$$\boxed{\mathrm{e}^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

για $\boxed{0 \le y < x < \infty}$

$$\rightsquigarrow \boxed{(x-y)\mathrm{e}^y < \mathrm{e}^x - \mathrm{e}^y < (x-y)\mathrm{e}^x}$$

$$\rightsquigarrow \mathrm{e}^y < \frac{\mathrm{e}^x - \mathrm{e}^y}{x-y} < \mathrm{e}^x$$

$$\Rightarrow (\mathrm{e}^x)'_- = \lim_{y \to x^-} \frac{\mathrm{e}^x - \mathrm{e}^y}{x-y} = \mathrm{e}^x$$

$$x \leftrightarrow y \rightsquigarrow (\mathrm{e}^x)'_+ = \mathrm{e}^x$$

$$\Rightarrow \boxed{\frac{d\,\mathrm{e}^x}{dx} = \mathrm{e}^x}$$

$$\rightsquigarrow (x-y)\mathrm{e}^y < \mathrm{e}^x - \mathrm{e}^y < (x-y)\mathrm{e}^x$$

$$\rightsquigarrow \boxed{(x-y)\mathrm{e}^{-x} < \mathrm{e}^{-y} - \mathrm{e}^{-x} < (x-y)\mathrm{e}^{-y}}$$

$$x > 0\,, y = \ln\, x = \exp^{-1} x \overset{1:1}{\longleftrightarrow} x = \exp\, y = \mathrm{e}^{y},\ y > 0$$

$$\frac{d}{dx}(\ln\, x) = \frac{d \exp^{-1} x}{dx} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}(\ln\, x)} = \lim_{x' \to x} \frac{\ln\, x - \ln\, x'}{x - x'} =$$

$$= \lim_{y' \to y} \frac{y - y'}{\exp\, y - \exp\, y'} =$$

$$= \lim_{y' \to y} \frac{1}{\frac{\exp\, y - \exp\, y'}{y - y'}} = \boxed{\frac{1}{\frac{d}{dy}(\exp\, y)}} = \frac{1}{\exp\, y} =$$

$$= \boxed{\frac{1}{x}}$$

Παράδειγμα $|x| \leq 1$

$$y = \arcsin\, x = \sin^{-1} x \overset{1:1}{\longleftrightarrow} x = \sin\, y, \qquad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\frac{d}{dx}(\arcsin\, x) = \frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}(\arcsin\, x)} = \lim_{x' \to x} \frac{\arcsin\, x - \arcsin\, x'}{x - x'} =$$

$$= \lim_{y' \to y} \frac{y - y'}{\sin\, y - \sin\, y'} =$$

$$= \lim_{y' \to y} \frac{1}{\frac{\sin\, y - \sin\, y'}{y - y'}} = \boxed{\frac{1}{\frac{d}{dy}(\sin\, y)}} =$$

$$= \frac{1}{\cos\, y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

Παράδειγμα $|x| \leq \infty$

$$y = \arctan x = \tan^{-1} x \overset{1:1}{\longleftrightarrow} x = \tan y, \; -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\frac{d}{dx}(\arctan x) = \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}$$

$$\boxed{\frac{d}{dx}(\arctan x)} = \lim_{x' \to x} \frac{\arctan x - \arctan x'}{x - x'} =$$

$$= \lim_{y' \to y} \frac{y - y'}{\tan y - \sin y'} =$$

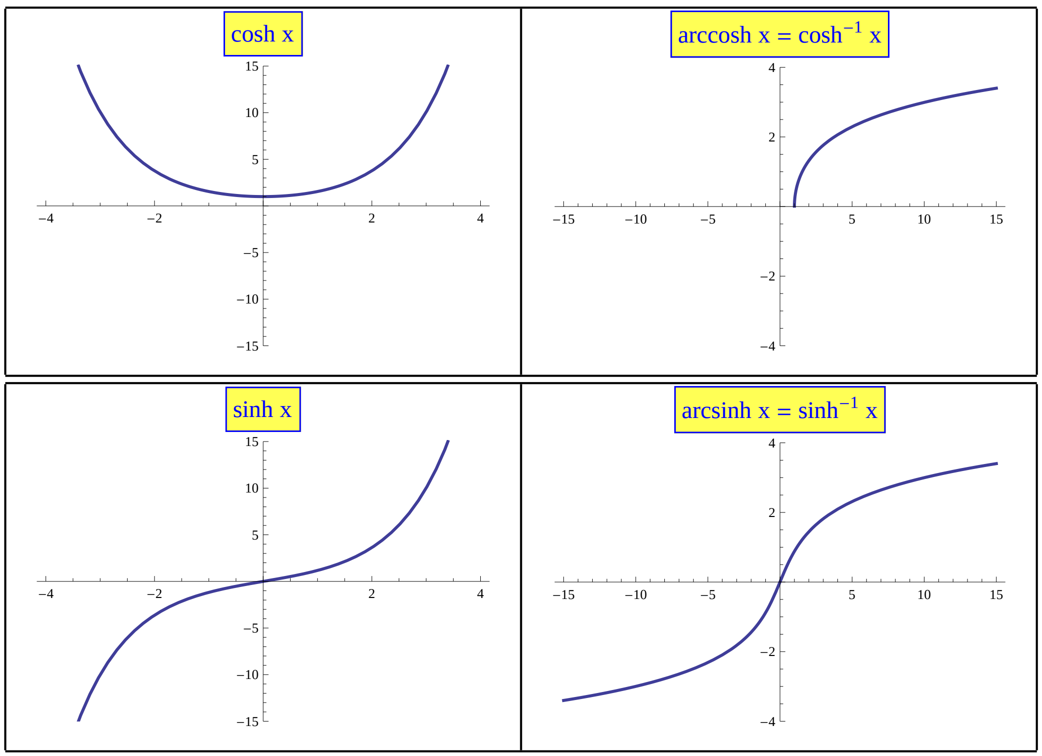
$$= \lim_{y' \to y} \frac{1}{\frac{\tan y - \tan y'}{y - y'}} = \boxed{\frac{1}{\frac{d}{dy}(\tan y)}} =$$

