to\infty}{\longrightarrow} 0\right\}$

Πρ. 9β: $\left\{x_n \neq 0, \; \left|\frac{x_{n+1}}{x_n}\right| \underset{n\to\infty}{\longrightarrow} k < 1\right\} \rightsquigarrow \left\{x_n \underset{n\to\infty}{\longrightarrow} 0\right\}$

ΜΗΔΕΝΙΚΕΣ ΑΚΟΛΟΥΘΙΕΣ

Ορισμός

$\{x_n\}_{n\in\mathbb{N}}$ είναι **μηδενική ακολουθία** $\Leftrightarrow \lim_{n\to\infty} x_n = 0$ ή $x_n \underset{n\to\infty}{\longrightarrow} 0$

$\Leftrightarrow$ $\forall\, \epsilon > 0,\ \exists\, N(\epsilon) > 0\ :\ \forall\, n > N(\epsilon)\ \rightsquigarrow\ |x_n| < \epsilon$

Πρ: $x_n \underset{n\to\infty}{\longrightarrow} 0 \Leftrightarrow |x_n| \underset{n\to\infty}{\longrightarrow} 0$

Πρ: $x_n \underset{n\to\infty}{\longrightarrow} 0 \Leftrightarrow \lambda\, x_n \underset{n\to\infty}{\longrightarrow} 0$

☐ *Απόδ*:

$$\left\{x_n \underset{n\to\infty}{\longrightarrow} 0\right\} \Leftrightarrow \left\{ \begin{array}{c} \forall\, \epsilon > 0,\ \exists\, N(\epsilon) > 0\ : \\ n > N(\epsilon)\ \rightsquigarrow\ |x_n| < \epsilon \end{array} \right\}$$

$$\left(\begin{array}{l} \text{Πρόχειρο:} \\ |x_n| < \epsilon\ \rightsquigarrow\ |\lambda\, x_n| = |\lambda|\,|x_n| < |\lambda|\epsilon = \epsilon' \end{array} \right)$$

$$\left\{ \begin{array}{c} \forall\, \epsilon' > 0, \\ \exists\, N'(\epsilon') = N\left(\frac{\epsilon'}{|\lambda|}\right) = N(\epsilon) > 0\ : \\ n > N'(\epsilon') = N(\epsilon)\ \rightsquigarrow\ |x_n| < \epsilon\ \rightsquigarrow\ |\lambda\, x_n| < \epsilon' \end{array} \right\}$$

$$\rightsquigarrow\ \left\{|\lambda\, x_n| \underset{n\to\infty}{\longrightarrow} 0\right\}$$

Πρ:

$$\left\{ \begin{array}{c} x_n \underset{n\to\infty}{\longrightarrow} 0 \\ \\ y_n \underset{n\to\infty}{\longrightarrow} 0 \end{array} \right\} \Rightarrow \; x_n + y_n \underset{n\to\infty}{\longrightarrow} 0$$

□ *Απόδ*:

$$\left\{x_n \underset{n\to\infty}{\longrightarrow} 0\right\} \Leftrightarrow \left\{ \begin{array}{c} \forall\,\epsilon > 0,\ \exists\,N_1(\epsilon) > 0\ : \\ n > N_1(\epsilon)\ \rightsquigarrow\ |x_n| < \epsilon \end{array} \right\}$$

$$\left\{y_n \underset{n\to\infty}{\longrightarrow} 0\right\} \Leftrightarrow \left\{ \begin{array}{c} \forall\,\epsilon > 0,\ \exists\,N_2(\epsilon) > 0\ : \\ n > N_2(\epsilon)\ \rightsquigarrow\ |y_n| < \epsilon \end{array} \right\}$$