$$= \frac{1}{\frac{1}{\cos^2 y}} = \frac{1}{1 + \tan^2 y} = \boxed{\frac{1}{1+x^2}}$$

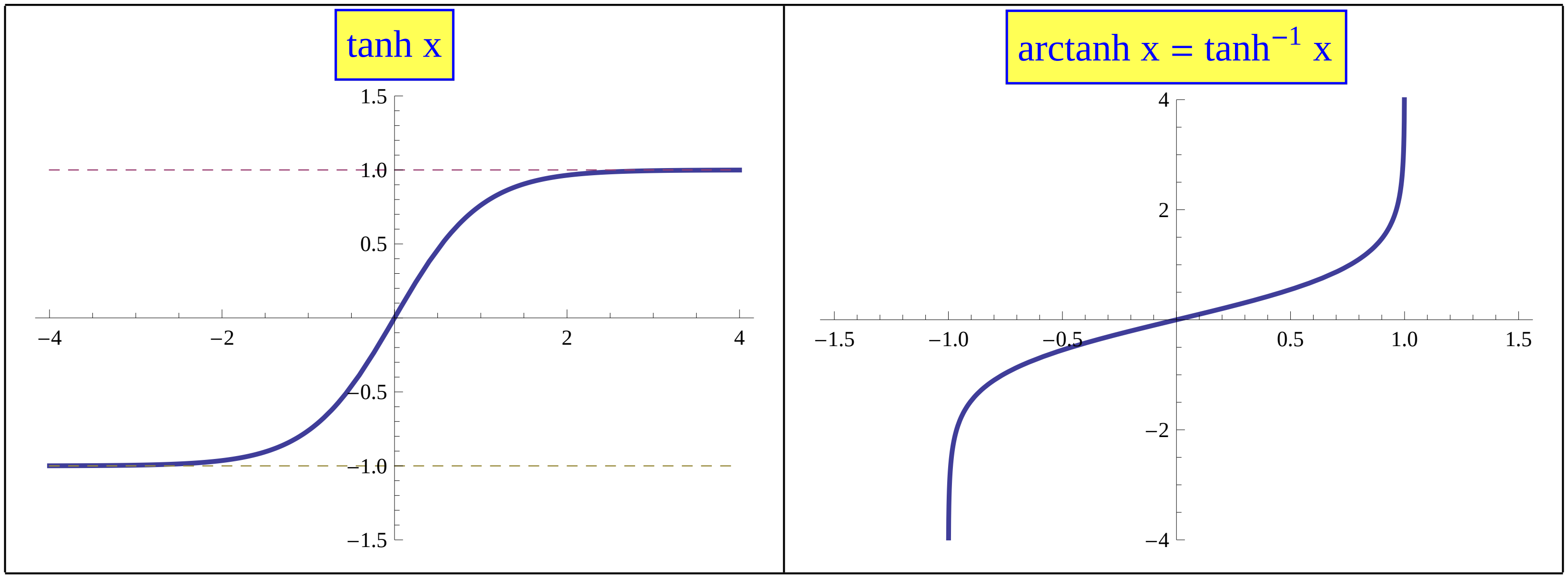
ΥΠΕΡΒΟΛΙΚΕΣ ΣΥΝΑΡΤΗΣΕΙΣ

$$\cosh x \equiv \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \sinh x \equiv \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh^2 x - \sinh^2 x = 1$$



$$\tanh\, x \equiv \frac{\sinh\, x}{\cosh\, x}$$



$$\frac{d\cosh\ x}{dx} = \sinh\ x, \quad \frac{d\sinh\ x}{dx} = \cosh\ x$$

$$\frac{d\cosh^{-1}\ x}{dx} = \frac{d\,\mathrm{arccosh}\,x}{dx} = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d\sinh^{-1}\ x}{dx} = \frac{d\,\mathrm{arcsinh}\,x}{dx} = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d\tanh\ x}{dx} = \frac{1}{\cosh^2 x}$$

$$\frac{d\tanh^{-1} x}{dx} = \frac{d\,\mathrm{arctanh}\,x}{dx} = \frac{1}{1-x^2}, \quad |x| < 1$$

$$\frac{d\coth\ x}{dx} = -\frac{1}{\sinh^2 x}$$

$$\frac{d\coth^{-1} x}{dx} = \frac{d\,\mathrm{arccoth}\,x}{dx} = \frac{1}{1-x^2}, \quad |x| >$$

ΤΥΠΟΣ *NEWTON*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \, y^{n-k}$$

ΤΥΠΟΣ *LEIBNITZ*

$$(f\,g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \, g^{(n-k)}$$