(Πρόχειρο $: \ |x_n + y_n| \leq |x_n| + |y_n| < 2\epsilon = \epsilon'$)

$$\left\{ \begin{array}{c} \forall\,\epsilon' > 0,\ \exists\,N'(\epsilon') = \max\left\{N_1\left(\frac{\epsilon'}{2}\right),\ N_2\left(\frac{\epsilon'}{2}\right)\right\} > 0\ : \\ n > N'(\epsilon') \geq N_1\left(\frac{\epsilon'}{2}\right) = N_1(\epsilon)\ \rightsquigarrow\ |x_n| < \frac{\epsilon'}{2} \\ n > N'(\epsilon') \geq N_2\left(\frac{\epsilon'}{2}\right) = N_2(\epsilon)\ \rightsquigarrow\ |y_n| < \frac{\epsilon'}{2} \\ \text{Αρα}\ \ |x_n + y_n| = |x_n| + |y_n| < \epsilon' \end{array} \right\}$$

Επομένως $x_n + y_n \underset{n\to\infty}{\longrightarrow} 0$ □

Πρ:

$$\left\{\begin{array}{c} x_n \underset{n\to\infty}{\longrightarrow} 0 \\ \\ y_n \underset{n\to\infty}{\longrightarrow} 0 \end{array}\right\} \Rightarrow x_n \cdot y_n \underset{n\to\infty}{\longrightarrow} 0$$

□ *Απόδ*:

$$\left\{x_n \underset{n\to\infty}{\longrightarrow} 0\right\} \Leftrightarrow \left\{\begin{array}{c} \forall\,\epsilon > 0,\ \exists\, N_1(\epsilon) > 0\ : \\ n > N_1(\epsilon) \ \leadsto\ |x_n| < \epsilon \end{array}\right\}$$

$$\left\{y_n \underset{n\to\infty}{\longrightarrow} 0\right\} \Leftrightarrow \left\{\begin{array}{c} \forall\,\epsilon > 0,\ \exists\, N_2(\epsilon) > 0\ : \\ n > N_2(\epsilon) \ \leadsto\ |y_n| < \epsilon \end{array}\right\}$$

(Πρόχειρο : $|x_n \cdot y_n| \le |x_n| \cdot |y_n| < \epsilon^2 = \epsilon'$)

$$\left\{\begin{array}{c} \forall\,\epsilon' > 0,\ \exists\, N'(\epsilon') = \max\left\{N_1\left(\sqrt{\epsilon'}\right),\, N_2\left(\sqrt{\epsilon'}\right)\right\} > 0\ : \\ n > N'(\epsilon') \ge N_1\left(\sqrt{\epsilon'}\right) = N_1(\epsilon) \ \leadsto\ |x_n| < \sqrt{\epsilon'} \\ n > N'(\epsilon') \ge N_2\left(\sqrt{\epsilon'}\right) = N_2(\epsilon) \ \leadsto\ |y_n| < \sqrt{\epsilon'} \\ \text{Άρα}\ \ |x_n \cdot y_n| = |x_n| \cdot |y_n| < \epsilon' \end{array}\right\}$$

Επομένως $x_n \cdot y_n \underset{n\to\infty}{\longrightarrow} 0$ □

Πρ:

$$\boxed{\left\{\begin{array}{c} |x_n| \leq |y_n| \\ \text{και} \\ y_n \underset{n\to\infty}{\longrightarrow} 0 \end{array}\right\} \Rightarrow \; x_n \underset{n\to\infty}{\longrightarrow} 0}$$

☐ *Απόδ*:

$$\left\{ y_n \underset{n\to\infty}{\longrightarrow} 0 \right\} \Leftrightarrow \left\{ \begin{array}{c} \forall\, \epsilon > 0,\; \exists\, N(\epsilon) > 0 \;: \\ n > N(\epsilon) \;\rightsquigarrow\; |x_n| \leq |y_n| < \epsilon \end{array} \right\}$$

☐

Πρ:

$$\boxed{\begin{array}{c} x_n \underset{n\to\infty}{\longrightarrow} 0 \;\Rightarrow \\ \{x_n\}_{n\in\mathbb{N}} \;\; \text{φραγμένη} \;\Leftrightarrow \\ \exists\, C > 0 \,:\, \forall\, n \;\rightsquigarrow\; |x_n| < C \end{array}}$$

☐ *Απόδ*: $x_n \underset{n\to\infty}{\longrightarrow} 0 \;\rightsquigarrow\; \epsilon = 1,\; \exists\, N(1) \;:\; n > N(1) \;\rightsquigarrow\; |x_n| < 1$

$$C = 1 + \max\{|x_k|,\; k \leq x_n\} \;\Rightarrow\; \forall\, n \;\rightsquigarrow\; |x_n| < C$$

☐

ΣΥΓΚΛΙΝΟΥΣΕΣ ΑΚΟΛΟΥΘΙΕΣ

Ορισμός

$$\{x_n\}_{n\in\mathbb{N}} \text{ συγκλίνει στο } x$$
$$\Updownarrow$$
$$\lim_{n\to\infty} x_n = x \text{ ή } x_n \underset{n\to\infty}{\longrightarrow} x$$
$$\Updownarrow$$
$$\forall \epsilon > 0,\ \exists N(\epsilon) > 0 : n > N(\epsilon) \rightsquigarrow |x_n - x| < \epsilon$$

Πρ. 1: $x_n \underset{n\to\infty}{\longrightarrow} x \Rightarrow |x_n| \underset{n\to\infty}{\longrightarrow} |x|$

□ *Απόδ*:

$$\left\{x_n \underset{n\to\infty}{\longrightarrow} x\right\} \Leftrightarrow \left\{ \begin{array}{c} \forall \epsilon > 0,\ \exists N(\epsilon) > 0 : \\ n > N(\epsilon) \rightsquigarrow |x_n - x| < \epsilon \end{array} \right\}$$

Επειδή $|\,(|x_n| - |x|)\,| \leq |x_n - x| < \epsilon$

$$\left\{ \begin{array}{c} \forall \epsilon > 0,\ \exists N(\epsilon) > 0 : \\ n > N(\epsilon) \rightsquigarrow |\,(|x_n| - |x|)\,| \leq |x_n - x| < \epsilon \end{array} \right\} \Rightarrow$$

$$\Rightarrow |x_n| \underset{n\to\infty}{\longrightarrow} |x|$$

□

Πρ. 2: $x_n \underset{n\to\infty}{\longrightarrow} x \Rightarrow \lambda x_n \underset{n\to\infty}{\longrightarrow} \lambda x$

Πρ. 3:
$$x_n \underset{n\to\infty}{\longrightarrow} x \Rightarrow \{x_n\}_{n\in\mathbb{N}} \text{ φραγμένη} \Leftrightarrow$$
$$\exists C, D > 0 \text{ και } N > 0 : \forall n > N \rightsquigarrow D < |x_n| < C$$

□ *Απόδ*:

$$\left\{x_n \underset{n\to\infty}{\longrightarrow} x\right\} \Rightarrow \left\{ \begin{array}{c} \text{για } \epsilon = \frac{|x|}{2},\ \exists N\left(\frac{|x|}{2}\right) > 0 : \\ n > N\left(\frac{|x|}{2}\right) \rightsquigarrow |x_n - x| < \frac{|x|}{2} \end{array} \right\}$$

Για $n > N\left(\frac{|x|}{2}\right)$ η τριγωνική ιδιότητα δίνει

$$|x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| < \frac{3}{2}|x| = C$$

$$D = \frac{|x|}{2} < |x| - |(x_n - x)| \leq |(x_n - x) + x| = |x_n|$$

□

Πρ. 4:
$$\left\{ \begin{array}{c} x_n \underset{n\to\infty}{\longrightarrow} x \\ \\ y_n \underset{n\to\infty}{\longrightarrow} y \end{array} \right\} \Rightarrow x_n + y_n \underset{n\to\infty}{\longrightarrow} x + y$$

Πρ.

5: $$\left\{\begin{array}{c} x_n \underset{n\to\infty}{\longrightarrow} x \\ \\ y_n \underset{n\to\infty}{\longrightarrow} y \end{array}\right\} \Rightarrow x_n \cdot y_n \underset{n\to\infty}{\longrightarrow} x \cdot y$$

□ *Απόδ*:

$$\left\{x_n \underset{n\to\infty}{\longrightarrow} x\right\} \Leftrightarrow \left\{\begin{array}{c} \forall\, \epsilon > 0,\ \exists\, N_1(\epsilon) > 0\ : \\ n > N_1(\epsilon)\ \rightsquigarrow\ |x_n - x| < \epsilon \end{array}\right\}$$

$$\text{Πρ. 3}\ \rightsquigarrow\ \forall\, n > N_1\left(\frac{|x|}{2}\right)\ \rightsquigarrow\ |x_n - x| < \frac{3|x|}{2}$$

$$\left\{y_n \underset{n\to\infty}{\longrightarrow} y\right\} \Leftrightarrow \left\{\begin{array}{c} \forall\, \epsilon > 0,\ \exists\, N_2(\epsilon) > 0\ : \\ n > N_2(\epsilon)\ \rightsquigarrow\ |y_n - y| < \epsilon \end{array}\right\}$$

Πρόχειρο:

$$\begin{aligned} |x_n y_n - xy| = &\ |x_n(y_n - y) + (x_n - x)y| \leq \\ \leq &\ |x_n| \cdot |y_n - y| + |x_n - x| \cdot |y| \leq \\ \leq &\ \tfrac{3|x|}{2}\epsilon + |y|\epsilon = \epsilon' \end{aligned}$$

$$\forall\, \epsilon'\ \exists\, N'(\epsilon') =$$
$$= \max\{N_1\left(\tfrac{\epsilon'}{|y|+3|x|/2}\right),\ N_2\left(\tfrac{\epsilon'}{|y|+3|x|/2}\right).N_1\left(\tfrac{|x|}{2}\right)\}$$
$$n > N'(\epsilon')\ \rightsquigarrow\ |x_n y_n - xy| < \epsilon'$$
$$\Rightarrow\ x_n y_n \underset{n\to\infty}{\longrightarrow} xy$$

□

Πρ.

6α: $$\left\{\begin{array}{c} 0 \le x_n \\ \text{και} \\ x_n \underset{n\to\infty}{\longrightarrow} x \end{array}\right\} \Rightarrow 0 \le x$$

□ *Απόδ*: Εστω $x < 0 \rightsquigarrow \exists N\left(\frac{|x|}{2}\right)$:

$n > N\left(\frac{|x|}{2}\right) \rightsquigarrow |x_n - x| < \frac{|x|}{2} \rightsquigarrow x_n < 0$ άτοπο. □

Πρ.

6β: $$\left\{\begin{array}{c} x_n \le y_n \\ \text{και} \\ x_n \underset{n\to\infty}{\longrightarrow} x, \quad y_n \underset{n\to\infty}{\longrightarrow} y \end{array}\right\} \Rightarrow x \le y$$

Πρ.

7: $$\left\{\begin{array}{c} z_n \le x_n \le y_n \\ \text{και} \\ z_n \underset{n\to\infty}{\longrightarrow} a, \quad y_n \underset{n\to\infty}{\longrightarrow} a \end{array}\right\} \Rightarrow x_n \underset{n\to\infty}{\longrightarrow} a$$

□ *Απόδ*: Είναι μια άμεση εφαρμογή της προηγούμενης πρότασης.
Αλλη απόδειξη με εψιλοντικό ορισμό:

$$\left\{y_n \underset{n\to\infty}{\longrightarrow} a\right\} \Leftrightarrow \left\{\begin{array}{c} \forall\,\epsilon > 0,\ \exists\,N_1(\epsilon) > 0\ : \\ n > N_1(\epsilon) \rightsquigarrow a - \epsilon < y_n < a + \epsilon \end{array}\right\}$$

$$\left\{z_n \underset{n\to\infty}{\longrightarrow} a\right\} \Leftrightarrow \left\{\begin{array}{c} \forall\,\epsilon > 0,\ \exists\,N_2(\epsilon) > 0\ : \\ n > N_2(\epsilon) \rightsquigarrow a - \epsilon < z_n < a + \epsilon \end{array}\right\}$$

$$n > \max\{N_1(\epsilon),\, N_2(\epsilon)\} \rightsquigarrow a - \epsilon < z_n \le x_n \le y_n < a + \epsilon$$

$$\Rightarrow x_n \underset{n\to\infty}{\longrightarrow} a$$

□

Πρ.

8: $x_n \underset{n\to\infty}{\longrightarrow} x$ και $x \neq 0 \rightsquigarrow \dfrac{1}{x_n} \underset{n\to\infty}{\longrightarrow} \dfrac{1}{x}$

Πρ. 9α: $|a| < 1 \Rightarrow a^n \underset{n\to\infty}{\longrightarrow} 0$

□ *Απόδ*: $|a| < 1 \rightsquigarrow |a| = \frac{1}{1+x},\ x > 0 \rightsquigarrow$
$(1+x)^n \geq 1 + nx \rightsquigarrow \frac{1}{|a|^n} \geq 1 + n(\frac{1}{|a|} - 1) \rightsquigarrow$
$|a|^n \leq \frac{1}{1+n(\frac{1}{|a|}-1)} \rightsquigarrow |a|^n \underset{n\to\infty}{\longrightarrow} 0$ □ Πρ.

9β: $x_n \neq 0,\ |\frac{x_{n+1}}{x_n}| \underset{n\to\infty}{\longrightarrow} k < 1 \rightsquigarrow x_n \underset{n\to\infty}{\longrightarrow} 0$

□ *Απόδ*:

$$
\begin{gathered}
|\tfrac{x_{n+1}}{x_n}| \underset{n\to\infty}{\longrightarrow} k < 1 \Rightarrow \\
\epsilon = \tfrac{1-k}{2},\ \exists N\left(\tfrac{1-k}{2}\right) : \\
n > N\left(\tfrac{1-k}{2}\right) > \left[N\left(\tfrac{1-k}{2}\right)\right] = m \rightsquigarrow \\
|\,|\tfrac{x_{n+1}}{x_n}| - k| < \tfrac{1-k}{2} \rightsquigarrow |\tfrac{x_{n+1}}{x_n}| < k + \tfrac{1-k}{2} = \tfrac{1+k}{2}
\end{gathered}
$$

$$
\begin{aligned}
&|x_{m+1}| < \tfrac{1+k}{2}|x_m| \\
&|x_{m+2}| < \tfrac{1+k}{2}|x_{m+1} < \left(\tfrac{1+k}{2}\right)^2 |x_m| \\
&\cdots\cdots \\
&|x_n| = |x_{m+(n-m)}| < \\
&< \left(\tfrac{1+k}{2}\right)^{n-m} |x_m| = \left(\tfrac{1+k}{2}\right)^n \cdot \left(\tfrac{1+k}{2}\right)^{-m} |x_m|
\end{aligned}
$$

$$
\lim_{n\to 0}|x_n| \leq \underbrace{\lim_{n\to 0}\left(\frac{1+k}{2}\right)^n}_{=0} \cdot \left(\frac{1+k}{2}\right)^{-m} |x_m| = 0
$$

□

Διάφορες ιδιότητες ακολουθιών

Ιδ. 1 : $a > 0, \ \sqrt[n]{a} \underset{n\to\infty}{\longrightarrow} 1$

Ιδ. 2 : $\sqrt[n]{n} \underset{n\to\infty}{\longrightarrow} 1$

Ιδ. 3 : $x_n > 0, \ x_n \underset{n\to\infty}{\longrightarrow} x \ \rightsquigarrow \ \sqrt[k]{x_n} \underset{n\to\infty}{\longrightarrow} \sqrt[k]{x}$

Ιδ. 4 : $P(n) = a_0 n^p + a_1 n^{p-1} + \cdots + a_{p-1} n + a_p,$

$Q(n) = b_0 n^q + b_1 n^{q-1} + \cdots + a_{q-1} n + a_q$

Αν $p = q$ τότε $\dfrac{P(n)}{Q(n)} \underset{n\to\infty}{\longrightarrow} \dfrac{a_0}{b_0}$

Αν $p < q$ τότε $\dfrac{P(n)}{Q(n)} \underset{n\to\infty}{\longrightarrow} 0$